How does the energy principle applied to fluid flow?

One may think of fluid flow as many particles moving along some path, which is made of a consecutive pipes smoothly joining together where each segment of the pipes can have different cross sections and are at various heights. Bernoulli's principle concerns a steady flow of ideal fluid, where the fluid is incompressible, i.e. the fluid density is constant throughout the flow, i.e.

$$\rho = \Delta m / \Delta V = const. \tag{1}$$

The flow is steady in that it is smooth, i.e. nonturbulent and the amount of fluid flow, either the amount of mass or the amount of volume, at any point along the flow is constant, i.e.

$$\Delta m / \Delta t = const, \quad \text{or} \quad \Delta V / \Delta t = const \tag{2}$$

The energy principle states that the the change of energy of a system is due to the work done by surrounding on the system. See MI text page 245 of fig 6.24-MI.

$$\Delta E_{sys} = W. \tag{3}$$

Let us see how the energy principle leads to Bernoulli's principle of eq(1)-manual.

We first consider the system taken to be just one horizontal segment. It is bordered by the two dotted lines, see Figure.

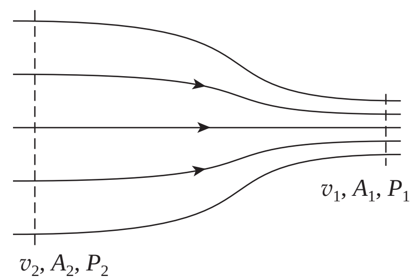

Figure 1: Figure: A horizontal segment of the flow, where a larger cross section is smoothly joined to a smaller cross section.

The surrounding is by definition the fluid which is in contact with the dotted lines. At the left end the surrounding exerts a force to push the fluid within the system to the right. At the right end, it exerts a counter force to slow down the motion. If the pressure at the two ends of the system is the same it will not move at all. To see what happens in detail let us introduce a set of symbols. Denote the left end pressure by P_2, the cross section A_2 and the local speed is v_2, at the right hand side, the corresponding quantities are: P_1, A_1 and v_1. At the left end there is a force pushing to the right, which is given by

$$F_2 = P_2 A_2 \tag{4}$$

where P_2 is the pressure pushing to from the left and A_2 the cross sectional area. During some time interval Δt, the work done in delivering new segement $\Delta x_2 = v_2 \Delta t$ into the sytem is given by

$$W_2 = \mathbf{F}_2 \bullet \Delta \mathbf{x} = P_2 A_2 \Delta x = P_2 \Delta V \tag{5}$$

where ΔV is the volume of the fluid entered into the system. At the right end, there is a counter force $F_1 = P_1 A_1$ which pushes the system to the left. Here the work by the surrounding to the system is

$$W_1 = \mathbf{F}_1 \bullet \Delta \mathbf{x}_1 = -P_1 A_1 \Delta x_1 = -P_1 \Delta V \tag{6}$$

Notice for this case the work done by the surrounding on the system is a negative quantity since force vector is to the left while the displacement vector is to the right. The net work which the surrounding acting on the system is given by

$$W = (P_2 - P_1)\Delta V \tag{7}$$

In the mean time the change of the kinentic energy content of the system is given by the difference in the kinetic energy gain due to the addition of the left segment to the system and the kinetic energy loss due to departure of the right segment

$$\Delta E_{sys} = \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 \tag{8}$$

Based on the energy principle the motion of the fluid system should satisfy the quation

$$\Delta E_{sys} = \frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 = W = (P_2 - P_1)\Delta V \tag{9}$$

Dividing both sides of the expression by ΔV gives

$$\frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2 = (P_2 - P_1)\Delta V \tag{10}$$

where fluid density is defined by eq(1). This is essentially simplified version of the Bernoulli's equation, eq(1)-manual, for the horizontal segment. If the two ends of the segment have different heights, say y_2 at the left end and v_1 at the right, the change of the energy of the system now involves the sum of the kinentic energy $\frac{1}{2}\Delta m v^2$ and potential energy $\Delta m g y$, i.e. this change will be given by

$$\Delta E_{sys} = \left[\frac{1}{2}\Delta m v_2^2 + \Delta m g y_2\right] - \left[\frac{1}{2}\Delta m v_1^2 + \Delta m g y_1\right] \tag{11}$$

We leave it as an exercise for the reader to show that this case leads to full expression for eq(1)-manual.