Fidelity of a Bose-Einstein Condensate

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We investigate fidelity, the Loschmidt echo, for a Bose-Einstein Condensate. It is found that the fidelity decays with time in various ways (exponential, Gaussian, and power-law), depending on the choice of initial coherent state as well as the parameters that determine properties of the underlying classical dynamics. Moreover, high fidelity is found for initial states lying in the regular region of a mixed-type phase space. A possible experimental scheme is suggested.

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The investigation of coherent manipulation of quantum state of matter and light has provided insights in many quantum phenomena and in quantum information processes [1]. The realization of Bose-Einstein condensation (BEC) in dilute gases has provided a new tool for such investigations [2]. Recently, it is found that the stability of quantum state has played a key role in many procedures for coherent manipulating and applying BEC[3, 4, 5, 6, 7]. In fact, how to sustain the coherence among the cooled atoms is very essential for the possible application of BEC to quantum information and quantum computation.

However, an important issue is still missing in the study of BEC, namely, the sensitivity of the quantum evolution of a BEC with respect to the small perturbation that may naturally arise from either the manipulation parameters or the interaction with environment. This type of stability of quantum motion is characterized by the so-called fidelity, or the Loschmidt echo, which is defined as the overlap of two states obtained by evolving the same initial state under two slightly different (perturbed and unperturbed) Hamiltonians. This quantity is of special interest in the fields of quantum information[8] and quantum chaos[9, 10, 11, 12, 13].

In this Letter, we propose a system of two-component BEC trapped in a harmonic potential [14], subject to a periodic coupling (successive kicks) between the two components. Our aim is two-fold: (1) To investigate the instability of the BEC system with a small perturbation on its system parameters; (2) To propose a possible experiment to directly detect the fidelity decay.

The system we propose is a two-component spinor BEC confined in a harmonic trap with two internal states coupled by a near resonant pulsed radiation field[14]. Within the standard rotating-wave approximation, the Hamiltonian can be cast into the form [15],

$$\hat{H} = \mu(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) + g(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)^2 + K\delta_T(t)(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) \tag{1}$$

where K is the coupling strength between the two internal states, g is the interaction strength, and μ is the difference between the chemical potentials of two components. $\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2$ and $\hat{a}_2^\dagger$ are boson annihilation and creation operators for the two components, respectively. $\delta_T(t) = \sum_n \delta(t - nT)$ means that the radiation field is only turned on at certain discrete moments, i.e., integral multiples of the period T. Writing the above Hamiltonian in terms of the angular momentum operators [16], $\hat{L}_x = \frac{\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1}{2}$, $\hat{L}_y = \frac{\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1}{2i}$, $\hat{L}_z = \frac{\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2}{2}$, we have $\hat{H} = \mu \hat{L}_z + g\hat{L}_z^2 + K\delta_T(t)\hat{L}_x$. The Floquet operator depicting the quantum evolution in one period takes the following form,

$$\hat{U} = \exp[-i(\mu \hat{L}_z + g\hat{L}_z^2)T] \exp(-iK\hat{L}_x). \tag{2}$$

The Hilbert space is spanned by the eigenstates of $\hat{L}_z$, $|l\rangle$, with $l = -L, -L+1, \ldots, L$, where $L = N/2$ and N is the total number of atoms. In the above expression and henceforth, the Planck constant is set to unit.

The above system has a classical counterpart in the limit $N \to \infty$, describing a spin on a Bloch sphere with $S_i = \frac{1}{L} < \hat{L}_i >, (i = x, y, z)$. The classical Hamiltonian takes the form, $H = \mu S_z + g_c S_z^2 + K\delta_T(t)S_x$, where $g_c = gL$. The equations $\dot{S}_i = [S_i, H]_{cl}, (i = x, y, z)$ determine the motion of the centers of coherent quantum wavepackets and the quantum fluctuation is ignored, (i.e., equivalent to the mean-field Gross-Pitaevskii equation without considering a total phase[7]). They can be solved analytically: the free evolution between two consecutive kicks corresponds to a rotation around S_z axis with the angle $(\mu + 2g_c S_z)T$, and the periodic kicks added at times nT give rotation around the S_x axis with the angle K.

Dynamic motion of the classical system is classified by the magnification of its initial deviation. An exponential increase in time of the deviation means dynamical instability or chaotic motion[3], causing rapid proliferation of thermal particles[7]. Quantitatively, one can calculate the (maximum) Lyapunov exponent, $\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta x(t)|}{|\delta x(0)|}$, with $|\delta x(t)|$ denoting distance in

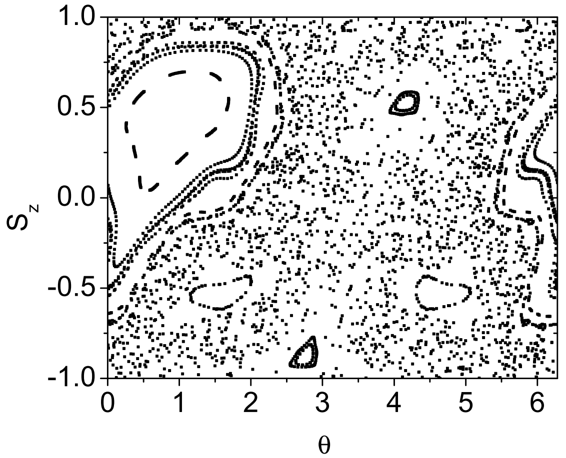

Figure 1: Stroboscopic plots of the orbits for $g_c = 1, K = 2$ where x-axis θ is the azimuthal angle. It shows one big island and four small islands. Inside the islands motions are stable, outside the islands motions are mainly unstable or chaotic. Here and in the following figures, $\mu = T = 1$.

phase space. The exponent is positive for unstable motion, and tends to zero if the orbit is stable. Usually, phase space is a mixture of chaotic orbits and quasi-periodic (stable) orbits, as is shown in Fig.1, where it is clearly seen one big island and four small islands; inside the islands motions are periodic or quasi-periodic, outside the islands motions are mainly chaotic.

The total relative area occupied by chaotic orbits, (as clearly seen in Fig.1), depends on system parameters (g_c, K), and can be used to characterize the degree of mixture. It has been obtained by calculating the Lyapunov exponent of orbits with initial points randomly scattered in the whole phase space. The result is shown

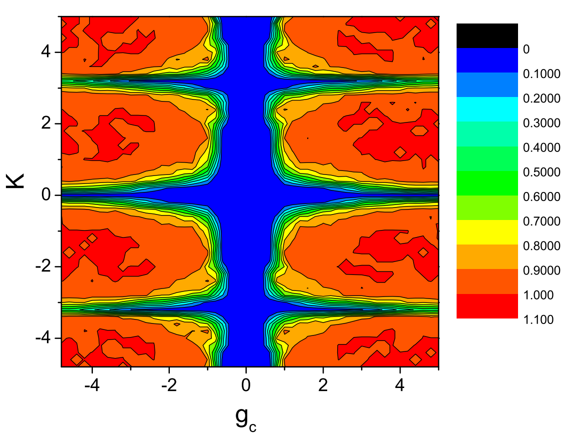

Figure 2: Contour plot of the fraction of chaotic orbits in phase space, with respect to system parameters.

in Fig.2. We see that the integrable cases mainly concentrate on the vertical line where the interaction strength vanishes, and on the horizontal lines where the coupling strength is a multiple of π. This fact indicates that both nonlinearity term and the kick strength are essential in inducing chaos. The deep red areas in Fig.2 give the parameter regime for the system where the phase space is full of unstable (chaotic) orbits.

Now we turn to the quantum system and trace the fidelity (Loschmidt echo) $M(t)$, defined as

$$M(t = nT) = \langle \Phi_0 | \left(\hat{U}_\epsilon^\dagger \right)^n \circ \left(\hat{U} \right)^n | \Phi_0 \rangle, \tag{3}$$

where the initial state $|\Phi_0\rangle$ is chosen as a coherent state, $|\Phi_0\rangle = e^{\alpha^* L_+ - \alpha L_-} | -L\rangle$, with $\alpha = \frac{\pi-\theta}{2} e^{-i\varphi}$. A small perturbation on the Hamiltonian is added by changing $K \to K + \varepsilon$, with $\hat{U} \to \hat{U}_\epsilon$. In this system, the effective Planck constant $\hbar_{\text{eff}} = 1/L$.

We discuss fidelity decay in three typical situations, in which the corresponding classical system is fully chaotic,

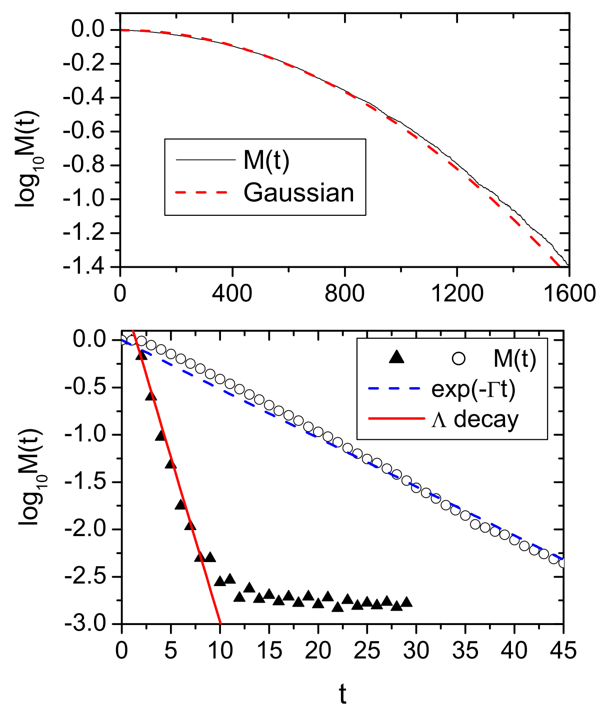

Figure 3: Fidelity decay in a fully chaotic case of $K = 2, g_c = 4$. Upper panel: An example of Gaussian decay at small perturbation, with $L = 100, \epsilon = 2 \times 10^{-4}$, obtained from one initial coherent state. The dashed curve is the Gaussian function of form $\exp\left(-1.3 \times 10^{-6} t^2\right)$. Lower panel: Circles: fidelity decay with an intermediate perturbation (above ϵ_p), where the average has been made over 20 initial coherent states chosen randomly. The dashed line is the exponential decay with $\Gamma = 0.12$. Triangles: Strong perturbation regime where the fidelity decays as $e^{-\Lambda t}$ for a short time ($t < 10$) and then saturates, with $L = 500, \epsilon = 1 \times 10^{-2}$, and average over 1000 initial coherent states. Here $\Lambda = 0.8$ is independent of perturbation strength[13].

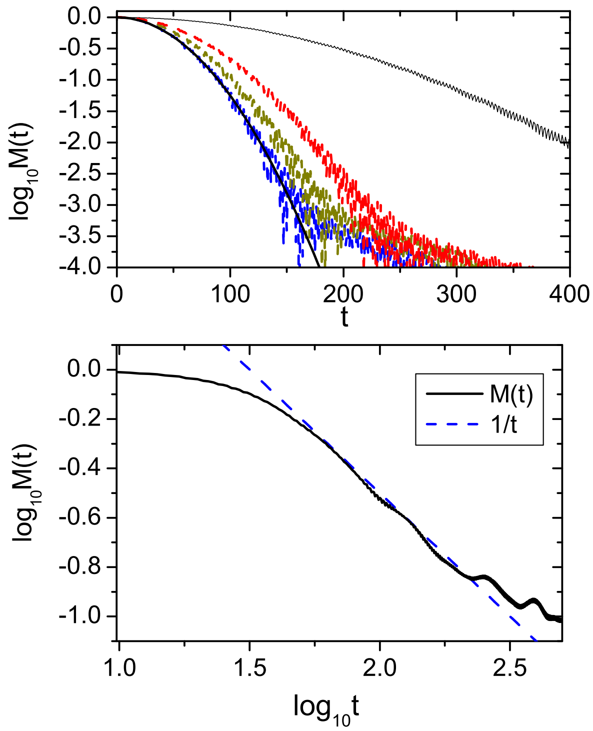

Figure 4: Fidelity decay in a classically nearly integrable case, with $K = 2, g_c = 0.2, L = 100$, and $\epsilon = 0.003$. Upper panel: Fidelity of four randomly chosen initial coherent states, with the smooth solid curve being the Gaussian fit to one of them. Lower panel: Averaged fidelity, with average performed over 50 initial coherent states.

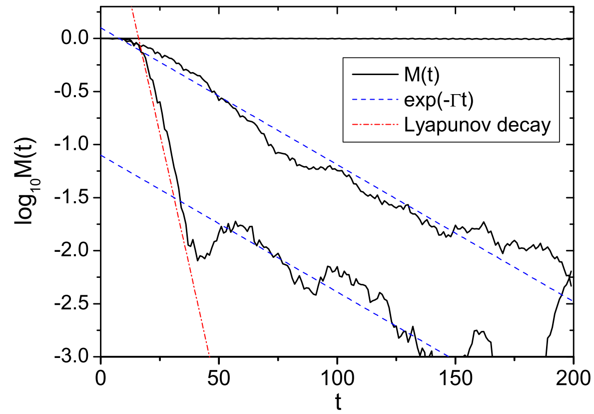

Figure 5: Fidelity decay in the mixed system whose classical phase space structure is shown in Fig. 1. $L = 500$ and $\epsilon = 6 \times 10^{-4}$, the intermediate perturbation. The non-decaying solid line is the fidelity of an initial coherent state lying within the largest regular region. The other two solid curves correspond to fidelity of two initial coherent states lying in the chaotic region of the classical system. One of them has an exponential decay with $\Gamma = 0.03$, as expected. Unexpectedly, the other one first has a fast Lyapunov decay $e^{-\lambda t}$, with λ being the Lyapunov exponent, then follows the exponential decay as the first one.

near-integrable, and mixed, respectively. The corresponding parameters are picked up from Fig.2.

For the parameters $K = 2, g_c = 4$, from Fig.2 we know that the phase space is fully chaotic. Because of the ergodicity of the chaotic orbits, fidelity decay is expected to be independent on the initial condition. However, it strongly depends on the perturbation strength. For a small perturbation, fidelity shows a slow Gaussian decay (upper panel in Fig.3). With increasing perturbation strength, one meets a border $\epsilon_p \sim 1/L^{3/2}$, at which the typical transition matrix element of perturbation between quasi-energy eigenstates becomes larger than the average level spacing. With the intermediate perturbation above the border (lower panel in Fig.3), the fidelity decays in an exponential way, where the decay rate Γ is the function of the interaction strength and the classical action diffusion constant[10]. With strong perturbation, the fidelity decays faster and finally saturates at some perturbation-independent decay rate [13] (lower panel in Fig.3).

From the above discussions and calculations we see, in practical applications of the BEC, the perturbation border ϵ_p gives a up-limit for the perturbation strength that is tolerable, in order to avoid low fidelity.

As we choose parameters as $K = 2, g_c = 0.2$, the classical system is nearly integrable where the phase space is full of periodic and quasi-periodic orbits. We found Gaussian decay for the fidelity of single initial coherent states, with a strong dependence of decaying rate on the choice of initial condition [12]. However, after averaging over the whole phase space, we found that the fidelity decay can be well fitted by a inverse power law $1/t$ (see Fig. 4). In this case, for the quantum evolution of initial coherent states, high fidelity can be expected because the fidelity has a power law decay on average.

Now we turn to the mixed case, which is more complicated than the previous two cases. It is usually expected that fidelity decay of initial coherent states lying in regular regions would be similar to that in a nearly integrable system, and that from chaotic regions be similar to that in a chaotic system. However, we found that this naive picture is not exact. As shown in Fig. 5, for initial states from both irregular and regular regions, the behavior of fidelity may be quite different from those in the fully chaotic case and in the nearly-integrable case as shown in Figs. 3 and 4. We concentrate our discussions on the case in which initial coherent states lie within the largest regular island. We found that their fidelity almost has no decay up to time $t = 200$, quite different from the initial-condition-dependent Gaussian decay shown in Fig. 4 for a nearly integrable system. Note that the quantum perturbation strength is chosen to be in the intermediate perturbation regime in Figs. 4 and 5. This phenomenon of high fidelity cannot be explained by means of expand-

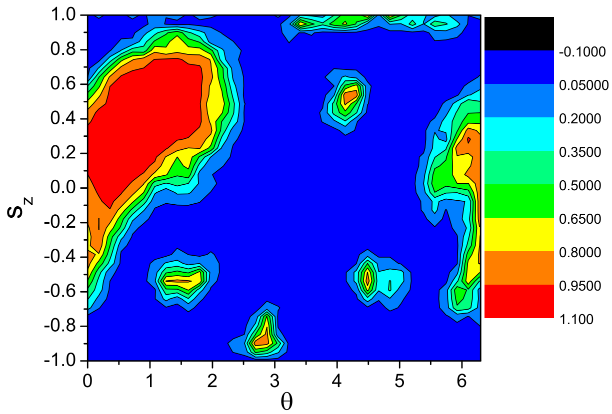

Figure 6: Contour plot of the fidelity $M(t)$ at $t = 200$, for $K = 2, g_c = 1, L = 500$, and $\epsilon = 0.0006$. The initial quantum states are cohenet states with corresponding (S_z, θ).

ing the coherent states in the eigenstates $|\alpha\rangle$ of the system [17], since the values of the participation function of the coherent states, defined by $1/(\sum_\alpha |\langle\alpha|\Phi_0\rangle|^4)$, is about 22. The principle of this way of sustaining high fidelity in quantum evolution of an coherent state may be useful in applying BEC in information processing.

In order to have a knowledge of the global situation of fidelity decay in a mixed system, in Fig.6 we show a contour plot for $M(t = 200)$, with respect to initial coherent states. With this figure at hand, in applying BEC to quantum information processing we may carefully choose the parameters to avoid the regimes of low fidelity. Moreover, in surprise we find the structure of fidelity plotting in Fig.6 quite similar to that of classical phase space in Fig.1. This similarity indicates a kind of connection between the dynamical instability of the classical meanfield equation and the fidelity of quantum boson system, i.e., the dynamical instability regime of the classical system usually corresponds to the low fidelity regime of the quantum system. Moreover, inside the islands (the large or small) where the classical motions are dynamical stable with zero Lyapunov exponent, the fidelity shows different behavior: The fidelity in the large island of a mixed-type phase space is higher than that in the small islands or even that in near-integrable case. This fact indicates that the fidelity contains more information about the system under a perturbation and therefore is a more general quantity to describe the stability of the BEC.

Experimentally, one can prepare two nearly identical two-component BECs by applying a strong blue detuning laser beam at the center of the two-component BEC with prepared initial state $\psi\chi$, where ψ is the external state and χ is the internal state [18]. The strength of laser beam is strong enough, so that the tunnelling between the two condensates is negligible. Then, two pulsed radiation fields with slight different strengths are applied to the two condensates and kick the internal state to χ_i ($i = 1, 2$) without important change of the external states ψ_i [14]. After certain numbers of kicks, the radiation field and the strong blue detuning laser beam are turned off simultaneously and two BECs begin to interfere. The visibility of the interference is governed by

$$I \propto |\psi_1\chi_1|^2 + |\psi_2\chi_2|^2 + 2Re\,(\psi_1^*\psi_2\chi_1^*\chi_2)\,. \qquad (4)$$

Clearly, high fidelity of the two internal states corresponds to high visibility of the interference.

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