

Transverse force on a quantized vortex in a superfluid

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Abstract

We have derived an exact expression for the total nondissipative transverse force acting on a quantized vortex moving in a uniform background. The derivation is valid for neutral boson or fermion superfluids, provided the order parameter is a complex scalar quantity. This force is determined by the one-particle density matrix far away from the vortex core, and is found to be the Magnus force proportional to the superfluid density. We conclude that contributions of the localized core states do not change this force.

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For a vortex moving in a superfluid there is a force transverse to its velocity which is the counterpart of the Magnus force in classical fluid mechanics [1]. This was used in the experiment by which Vinen [2] established the quantization of circulation round a wire in ^4He . The obvious generalization of the form of the Magnus force to the situation in superfluid helium is to take the force per unit length to $\mathbf{F}_M = \rho_s \mathbf{K} \times (\mathbf{v}_V - \mathbf{v}_s)$, where \mathbf{K} is the circulation, \mathbf{v}_V is the vortex velocity, and ρ_s is the superfluid density and \mathbf{v}_s is the superfluid velocity in the neighborhood of the vortex. Even for the case of ^4He there has been some controversy about the form of this transverse force, despite the experimental measurements of Vinen [2] and of Whitmore and Zimmermann [3] at temperatures where there is an appreciable difference between ρ and ρ_s . This controversy is partly based on the observation by Iordanskii [4] that excitations such as phonons are asymmetrically scattered by a vortex, and this should lead to a transverse force proportional to ρ_n and to the velocity $\mathbf{v}_n - \mathbf{v}_V$ of the normal fluid component relative to the vortex. The magnitude of this term is not immediately obvious, nor is it obvious whether this contribution should be added to the Magnus force as we have written it here, or to the expression for \mathbf{F}_M with the superfluid density replaced by the total fluid density. It has been convincingly argued [5] that the correct form of the transverse force is

$$\mathbf{F} = \rho \mathbf{K} \times (\mathbf{v}_V - \mathbf{v}_s) + \rho_n \mathbf{K} \times (\mathbf{v}_n - \mathbf{v}_V), \quad (1)$$

which can also be written as \mathbf{F}_M plus a term proportional to $\mathbf{v}_n - \mathbf{v}_s$. The problem of determining the correct form of the force is discussed in detail by Sonin [6].

For a fermion superfluid there appear to be additional complications introduced by the spectrum of low-lying fermion excitations associated with the vortex core. Volovik [7,8] has identified an additional contribution proportional to $\mathbf{v}_n - \mathbf{v}_V$ associated with these excitations, which, in some estimates, cancels most of the Magnus force. However, measurements of the circulation quantization in the B phase of superfluid ^3He [9] show a Magnus force of the expected size. In superconductors there seems to be little unambiguous evidence about

the existence or magnitude of this transverse force.

In earlier work we have shown the close connection between the Magnus force and a Berry phase [10], but our arguments were dependent on a detailed model of the vortex structure. In this paper we consider a single, isolated vortex forced to move through an infinite superfluid with uniform velocity \mathbf{v}_V by some moving pinning potential. We are able to make an exact evaluation of the coefficient of \mathbf{v}_V in the force which has to be applied to the vortex in terms of the integral of the momentum density round a large loop that surrounds the vortex, and this is essentially the term proportional to \mathbf{v}_V in eq. (1). The first step in the argument involves a perturbative calculation of the effect of the motion of the vortex in terms of the instantaneous eigenstates of the Hamiltonian. In the case of a system with no inhomogeneous substrate this expression can be written in terms of a Berry phase [11]. This expression can be written in terms of the derivatives of one- and two-particle Dirac density matrices, or, alternatively, as the commutator of two components of the total momentum operator. This can be turned into an integral over a surface (or a loop for the two-dimensional case) surrounding the vortex and at a large distance from it. From this we can conclude that there are no corrections to the term proportional to \mathbf{v}_V in eq. (1).

For simplicity we work with a two-dimensional system, but the argument can readily be applied to a three-dimensional system. We assume that the system is homogeneous and infinite. The isolated vortex in the superfluid can be pinned to a position \mathbf{r}_0 by applying a potential $\sum_i V(\mathbf{r}_0 - \mathbf{r}_i)$ which acts on all the particles in the superfluid. For a static problem this potential can be arbitrarily weak, but when the pinning potential is made to move through the superfluid it must have a strength sufficient that the vortex does not get detached from the pinning center, either by tunneling with the aid of the Magnus force, or by acquiring enough energy from the phonon system. The potential has to be strong enough to break the degeneracy of this broken symmetry vortex state and allow us to use a perturbative treatment of the velocity. The situation is somewhat reminiscent of the theory of the Stark effect in atoms, where a perturbative treatment of a weak uniform electric field is possible, despite the fact that the electric field connects the bound levels of the electrons

in the atom to a continuum of ionized states of the same energy. The pinning potential may also be very strong, as it is in the Vinen experiment, where the circulation is pinned to a solid wire. Our results are quite independent of the details of the pinning potential, provided that it is strong enough to allow perturbative methods to be used, and does not break cylindrical symmetry.

In terms of the instantaneous eigenvalues $E_\alpha(t)$ and eigenstates $|\Psi_\alpha(t)\rangle$, for which we choose phases such that $\langle\Psi_\alpha|\dot{\Psi}_\alpha\rangle = 0$, the time-dependent solutions of the Schrödinger equation can be written as

$$|\Psi(t)\rangle = a_\alpha(t)e^{-i\int^t E_\alpha(t')dt'/\hbar}|\Psi_\alpha(t)\rangle + \sum_{\beta\neq\alpha} a_\beta(t)e^{-i\int^t E_\beta(t')dt'/\hbar}|\Psi_\beta(t)\rangle, \quad (2)$$

where, to first order in the velocity, $a_\alpha(t) = 1$, and

$$\begin{aligned} a_\beta(t) &= -\int^t dt' \langle\Psi_\beta|\dot{\Psi}_\alpha\rangle e^{-i\int^{t'} (E_\alpha - E_\beta)dt''/\hbar} \\ &= -\int^t dt' \langle\Psi_\beta|\mathbf{v}_V \cdot \nabla_0 \Psi_\alpha\rangle e^{-i\int^{t'} (E_\alpha - E_\beta)dt''/\hbar}. \end{aligned} \quad (3)$$

Here ∇_0 denotes the partial derivative with respect to the position \mathbf{r}_0 of the pinning potential.

This gives the expectation value of the force on the pinning potential as

$$\begin{aligned} \mathbf{F} &= -\sum_\alpha f_\alpha \langle\Psi_\alpha|\nabla_0 H|\Psi_\alpha\rangle \\ &+ \sum_\alpha f_\alpha \langle\Psi_\alpha|\nabla_0 H e^{i\int^t (E_\alpha - H)dt'/\hbar} \mathcal{P}_\alpha \int^t dt' e^{-i\int^{t'} (E_\alpha - H)dt''/\hbar} \mathbf{v}_V \cdot \nabla_0 + \text{h.c.}|\Psi_\alpha\rangle, \end{aligned} \quad (4)$$

where H is the total Hamiltonian of the system, which depends on \mathbf{r}_0 only through the pinning potential, f_α is the occupation probability of the state α , \mathcal{P}_α is the projection operator off the state α , and h.c. denotes the Hermitian conjugate term. The first term on the right of this equation includes any forces on the vortex proportional to the normal and superfluid velocities, and we have not attempted to evaluate these. We concentrate on the second term, which gives those forces proportional to the vortex velocity. There are energy-conserving dissipative forces included in this expression, but, as with the Kubo

formula for longitudinal conductivity, their evaluation requires a careful limiting process, and we have not done this here. We concentrate on the contributions from intermediate states whose energy is different from that of the initial state, and it is these that give rise to the transverse force perpendicular to the vortex velocity.

For the homogeneous system that we are considering the eigenvalues are independent of time, so eq. (4) can be written in the simpler form

$$\mathbf{F} = - \sum_{\alpha} f_{\alpha} \langle \Psi_{\alpha} | \nabla_0 H | \Psi_{\alpha} \rangle + \sum_{\alpha} f_{\alpha} \left\langle \Psi_{\alpha} \left| \nabla_0 H \frac{i\hbar \mathcal{P}_{\alpha}}{E_{\alpha} - H} \mathbf{v}_V \cdot \nabla_0 + \text{h.c.} \right| \Psi_{\alpha} \right\rangle . \quad (5)$$

Since $\nabla_0 H$ is the commutator of the operator ∇_0 with H , the commutator cancels the energy denominator, and the part linear in the vortex velocity can be written as

$$\mathbf{F} \times \hat{\mathbf{n}} = -i\hbar \mathbf{v}_V \sum_{\alpha} f_{\alpha} \left(\left\langle \frac{\partial \Psi_{\alpha}}{\partial x_0} \left| \frac{\partial \Psi_{\alpha}}{\partial y_0} \right\rangle - \left\langle \frac{\partial \Psi_{\alpha}}{\partial y_0} \left| \frac{\partial \Psi_{\alpha}}{\partial x_0} \right\rangle \right) , \quad (6)$$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plane. The Berry phase associated with a closed loop in \mathbf{r}_0 can be written as the integral over the area enclosed by the loop of $F_M/\hbar v_V$. Equation (6) corresponds to the familiar form for the Berry phase [11]. Since we can choose the wave functions in such a way that the dependence on \mathbf{r}_0 is entirely through $(\mathbf{r} - \mathbf{r}_0)$, the partial derivatives with respect to \mathbf{r}_0 can be replaced by a sum over partial derivatives with respect to the particle coordinates \mathbf{r}_j . Upon thermal average, this expression can now be written in terms of the Dirac density matrices as

$$\begin{aligned} F/v_V &= -i\hbar \sum_{\alpha} f_{\alpha} \sum_{i,j} \left(\left\langle \frac{\partial \Psi_{\alpha}}{\partial x_i} \left| \frac{\partial \Psi_{\alpha}}{\partial y_j} \right\rangle - \left\langle \frac{\partial \Psi_{\alpha}}{\partial y_j} \left| \frac{\partial \Psi_{\alpha}}{\partial x_i} \right\rangle \right) \\ &= -i\hbar \hat{\mathbf{n}} \cdot \int \int d^2 r [\nabla \times \nabla' \rho(\mathbf{r}', \mathbf{r})]_{r=r'} \\ &\quad - i\hbar \hat{\mathbf{n}} \cdot \int \int d^2 r_1 \int \int d^2 r_2 [2\nabla_1 \times \nabla'_2 \Gamma(\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{r}_1, \mathbf{r}_2)]_{r=r'} , \end{aligned} \quad (7)$$

where ρ and Γ are the one- and two-particle Dirac density matrices for the system, and the sum over i and j denotes a sum over all the particles in the system.

The integral over the two-particle density matrix Γ vanishes. Because of the symmetry between \mathbf{r}_1 and \mathbf{r}_2 , the integrand in this integral can also be written as

$$[(\nabla_1 + \nabla'_1) \times (\nabla_2 - \nabla'_2)\Gamma(\mathbf{r}'_1, \mathbf{r}'_2; \mathbf{r}_1, \mathbf{r}_2) - \rho(\mathbf{r}'_1, \mathbf{r}_1)\rho(\mathbf{r}'_2, \mathbf{r}_2)]_{r=r'} ; \quad (8)$$

the replacement of Γ by $\Gamma - \rho\rho$ is also obvious, since the integral over \mathbf{r}_1 of $\mathbf{grad}_1\rho(\mathbf{r}_1)$ vanishes. For any value of \mathbf{r}_2 this can be integrated over \mathbf{r}_1 to give a correlation function between distant points which should vanish if the range of integration is sufficiently large.

A more formal way of getting the same result is to say that the first line of eq. (7) gives the transverse force in terms of the expectation value of the commutator of the x and y components of the total momentum of the particles. This commutator is a one-particle operator given by

$$[P_x, P_y] = \int \int dx dy \left(\frac{\partial\psi^\dagger}{\partial x} \frac{\partial\psi}{\partial y} - \frac{\partial\psi^\dagger}{\partial y} \frac{\partial\psi}{\partial x} \right), \quad (9)$$

where the ψ^\dagger, ψ are creation and annihilation operators for fermions or bosons. Since it is a one-particle operator, its expectation value is given by the one-particle Dirac density matrix.

The integrand of the first term in eq. (7) can be written as half the curl of $(\nabla' - \nabla)\rho(\mathbf{r}', \mathbf{r})$ evaluated at $\mathbf{r} = \mathbf{r}'$. Now divide the integrals over \mathbf{r} and \mathbf{r}_1 into a sum over finite areas labeled with an index σ ; the first of these areas will be centered on the position of the vortex, and the others are all well away from the vortex. Stokes' theorem can be used to write the result as

$$F/v_V = \sum_{\sigma} \oint_{\sigma} \frac{i\hbar}{2} d\mathbf{r} \cdot [(\nabla - \nabla')\rho(\mathbf{r}', \mathbf{r})]_{r=r'}. \quad (10)$$

This result is exact, and there are no contributions to the integrals from the neighborhood of the vortex core, since we have chosen the boundaries of the regions of integration to be well away from the core; any contributions from the core states or from the properties of the core must be reflected in the density matrices well away from the core. In particular, because localized states inside the vortex core, such as occur for an s-wave superconductor [12], do not influence the one-particle density matrix far away from the core, their contributions to the transverse force must be zero. It should be pointed out that this property is not transparent in eq. (7), where both detailed forms of density matrices and explicit integrations are needed to recover it.

For a neutral superfluid the integrand in eq. (10) is just the momentum density, equal, by definition, to $\rho_s \mathbf{u}_s + \rho_n \mathbf{u}_n$, where \mathbf{u}_s and \mathbf{u}_n are the local values of the superfluid and normal velocities. If the circulation of the normal fluid is zero, this term gives $\mathbf{F} = \rho_s \mathbf{K} \times \mathbf{v}_V$, the Magnus force \mathbf{F}_M [2,9,10]. From the Galilean invariance of the problem we can deduce that \mathbf{v}_V should be replaced by $\mathbf{v}_V - \mathbf{v}_s$, but we cannot determine the coefficient of the term proportional to $\mathbf{v}_n - \mathbf{v}_s$ without explicitly putting in different background velocities for the normal fluid and the superfluid components.

At first sight it is surprising that we can obtain an exact result for this problem, since there are very few problems to which quantum many-body theory gives exact answers, but there is a similar argument that can be used in classical hydrodynamics. If fluid is streaming past a fixed vortex with velocity \mathbf{v} , one can calculate the momentum balance on a very large cylinder centered on the vortex. The net pressure force per unit length on the interior of this cylinder is $\rho_n \mathbf{K} \times \mathbf{v}/2$, the momentum flow out from the cylinder is $-\rho_n \mathbf{K} \times \mathbf{v}/2$, and these must be balanced by a force $-\rho_n \mathbf{K} \times \mathbf{v}$ acting on the vortex core. It is by an elaboration of this argument that Barenghi *et al.* [5] obtain their expression for the contribution of the flow of the normal fluid past the vortex.

Our conclusion is that for any infinite, homogeneous, neutral superfluid with a scalar order parameter the transverse force has the form $\rho_s \mathbf{K} \times \mathbf{v}_V$, and those mechanisms which have been suggested as giving corrections must actually give alternative ways of looking at the same thing. Because our argument is, like the classical argument quoted in the previous paragraph, basically an argument that balances the forces on the vortex core with forces and momentum flow at large distances for the vortex, there is no obvious generalization that allows for the effect of the disordered background potential which is normally important for real superconductors.

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