Statistical Mechanics of an Optical Phase Space Compressor

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Abstract. – We describe the statistical mechanics of a new method to produce very cold atoms or molecules. The method results from trapping a gas in a potential well, and sweeping through the well a semi-permeable barrier, one that allows particles to leave but not to return. If the sweep is sufficiently slow, all the particles trapped in the well compress into an arbitrarily cold gas. We derive analytical expressions for the velocity distribution of particles in the cold gas, and compare these results with numerical simulations.

Introduction. – Evaporative cooling was originally suggested as a means to achieve Bose-Einstein condensation in atomic hydrogen [1–3]. Its application to magnetically trapped alkali atoms [4,5] culminated in the first observation of Bose–Einstein condensation in atomic vapors [6–8]. Since then it has been the essential process by which to obtain degenerate quantum gases. Nevertheless it has shortcomings. The main two are

1. Almost all atoms originally trapped to produce the condensate are lost during the evaporation process.

2. The time scale for collisions leading to thermal equilibrium can be short compared to the time employed to form the condensate.

The latter shortcoming is especially severe for fermionic atoms, since for two fermions in the same state, s-wave scattering is forbidden by the Pauli exclusion principle. Currently, degenerate fermionic gases can only be obtained by a combination of evaporative and sympathetic cooling [9] in the presence of bosonic atoms or different states of the fermionic atoms [10].

Recently, procedures to construct semi-permeable barriers for ultra–cold atoms have been suggested [11, 12]. Such barriers transmit atoms coming from one side and reflect them from another. Their operation relies on different optical shifts for different internal states. In principle these barriers may be constructed for many different atoms and molecules. We have shown that by placing such a wall into a box-shaped potential one may achieve a substantial increase in phase space density [11], or equivalently, substantial cooling.

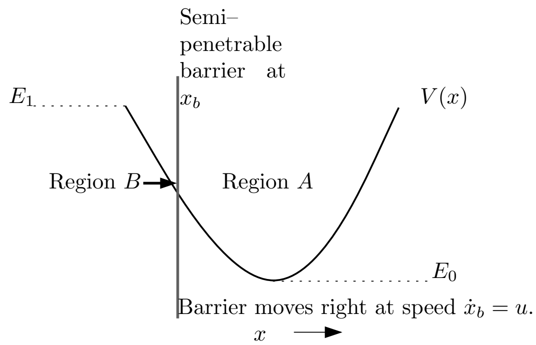

Fig. 1 – Schematic view of the optical compressor. Particles begin in region A with characteristic energies $k_B T$. Particles arriving at the semi–permeable barrier from the right travel through it, while those in region B arriving from the left are reflected. The semi–permeable barrier moves to the right at speed $u = \dot{x}_b$, transferring particles from region A to region B, where their maximum kinetic energy is much less.

Our goal in this letter is to demonstrate that by slowly sweeping a semi–permeable wall through a general trapping potential, the particles naturally compress into a state of very low energy. The basic idea is illustrated in Figure 1. At any given time, all particles remaining in region A to the right of the potential have energy less than $V(x_b)$. When the wall moves slightly to the right, the particles that reach it are at their turning point, and have very small kinetic energy. As the semi–permeable barrier continues to move to the right, one might think that particles on the left in region B gain back their energy. However, for a convex potential, as the particles bounce off the moving wall, they lose more energy in the collision than they gain otherwise. In this way, a slow sweep of the semi–permeable barrier through the convex well reduces particle energies to very low values set by the speed of the sweep.

The conditions needed for this optical compressor to work are quite different from those required for the effectiveness of evaporative cooling. The optical compressor tolerates the existence of a nonequilibrium distribution of particles to the right of the wall. In particular, the velocities of particles that reach the semi–permeable barrier from the right are all very low, rather than being given by the Maxwell–Boltzmann distribution that would describe them in equilibrium. Thus the process of compression may be fast compared to the thermal equilibration time of the particles. On the other hand, the sweep of the wall cannot occur too quickly, because the kinetic energy of particles after they traverse the wall is given by a positive power of the wall velocity.

Thus, the optical compressor provides a process completely complementary to evaporative cooling:

1. No atoms are lost during the compression process.

2. The time scale for collisions leading to thermal equilibrium must be long compared to the time spent sweeping the semi–permeable barrier.

We note that as the equilibration time becomes comparable to the time of the sweep, phase space compression will occur due to combination of evaporative cooling in region A and the process discussed here.

Model. – We consider an ideal collisionless gas trapped in a one-dimensional potential $V(x)$; two other dimensions are either untrapped or confined in a box-shaped potential. The gas is originally in thermal equilibrium at temperature $T = 1/k_B\beta$. An ideal infinitely thin semi–permeable barrier is located at position x_b, which is originally far to the left of any particles. The barrier moves to the right with velocity $u = \dot{x}_b$, its intersection point with the well moving from E_1 to E_0 and eventually passing through the whole well and out the right hand side.

In the limit of slow wall velocities it is possible to obtain analytical results. We first focus on the distribution of velocities with which particles cross the barrier, and then consider the question of how their velocities change after they have crossed the barrier.

Particles with energy E are not affected by the wall until the wall reaches the point where $V(x_b) = E$. Let the period of oscillation of a particle of energy E in region A be $\mathcal{T}(E)$. We assume that there are

$$n(E)dE = N\beta e^{-\beta E}dE \tag{1}$$

particles near energy E, and that their positions in the trap are random. Therefore, from the time the first particle of this energy passes through the barrier, until the last one leaves, there passes a time $\mathcal{T}(E)$. The last particle to be captured is one that had just passed the turning point and was headed to the right as the barrier reached energy E. Particles of energy E will pass at a uniform rate through the barrier during the time interval $\mathcal{T}(E)$. The first particle to pass the barrier will have no kinetic energy, while the last one through will have kinetic energy

$$K = -\frac{\partial V}{\partial x_b} u\mathcal{T}(E) \equiv \dot{E}\mathcal{T}. \tag{2}$$

Here $\dot{E}$ gives the rate at which the intersection point of the barrier with the potential well decreases in energy per time. We are using here the assumption that motion of the semi–permeable barrier through the well is fast compared to the thermal equilibration time, or else the kinetic energies of particles escaping the trap would be described by a Maxwell–Boltzmann distribution with temperature T. We note that even if the semi–permeable barrier moved so slowly through the trap that thermal equilibrium obtained in region A, there would still be some cooling in region B, as we now describe.

Once particles have passed the semi–permeable barrier, they collide repeatedly with the barrier as it moves to the right and reflect from it. They lose energy to the barrier in this process. The final energy of each particle can be determined by observing that the process is adiabatic in the sense of mechanics, so that the action $I = \oint pdq$ is conserved [13]. Consider a particle that has kinetic energy K and total energy E as it passes through the barrier. If the kinetic energy K is not too large, the potential in region B can be treated as linear, and one computes that the particle has action

$$I = \frac{2}{3}\frac{(2mK)^{3/2}}{m\,|V'(E)|}, \tag{3}$$

where m is the particle mass, and V' is the slope of the potential. As the wall continues to move to the right, this action is preserved, allowing one to determine the final energy e of the particle once the barrier has swept all the way through the trap. We define in particular the function

$$K(e, E). \tag{4}$$

which gives the initial kinetic energy K of the particle in terms of its total final energy e, and its initial energy E when it crossed the barrier.

Thus we have the following expression for the distribution of particle energies $f(e)$ in region B at the end of the compression process:

$$f(e) = \int_{E_0}^{E_1} dE \frac{dK}{de} \frac{\theta(K(e,E))\theta(\dot{E}\mathcal{T} - K(e,E))N\beta e^{-\beta E}}{\dot{E}\mathcal{T}}, \tag{5}$$

Here θ is a Heaviside step function. This expression follows by noting that the $n(E)dE$ particles with potential energy E cross the barrier with kinetic energies K evenly distributed between 0 and $\dot{E}\mathcal{T}$. The energies E_0 and E_1 are the minimum and maximum intersection points of the semi–permeable barrier with the potential well, as indicated in Figure 1. The factor dK/de accounts for changes in the energy distribution of particles due to adiabatic expansion in region B.

The distribution $f(e)$ in Eq. (5) does not describe thermal equilibrium. Once the compression process has terminated, we expect that the gas will be maintained for times long compared with the thermal equilibration time. The total energy of particles in the trap will be conserved in this process. Thus the end result will be a thermal distribution of particles with average energy $\bar{e}_f = E/N$ and temperature T_f that may be found from the system of three equations with three unknowns (entropy S, free energy F, and temperture T) [14]:

$$\begin{aligned} F &= -NT \ln \frac{e}{N} \int \exp\left[-\beta\left(\frac{p^2}{2m} + V(x)\right)\right] \frac{dxdp}{2\pi\hbar}, \\ S &= -\frac{\partial F}{\partial T}, \\ E &= F + TS. \end{aligned} \tag{6}$$

We characterize this final equilibrium distribution by the efficiency γ, defined to be the ratio of phase space density before and after compression [15]:

$$\gamma = \frac{\Gamma_f}{\Gamma_i} = \exp\left(\frac{S_i - S_f}{k_B N}\right). \tag{7}$$

Note that for a power–law potential $V(x) = Ax^n$, moving from initial average energy $\bar{e}_i$ to final average energy $\bar{e}_f$ the solution of the system (7) above gives the compression

$$\gamma = \frac{\Gamma_f}{\Gamma_i} = \left(\frac{\bar{e}_i}{\bar{e}_f}\right)^{\frac{1}{2}+\frac{1}{n}}. \tag{8}$$

Examples. – We now provide examples of two different trapping potentials, and calculate their effectiveness in cooling dilute gases.

First, consider the *gravitational trap*, defined by

$$V(x) = \begin{cases} -Ax & \text{for } x < 0 \\ \infty & \text{else.} \end{cases} \tag{9}$$

As the semi–permeable barrier moves through this potential, the shape of region B does not change, and therefore the kinetic energy of a particle when it passes the barrier precisely equals its final total energy; that is, $K(e,E) = e$. Carrying out a computation involving the period of motion in such a potential, we find

$$\begin{aligned} f(e) &= B_1 \mathrm{erfc}(e/e_0), && (10)\\ \text{where } e_0 &= 2\sqrt{2}u\sqrt{mk_BT}. && (11) \end{aligned}$$

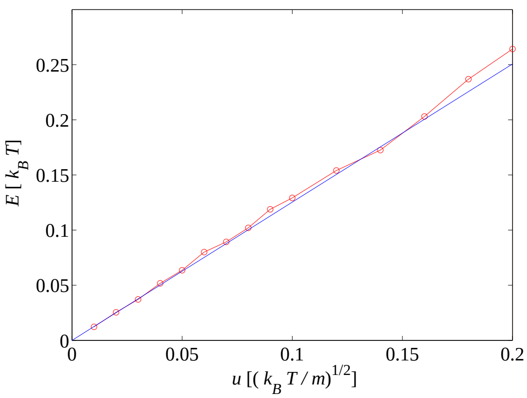

Fig. 2

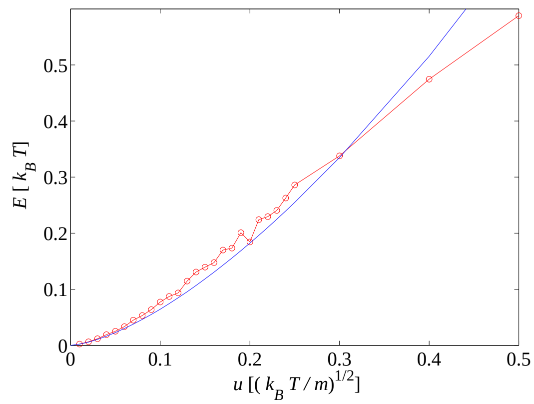

Fig. 3

Fig. 2 – Final energy in gravitational trap. The straight line is the analytical result, Eq. (12). Connected dots come from numerical simulations. The wall is initially placed at $E = 7k_BT$. Each point is an average over $N = 1000$ particles.

Fig. 3 – Final energy in parabolic trap. The straight line is the analytical result, Eq. (19). Connected dots come from numerical simulations. The wall is initially placed at $E = 3k_BT$. Each point is an average over $N = 1000$ particles.

Here erfc(x) is the complementary error function and B_1 is a normalization coefficient. From this distribution we obtain the average energy after compression,

$$\bar{e}_f = \sqrt{\pi/2}u\sqrt{mk_BT}. \tag{12}$$

and the efficiency

$$\gamma = \left(\frac{\bar{e}_i}{\bar{e}_f}\right)^{3/2} = \left(\frac{9}{2\pi}\frac{k_BT}{m}\right)^{3/4}\frac{1}{u^{3/2}}. \tag{13}$$

The average energy vanishes as velocity of the wall goes to zero.

Next consider the *parabolic trap*

$$V(x) = \frac{1}{2}Ax^2. \tag{14}$$

Employing Eq. (3) we find that

$$K(e,E) = \left[\frac{3\pi}{2}\sqrt{E}e\right]^{2/3}. \tag{15}$$

In this case the energy distribution is given by

$$f(e) = B_2\left(\frac{e_0}{e}\right)^{1/3}\Gamma\left[\frac{5}{6}\left(\frac{e}{e_0}\right)^4\right], \tag{16}$$

$$\tag{17}$$

$$\text{where}\quad e_0 = \epsilon_0\left(u\sqrt{\frac{m}{k_BT}}\right)^{3/2}k_BT, \tag{18}$$

and $\Gamma[a,x] = \int_x^\infty dte^{-t}t^{a-1}$ is an incomplete Gamma function [16], B_2 is another normalization constant and $\epsilon_0 = 2 \cdot 2^{3/4}(2\pi)^{3/2}/3\pi$. The average energy after the process is

$$\bar{e}_f = Cm^{3/4}u^{3/2}(k_BT)^{1/4} \tag{19}$$

where $C = \epsilon_0 \frac{2}{5}\Gamma[\frac{5}{4}] \approx 2.038$. The efficiency thus depends upon the wall velocity u just as in the previous example, but with a different numerical prefactor

$$\gamma = \frac{\bar{e}_i}{\bar{e}_f} = \frac{1}{C}\left(\frac{k_BT}{m}\right)^{3/4}\frac{1}{u^{3/2}}. \tag{20}$$

We performed numerical simulations of the process by randomly preparing particles with various energies in gravitational and harmonic potentials. We solved the equations of motion while moving a semi–permeable barrier slowly through the potential. This procedure was repeated for N particles with average energy corresponding to the temperature. The results of these simulation are shown in Figs. 2 and 3. They are in good agreement with the analytical formulas for small velocities.

Comparisons and limitations. – Because the one-way wall for an atomic barrier relies upon different internal states, it truly diminishes the system entropy as a Maxwell demon would, except for the unavoidable heating due to recoil of a photon motion. This can be captured as the cooling effect as described in this paper. By comparison, in a plasma, where analogous one-way walls were proposed in the radio frequency regime [17], there is no opportunity to change internal states of the plasma ions. Instead, the one-way ponderomotive-effect wall operates through Hamiltonian forces only, thereby conserving phase space. Thus for plasmas, no matter how the wall is moved, no real cooling can take place. In the end, if the plasma ions occupy the same volume in space, they would of necessity occupy the same volume in velocity space – and hence not achieve a cooling effect. Note, however, that while the one-way radio-frequency wall does not cool plasma, it can force ions or electrons to move in one direction only. Thus, plasma currents can be driven by plasma waves, which can be useful for a variety of plasma applications [18].

The limitation of the semi-permeable wall we suggested [11] is that it results in heating of atoms to a single photon recoil $mv_r = \hbar k_L$. As the wall velocity diminishes, the process becomes inefficient. If the temperature of the gas is originally n_r recoils; i.e. $k_BT = n_r^2 E_r$ where $E_r = \hbar^2 k_L^2/2/m$, then assuming that the final energy is E_r we find the slowest velocity with which it is still advantageous to move the wall in case of the parabolic trap is

$$u \approx \frac{0.15}{n_r^{1/3}} v_r. \tag{21}$$

In particular, if we start with a temperature of 10 recoils, the minimum wall velocity comes out to be $u = 0.05v_r$. If velocity relaxation happens on time scale τ, the size of the trap can then be $u\tau$. For alkalies, τ can be as long as tens of seconds; hence in this case the size of the cloud is on the order of centimeters.

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