Vortex Dynamics in Superfluids: Cyclotron Type Motion

E. Demircan[1], P. Ao[2], and Q. Niu[1]

[1] Department of Physics

The University of Texas at Austin

Austin, TX 78712

[2] Department of Theoretical Physics

Umeå University, S-901 87, Umeå, SWEDEN

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Abstract

Vortex dynamics in superfluids is investigated in the framework of the non-linear Schrödinger equation. The natural motion of the vortex is of cyclotron type, whose frequency is found to be on the order of phonon velocity divided by the coherence length, and may be heavily damped due to phonon radiation. Trapping foreign particles into the vortex core can reduce the cyclotron frequency and make the cyclotron motion underdamped. The density fluctuations can follow the vortex motion adiabatically within the phonon wave length at the cyclotron frequency, which results in a further downward renormalization of the cyclotron frequency. We have also discussed applications on the dynamics of vortices in superconducting films.

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I. INTRODUCTION

Besides collective excitations of density fluctuations, superfluid systems can also have topological excitations called vortices. Vortices are associated with many important phenomena, such as quantization of circulation [1], Kosterlitz-Thouless phase transition [2], mutual friction [3,4], and flux creep [4,5]. Although the static properties of vortices are relatively well understood, the dynamical side is still wide open to investigations.

It has been a common practice to treat a vortex in two dimensions as a particle, and to describe its motion by Newton's law in the following form

$$M_v \ddot{\mathbf{r}}_0 = -2\pi\hbar\bar{\rho}\dot{\mathbf{r}}_0 \times \hat{\mathbf{z}} - \eta\dot{\mathbf{r}}_0, \tag{1}$$

where $\mathbf{r}_0$ denotes the vortex position, and $\bar{\rho}$ is the 2D superfluid number density. The terms on the right hand side represent Magnus and frictional forces respectively, and M_v stands for the vortex mass. The Magnus force term was motivated from the behavior of vortices in classical fluids, and has recently been related to the Berry phase of the many-body wave function [6–8]. Its existence has been experimentally established for superfluid ^{4}He, but this has been a controversial subject for magnetic fluxes in superconductors [9]. The friction term is considered to be a result of the interaction of the vortex with collective excitations, or with the core excitations in the case of superconductors [3,4]. However, very little experimental information is available to test the theories on dissipation at low enough temperatures. The vortex mass has also been a topic of debate. For incompressible classical fluids, it has been customary to regard the vortex as massless. This point of view has also been adopted in some calculations for nucleation and motion of quantized vortices [10]. Another point of view is that quantized vortices should have finite masses, roughly equal to the mass of the vortex core [11]. In references [12–17], the vortex mass was found to be renormalized by the condensate motion to a value logarithmically divergent with the system size in neutral superfluids. For charged superfluids, the vortex mass turns out to be finite, because the screening currents effectively replace the system size by the London penetration length [12–17]. Some recent discussion on the nature of the vortex mass can be found in [8]. The size of the vortex mass may affect tunneling and specific heat of a vortex lattice [18].

In this work, we study the vortex dynamics based on the nonlinear Schrödinger equation, which has been used to model superfluids in a semi-microscopic manner. This equation has been derived by Gross and Pitaevskii for a weakly interacting superfluid [19,20]. In the appendix, we derive a nonlocal version of the nonlinear Schrödinger theory from Feynman's many-body trial wave function, which can take into account strong correlations in a superfluid such as ^{4}He. The derivation of the nonlinear Schrödinger equation for superconductors in the clean limit is given in Refs. [21,22]. This equation contains both solutions for collective excitations of density fluctuations and for topological excitations of vortices. It provides a useful starting point for the study of vortex motion and its coupling to the collective excitations. An effective Lagrangian will be derived for the vortex coordinate and for the density and phase fields of the superfluid condensate. As we will see, the equation of motion for a vortex naturally contains the Magnus force, and the effect of the condensate motion.

We will concentrate our attention on the cyclotron motion of the vortex. Such a motion is a natural solution of the phenomenological equation (1), with the cyclotron frequency given by

$$\omega = \frac{2\pi\hbar\bar{\rho}}{M_v} \tag{2}$$

if damping is ignored. We found that the cyclotron motion is also a natural solution of the equations of motion of the vortex based on the nonlinear Schrödinger equation. We found that the cyclotron motion is damped by phonon radiation from the vortex. For a bare vortex, the damping rate is found to be about the same as the cyclotron frequency. For a vortex with trapped particles in it, the damping can be much smaller than the cyclotron frequency, although the cyclotron frequency itself is also reduced. With the cyclotron frequency determined from the equations of motion for the vortex coupled to the condensate, we can use equation (2) to define the vortex mass. The frictional coefficient η in equation (1) can be calculated from the rate of damping due to phonon radiation.

We have carefully examined the adiabaticity assumption used in references [12–14,16,17] that the field of condensate phase follows rigidly with the moving vortex. We found that this is possible within a length scale of $\lambda \approx \xi(1 + M_e/M_c)$, where ξ is the coherence length, and M_c and M_e are the masses of the vortex core and of the trapped particles. Outside this length scale, the condensate cannot follow the vortex motion. Because of this, we found that the logarithmic divergence of the vortex mass with the system size, which was found in references [12–14,16,17], is cut off by the scale of phonon wavelength,.

The organization of the paper is as follows. In Sec. II, we will give the basic ingredients of the nonlinear Schrödinger theory, and obtain the static vortex solution. In Sec. III, we will introduce vortex motion, derive the effective Lagrangian and the dynamical equations for the vortex and condensate. In Sec. IV, we will solve for the condensate response to the cyclotron motion, and find the conditions when the adiabaticity assumption is valid. In Sec. V, we will study the low frequency motion of the vortex, which is possible when the mass of the trapped particles is large. In Sec. VI, we present our numerical results for finite frequency of the cyclotron motion. In Sec. VII we will present our conclusions.

II. NONLINEAR SCHRÖDINGER LAGRANGIAN

Our starting point is the nonlinear Schrödinger equation which may be derived from the following Lagrangian:

$$L = \int d^2r \left[i\hbar\psi^* \frac{\partial}{\partial t}\psi - \frac{\hbar^2}{2m} |\nabla\psi|^2 - \frac{1}{2} V \left[|\psi|^2 - \bar{\rho} \right]^2 \right], \tag{3}$$

where m stands for the mass of the superfluid atom, V represents the interaction potential between the atoms, and $\bar{\rho}$ is the background superfluid number density. The sign of V is positive to represent a repulsive interaction.

The natural length and time scales of the nonlinear Schrödinger equation are $\xi = \hbar/(mV\bar{\rho})^{1/2}$ and $\tau = \hbar/(V\bar{\rho})$. As will be seen later, ξ gives the length scale of a vortex, and τ gives the time scale of a bare vortex in cyclotron motion. If we scale the Lagrangian by the energy $\hbar^2\bar{\rho}/m$, then the Madelung transformation $\psi = \sqrt{\rho}e^{iS}$ puts the Lagrangian in the dimensionless form

$$L = -\int \left[\rho\dot{S} + \frac{1}{2}\rho\left|\nabla S\right|^2 + \frac{1}{8\rho}\left|\nabla\rho\right|^2 + \frac{1}{2}\left[\rho - 1\right]^2\right] d^2r. \tag{4}$$

The dynamical equations of the condensate follow from the variation of the action with respect to the phase and the density

$$\dot{\rho} + \nabla\cdot[\rho\nabla S] = 0, \tag{5}$$

$$\dot{S} + \frac{1}{2}|\nabla S|^2 + \frac{|\nabla\rho|^2}{8\rho^2} - \frac{\nabla^2\rho}{4\rho} + \rho - 1 = 0. \tag{6}$$

The first equation is nothing but the continuity equation, whereas the second resembles the Euler equation of hydrodynamics of classical fluids. The normal modes of the linearized equations of the fluctuations around the uniform condensate describe the collective excitations of the system. The low frequency part of the spectrum is phonon like with velocity $c_s = \xi/\tau = \sqrt{V\bar{\rho}/m}$ [3,20]. However, the roton minimum is absent in the spectrum, because the short range atomic repulsion is not appropriately taken into account. In the Appendix, we give the derivation of an effective Lagrangian that may overcome this problem.

The equations also allow vortex solutions. For simplicity, we consider a single vortex with unit circulation around the center $\mathbf{r}_0$ in the $x - y$ plane, with the phase S of the condensate wave function given by

$$S = S_0 \equiv \Theta(\mathbf{r} - \mathbf{r}_0), \tag{7}$$

where $\Theta = \arctan[(y - y_0)/(x - x_0)]$ is the polar angle of $\mathbf{r} - \mathbf{r}_0$. The density ρ_0 satisfies

$$\frac{|\nabla\rho_0|^2}{8\rho_0^2} - \frac{\nabla^2\rho_0}{4\rho_0} + \rho_0 - 1 + \frac{1}{2|\mathbf{r} - \mathbf{r}_0|^2} = 0. \tag{8}$$

The asymptotic forms are easy to find:

$$\rho_0(r) = \begin{cases} 2r^2 & , \; r \ll 1; \\ 1 - \frac{1}{2r^2} & , \; r \gg 1, \end{cases} \tag{9}$$

A simple analytical expression that interpolate between the above asymptotic forms is given by [23]

$$\rho_0(r) = \frac{2r^2}{1+2r^2}. \tag{10}$$

III. MOVING VORTEX

In this section we will derive the equations of motion for a moving vortex. We assume that the motion of the vortex has a small amplitude (not necessarily slow), such that the fields of density and phase of the condensate may be expanded as

$$\begin{aligned} S &= S_0(\mathbf{r}-\mathbf{r}_0(t)) + S_1(\mathbf{r},t), \\ \rho &= \rho_0(\mathbf{r}-\mathbf{r}_0(t)) + \rho_1(\mathbf{r},t), \end{aligned} \tag{11}$$

where at any instant of time, S_0 and ρ_0 satisfy the static vortex equations (7) and (8). S_1 and ρ_1 represent small corrections caused by the motion of the vortex. By substituting Eq.(11) into the Lagrangian (4), we arrive at an effective Lagrangian for $S_1(\mathbf{r},t)$ and $\rho_1(\mathbf{r},t)$ as well as the vortex coordinates $\mathbf{r}_0(t)$. There is no problem of redundancy in the dynamic variables; unlike the original phase field $S(\mathbf{r},t)$, $S_1(\mathbf{r},t)$ is required to be single valued [24]. Keeping up to second order terms in $\mathbf{r}_0$, S_1 and ρ_1, we find the new Lagrangian as

$$\begin{aligned} L = -\int d^2r \Big[& \rho_0\dot{S}_0 + \frac{1}{2}\rho_0|\nabla S_0|^2 + \frac{1}{2}[\rho_0-1]^2 + \rho_0\dot{S}_1 + \rho_1\dot{S}_0 + \rho_0\nabla S_0\cdot\nabla S_1 + \\ & + \frac{1}{2}|\nabla S_0|^2\rho_1 + [\rho_0-1]\rho_1 + \rho_1\dot{S}_1 + \frac{1}{2}\rho_0|\nabla S_1|^2 + \rho_1\nabla S_0\cdot\nabla S_1 + \frac{1}{2}\rho_1^2 \\ & + \frac{1}{8\rho_0}|\nabla\rho_0|^2 + \frac{1}{4\rho_0}\nabla\rho_0\cdot\nabla\rho_1 - \frac{|\nabla\rho_0|^2}{8\rho_0^2}\rho_1 + \frac{|\nabla\rho_0|^2}{8\rho_0^3}{\rho_1}^2 - \frac{\rho_1}{4\rho_0^2}\nabla\rho_0\cdot\nabla\rho_1 + \frac{|\nabla\rho_1|^2}{8\rho_0}\Big]. \end{aligned} \tag{12}$$

We will add an extra kinetic energy term of the form

$$\frac{1}{2}M_e\dot{\mathbf{r}}_0^2, \tag{13}$$

to simulate the situation with particles trapped inside the vortex core. It is understood that this may be an over simplification of the real interactions between the trapped particles and the vortex [25], but the main purpose of introducing this term is to provide a mechanism for controlling the time scale of the vortex motion. The external mass M_e is measured in units of $(m\xi^2\bar{\rho})$, and a bare vortex is described by $M_e = 0$.

The resulting Lagrangian can be simplified considerably by using the equations for the static vortex, integration by parts and dropping the constant terms:

$$
\begin{aligned}
L = & -\int d^2r \Big[\rho_0\dot{S}_0 + \rho_0\dot{S}_1 + \rho_1\dot{S}_0 + \rho_1\dot{S}_1 + \frac{1}{2}\rho_0|\nabla S_1|^2 + \rho_1\nabla S_0\cdot\nabla S_1 + \\
& + \frac{|\nabla\rho_0|^2}{8\rho_0^3}{\rho_1}^2 - \frac{\rho_1}{4\rho_0^2}\nabla\rho_0\cdot\nabla\rho_1 + \frac{1}{8\rho_0}|\nabla\rho_1|^2 + \frac{1}{2}\,{\rho_1}^2\Big] - \\
& - \oint S_1\rho_0\nabla S_0\cdot\hat{\mathbf{n}}d\ell + \frac{1}{2}M_e\dot{\mathbf{r}}_0^2.
\end{aligned}
\tag{14}
$$

The boundary conditions will be taken as $\rho_1 = 0$ and $\nabla S_1 = 0$ as well as $\dot{S}_1 = 0$ at infinity. Then, the line integral, which is taken around the boundary, only adds a constant to the Lagrangian and can be dropped. The dynamical equation of the vortex is then obtained by variation of the action with respect to $\mathbf{r}_0$. The equation, linearized in $\dot{\mathbf{r}}_0$, ρ_1 and S_1, is

$$
-2\pi\dot{\mathbf{r}}_0\times\hat{\mathbf{z}} + \int\left[\dot{S}_1\nabla\rho_0 - \dot{\rho}_1\nabla S_0\right]d^2r - M_e\ddot{\mathbf{r}}_0 = 0,
\tag{15}
$$

where we have used the fact that $\nabla_{\mathbf{r}_0}$ can be replaced by $-\nabla$ when it acts on $(\mathbf{r}-\mathbf{r}_0)$. The first term represents the well-known Magnus force. The second term shows the coupling between the vortex and the condensate, and the last term is the usual inertial force on the particles trapped at the core. Note that by comparing Eq.(15) to Eq.(1), one can extract the phenomenological parameters, such as the vortex mass and coefficient of viscosity. Also note that, it is not possible to identify the vortex mass immediately from this equation without knowing the perturbations ρ_1 and S_1.

The linearized equations for ρ_1 and S_1 are

$$
\dot{S}_1 + \nabla S_0\cdot\nabla S_1 + \frac{1}{4\rho_0^2}\nabla\rho_0\cdot\nabla\rho_1 - \frac{1}{4\rho_0}\nabla^2\rho_1 - \frac{|\nabla\rho_0|^2}{4\rho_0^3}\rho_1 + \frac{\nabla^2\rho_0}{4\rho_0^2}\rho_1 + \rho_1 = \dot{\mathbf{r}}_0\cdot\nabla S_0,
\tag{16}
$$

$$
\dot{\rho}_1 + \nabla\rho_0\cdot\nabla S_1 + \rho_0\nabla^2 S_1 + \nabla S_0\cdot\nabla\rho_1 = \dot{\mathbf{r}}_0\cdot\nabla\rho_0.
\tag{17}
$$

From these two equations, we notice that the dynamics of the condensate is driven by the motion of the vortex. In other words, the vortex is accompanied with backflow-like corrections, ρ_1 and S_1, which on the other hand, act back on the vortex in a way determined via Eq.(15).

In the rest of the paper, we choose the origin of $\mathbf{r}$ to be instantaneously at $\mathbf{r}_0$. The dynamical equations (15), (16) and (17) remain unchanged up to second order terms in $\mathbf{r}_0$, ρ_1 or S_1. Also, the boundary conditions are such that ρ_1 and S_1 are regular at the origin, while $\rho_1 \to 0$, $\nabla S_1 \to 0$ at infinity (boundary).

IV. THE EQUATIONS FOR CYCLOTRON MOTION

We would like to consider solutions of the following form:

$$\mathbf{r}_0 = \mathrm{Re}\left[b e^{-i\omega t} (\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \right], \tag{18}$$

$$\rho_1 = \mathrm{Re}\left[bF(r) e^{-i(\omega t - \theta)} \right], \tag{19}$$

$$S_1 = \mathrm{Re}\left[ibG(r) e^{-i(\omega t - \theta)} \right]. \tag{20}$$

The constant parameters ω and b, and the functions $F(r)$ and $G(r)$, are yet to be determined. The first expression shows that the motion of the vortex is of cyclotron type with size b and frequency ω. The frequency is allowed to have an imaginary part to describe a damped cyclotron motion. A complex phase factor in b is immaterial, because it only affects the initial angle of $\mathbf{r}_0$. A simple angular harmonic analysis shows that the angular dependence of the condensate response has to be in the given form. In a general sense, this motion is the analogue of the massive branch of the helical vortex waves in classical fluids [26,27]. For simplicity the expression "Re" will be dropped in the rest of the discussion.

Using the following expressions for the right hand sides of equations (16) and (17)

$$\begin{aligned} \dot{\mathbf{r}}_0 \cdot \frac{\hat{\boldsymbol{\theta}}}{r} &= \frac{b\omega}{r} e^{-i(\omega t - \theta)}, \\ \dot{\mathbf{r}}_0 \cdot \hat{\mathbf{r}} \rho_0' &= -ib\omega \rho_0' e^{-i(\omega t - \theta)}, \end{aligned} \tag{21}$$

we find that Eqs.(15),(16) and (17) imply the following equations for ω, F, and G:

$$\frac{1}{2}\int_0^\infty [r\rho_0' G + F]\, dr + \frac{M_e}{2\pi}\omega = 1, \tag{22}$$

$$\frac{\rho_0'}{4\rho_0^2}F' - \frac{1}{4\rho_0}\left[F'' + \frac{F'}{r} - \frac{F}{r^2}\right] - \left[\frac{{\rho_0'}^2}{4\rho_0^3} - \frac{\rho_0'' + \rho_0'/r}{4\rho_0^2} - 1\right]F + \left[\omega - \frac{1}{r^2}\right]G = \frac{\omega}{r}, \tag{23}$$

$$\rho_0' G' + \rho_0\left[G'' + \frac{G'}{r} - \frac{G}{r^2}\right] - \left[\omega - \frac{1}{r^2}\right]F = -\omega\rho_0', \tag{24}$$

where a prime denotes differentiation with respect to r. Note that the parameter b has been factored out from the above equations, and it is only to be fixed by the initial velocity of the vortex.

V. CONDENSATE RESPONSE TO THE CYCLOTRON MOTION

In this section, we consider the response of the condensate density and phase to the vortex motion. Analytic and semiquantitative results may be obtained by studying the limits of large ($r \gg 1$) and small distances ($r \ll 1$). The large r limit carries the information on how the system size may affect the dynamics, and the small r limit gives the contribution of the core where most of the variation in ρ_0 occurs. These two limits cover the essential features of the dynamics.

When $r \ll 1$, we can neglect ω and replace ρ_0 by $2r^2$ (cf.Eq.(9)) on the left hand side of Eqs.(23) and (24). After a straightforward but tedious calculation one can show that the solutions are of the following form

$$F(r) = 4A\omega r^3, \tag{25}$$

$$G(r) = B\omega r. \tag{26}$$

It is not difficult to find $A + B = -1$, although the individual values of A and B are undetermined yet. Our knowledge of their sum will be enough for the purpose of evaluating the integrand in Eq.(22).

When $r \gg 1$, we may use the approximation of ρ_0 for large r as given in Eq.(9). Also, the first three terms on the left hand side of Eq.(23) is dominated by F, yielding

$$F + \left[\omega - \frac{1}{r^2}\right] G = \frac{\omega}{r}. \tag{27}$$

Substituting this relation into Eq.(24), we arrive at the following equation for G

$$G'' + \frac{1}{r}G' - \frac{1 + 2\omega}{r^2}G + \omega^2 G = \frac{\omega^2}{r} - \frac{2\omega}{r^3}. \tag{28}$$

It is easily verified that a special solution is $G = \frac{1}{r}$. To obtain the general solution, we note that the corresponding homogeneous equation for G is that of the Bessel functions of order $\nu = \sqrt{1 + 2\omega}$. We only keep the outgoing wave, which represents the radiated phonons, then the general solution is given by

$$G = \frac{1}{r} + CH_\nu^{(1)}(\omega r), \tag{29}$$

where C is a constant and $H_\nu^{(1)}$ is the first kind Hankel's function. This implies via Eq.(27)

$$F = \frac{1}{r^3} - C\left[\omega - \frac{1}{r^2}\right] H_\nu^{(1)}(\omega r), \tag{30}$$

A similar radiation process is also found for an oscillating object in a classical fluid [28].

The constants B and C must be determined by matching the solutions in the intermediate region $r \approx 1$. For semiquantitative purposes, we may regard the above solutions for small and large r to be valid up to the coherent length $r = 1$ (where $\rho_0 = 2/3$). Then the continuity of G and its derivative yield

$$C = \frac{2}{\omega H_\nu^{(1)\prime}(\omega) - H_\nu^{(1)}(\omega)} \tag{31}$$

and

$$B = \frac{1}{\omega} \frac{\omega H_\nu^{(1)\prime}(\omega) + H_\nu^{(1)}(\omega)}{\omega H_\nu^{(1)\prime}(\omega) - H_\nu^{(1)}(\omega)}, \tag{32}$$

where a prime denotes differentiation with respect to the argument.

VI. LOW FREQUENCY SOLUTIONS

It will be instructive to consider the case of small ω, which, as we will see later, may be achieved in the limit of large mass of the trapped particles. Using the expansion of the Hankel function for the limit of small argument, we can find the constants as $C = -i\frac{\pi}{2}\omega$, and $B = \frac{1}{2}$.

Depending on the assumption that the asymptotic forms of the solutions are valid in the whole regions above and below $r = 1$ respectively, the cyclotron frequency can be determined from Eq.(22). Obviously, this constitutes a rough estimate of the frequency, but we expect the qualitative behavior to be well reflected. Then, to lowest order in ω we have

$$\left[\frac{1}{2}\ln(1/\omega) + \frac{\pi}{4}i\right]\omega + \frac{M_e}{2\pi}\omega = 1. \tag{33}$$

In the large M_e limit, we can solve for ω as

$$\omega = \frac{1}{\frac{M_e}{2\pi} + \frac{1}{2}\ln(\frac{M_e}{2\pi}) + \frac{\pi}{4}i}. \tag{34}$$

This gives a vortex mass via Eq.(2)

$$M_v = M_e + \ln(\frac{M_e}{2\pi}) + \frac{\pi^2}{2}i. \tag{35}$$

Apart from the mass of the trapped particles, there is a hydrodynamic correction (the second term) which diverges logarithmically with M_e. The imaginary part represents a decay rate of the cyclotron motion due to the radiation of phonons. It is seen that this decay becomes unimportant when $M_e >> \pi^2/2$.

To understand the logarithmic correction, we divide the $r > 1$ region further into two regimes: $1 < r < 1/\omega$, and $r > 1/\omega$. In the former regime, we can use the asymptotic form of the Hankel's function, with the result:

$$G = \frac{0.575\omega}{r}, \tag{36}$$

and

$$F = \frac{\omega}{r}. \tag{37}$$

This gives a density response ρ_1 exactly the same as found using the adiabaticity assumption [12–14,16]. We will therefore call this regime as the adiabatic regime. This regime has a size of one phonon wavelength $\lambda = \frac{1}{\omega}$, and can be very large when M_e is large. The adiabatic following of the condensate with the vortex in this region is responsible for the logarithmic correction to the vortex mass.

For $r >> 1/\omega$ we can write the solutions as

$$G = \frac{1}{r} - i\sqrt{\frac{\pi\omega}{2r}} e^{i(\omega r + \frac{\pi}{4})}, \tag{38}$$

and

$$F = \frac{1}{r^3} + i\omega\sqrt{\frac{\pi\omega}{2r}} e^{i(\omega r + \frac{\pi}{4})}. \tag{39}$$

It is not difficult to show that to first order in $\mathbf{r_0}$, the condensate phase and density becomes

$$S(\mathbf{r}, t) = S_0(\mathbf{r}) + b\sqrt{\frac{\pi\omega}{2r}} e^{i(\omega r + \frac{\pi}{4})} e^{-i\omega t}, \tag{40}$$

and

$$\rho(\mathbf{r}, t) = \rho_0(\mathbf{r}) + ib\omega\sqrt{\frac{\pi\omega}{2r}} e^{i(\omega r + \frac{\pi}{4})} e^{-i\omega t}. \tag{41}$$

The first terms on the right hand sides are the phase and density of a static vortex at the origin, and the other terms are oscillations in the condensate that represent the phonons radiated by the motion of the vortex. Therefore, at distances larger than λ the motion of the condensate is not an adiabatic following of the vortex.

We now calculate the coefficient of viscosity corresponding to the damping. It is given by

$$\eta = \frac{2\pi\hbar\bar{\rho}}{Q}, \tag{42}$$

where Q is the quality factor, and we have restored the real units. Using the previous results, we find $Q = M_c/M_e$, where $M_c = \frac{\pi^2}{2}\xi^2 m\bar{\rho}$ is roughly the mass of the superfluid that is expelled from the vortex core. Using the value of the three dimensional density $\bar{\rho}_{3D} \approx 10^{28}\text{m}^{-3}$ for superfluid He^4, the viscosity per unit length of a vortex line becomes

$$\frac{\eta}{d} \approx 7 \times 10^{-6} \left[\frac{M_c}{M_e}\right] \text{kg/(m} \cdot \text{s)}. \tag{43}$$

In comparison to the viscosity induced by the scattering of excitations in superfluid ^{4}He at temperature 1 K [4], the numerical factor is on the same order, but the factor (M_c/M_e) can reduce it further.

VII. CYCLOTRON MOTION OF FINITE FREQUENCY

In order to extend our conclusions to the small M_e as well as to the bare vortex case, we adopt a semi-numerical method. We use the approximate solutions found in Eqs.(25),(26),(29) and (30) to find a numerical solution of the cyclotron frequency from Eq.(22). Our results are contained in Fig.(1), where solid lines are the real and imaginary parts of the cyclotron frequency from the numerical calculation, and the dashed lines are from the approximate expression in Eq.(33).

The agreement between the approximate and numerical solution for large M_e is quite satisfactory. The estimate, $M_e >> \pi^2/2$, for the radiative damping to be negligible is seen to be appropriate. In general we see that as M_e decreases, the magnitudes of the imaginary and real parts increase, and for $M_e = 0$ they are both approximately given by the time scale of the nonlinear Schrödinger Lagrangian

$$\omega_{r,i} \approx \tau^{-1} = \frac{\hbar}{m\xi^2}. \tag{44}$$

Thus, the motion of a bare vortex is heavily damped. Also, we notice that the imaginary part is smooth over the whole range, but there is a peak in the real part near $M_e/2\pi \approx 0.81$. This can either be a superficial defect caused by the inappropriate handling of the solutions at $r \approx \xi$, or a real physical phenomenon, the nature of which can only be substantiated by a numerical solution of the complete dynamical equations. Such an effort is currently underway.

VIII. DISCUSSION AND CONCLUSIONS

In summary, we have tried to give a better understanding of the dynamics of vortices in superfluid systems. We have derived an effective Lagrangian from a nonlinear Schrödinger Lagrangian, and obtained the dynamical equations for the vortex coupled with the condensate phase and density.

We showed that the natural motion of the vortex turns out to be of cyclotron type just like the one predicted by the phenomenological vortex equation in (1). There are three qualitatively different regimes for the condensate response: the core regime ($r < \xi$), the adiabatic regime ($\xi < r < \lambda$), where λ is the phonon wavelength at the cyclotron frequency, and the radiation regime ($r > \lambda$). Combined with the numerical result, this wavelength is roughly given by $\lambda \approx \xi(1 + M_e/M_c)$. When the mass of the trapped particles is large compared to the core mass, the cyclotron frequency is low, and there is a logarithmic correction to the vortex mass due to the adiabatic following of the density and phase fluctuations in the large region of adiabatic regime [12]. Similar results have been obtained by Wexler and Thouless [29], and Arovas and Freire [30] using different methods. The phonon radiation damping is negligible here as was expected in an earlier work of Niu, Ao and Thouless [8].

For a bare vortex, the cyclotron period is on the order of the time that a phonon travels a coherence length, and the vortex mass is on the order of the mass of the fluid that can occupy the core. The adiabatic regime is essentially empty, and the logarithmic correction is absent. However, due to the large density of states of the phonons at the enhanced frequency, radiation damping is heavy and may completely overshadow the cyclotron motion.

In order to experimentally observe the vortex cyclotron motion, one has to create a situation of small damping. The present theory is too crude to exactly tell whether a bare vortex in superfluid ^{4}He should be overdamped or underdamped, because the nonlinear Schrödinger equation does not treat the core structure accurately. One needs to carry out a microscopic calculation in order to pin down this issue. In the appendix, we briefly outline one such method.

Ion trapping in a vortex core can be a useful way of reducing the cyclotron frequency and the damping. We emulated this effect crudely by adding an external mass term to the Lagrangian. In reality, a trapped ion also expel the superfluid particles, making the core much bigger than the coherence length ξ. For example, for negative ions (electrons) the expanded core size can be as large as $16\mathring{A}$ compared to a bare core size of $1\mathring{A}$ in superfluid ^{4}He. In such cases the external mass in our theory should also include the expelled superfluid mass by the ions. The vortex can be driven either by an oscillating superflow, or by an ac electric field which acts on the ions. Resonances should be observed at the cyclotron frequency, which is on the order of $\frac{\hbar}{ma^2}$ where a is the radius of the hollow core. For the case of a negative ion, this gives $\omega \approx 6$ GHz.

When we consider charged superfluids, we must deal with the complication due to the existence of yet another length scale, the London penetration length. However, for thin superconductor films, the London penetration length is quite large, and vortices are very similar to their counterparts in ^{4}He. For a bare vortex, the adiabatic length is about the same as the coherence length, so that there is no logarithmic correction due to the adiabatic following of the condensate. The cyclotron frequency should be given by the flux quantum times the background 2D charge density divided by the vortex mass:

$$\omega = \frac{\hbar}{m_e \xi^2}, \tag{45}$$

which only depends on the electron mass and the coherence length. Using the relation between the coherence length and the superconducting gap Δ, $\xi = \frac{\hbar v_f}{\pi \Delta}$, the cyclotron energy is found to be smaller than the gap by a small factor equal to Δ/ϵ_f, where ϵ_f is the normal-electron Fermi energy. The cyclotron motion may be resonantly excited by an ac superflow as we discussed above for the case of superfluid ^{4}He. For Al thin films, in the very clean limit, $\xi = 1.6$ μm, we have $\omega = 45$ MHz. The resonance should show itself as a peak in the resistance-frequency plot.

The finite temperature effects can also be foreseen qualitatively along the lines of the present analysis. Primarily, since the core size increases with the temperature, the vortex mass should become larger, and one expects to see the resonances at lower cyclotron frequencies. Another fact that contribute to this effect is the reduction of the Magnus force due to the reduction of the superfluid density at finite temperature.

Similar to the mechanism of phonon radiation damping in neutral superfluids, there is plasmon radiation damping in the superconductor case, but the effect is not strong because of low (zero for 3D) density of states of plasma at the cyclotron frequency. The cyclotron motion may also be damped by the excitations of normal electrons in the core. This effect might be strong due to the coincidence of the cyclotron energy with the excitation energies of normal electrons in the core, but the exact nature of this coupling has to be established in a microscopic calculation.

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X. APPENDIX: A NONLOCAL NONLINEAR SCHRÖDINGER THEORY

Assume that we are given the correct ground state $|0\rangle$ of a superfluid in the absence of vortices and collective excitations. Then to a good approximation, the dynamics of the superfluid may be described by the Feynman wave function [31]:

$$|\Psi_F\rangle = \frac{\exp\left(\sum_j (iS(\mathbf{r}_j) + \alpha(\mathbf{r}_j))\right)|0\rangle}{\left[\langle 0|\exp(2\sum_j \alpha(\mathbf{r}_j))\,|0\rangle\right]^{1/2}}, \tag{46}$$

where S and α are time-dependent real functions, and the denominator is a normalization factor. The essential feature of the Feynman wave function lies in the factor of products of single particle wavefunctions, whose position and time dependence may generate superflow, density fluctuations, and quantized vortices.

The dynamical equations for S and α may be obtained variationally. The time-dependent Schrödinger equation can be regarded as the Euler-Lagrangian equations of the following Lagrangian

$$L = i \langle\Psi| \frac{\partial}{\partial t} |\Psi\rangle - \langle\Psi| H |\Psi\rangle . \tag{47}$$

where H is the full microscopic Hamiltonian containing the kinetic as well as interaction energies of the superfluid particles. If we substitute the Feynman wave function into the above expression, then, after some mathematical manipulations, the following Lagrangian may be obtained for the fields S and α:

$$L = -\int \rho(\mathbf{r}) \left[\dot{S}(\mathbf{r}) + \frac{1}{2} |\nabla S(\mathbf{r})|^2 + \frac{1}{2} |\nabla \alpha(\mathbf{r})|^2 \right] d^2r, \tag{48}$$

where the density $\rho(\mathbf{r})$ is defined by

$$\rho(\mathbf{r}) = \frac{\langle 0| \exp\left(2 \int \alpha(\mathbf{r}')\hat{\rho}(\mathbf{r}') d^2r'\right) \hat{\rho}(\mathbf{r}) |0\rangle}{\langle 0| \exp\left(2 \int \alpha(\mathbf{r})\hat{\rho}(\mathbf{r}) d^2r\right) |0\rangle} \tag{49}$$

with $\hat{\rho}$ being the density operator. The last two equations provide a self-contained set. Incorporated with the works on the ground state wavefunctions [32], this procedure constitutes a microscopic derivation of an effective Lagrangian that takes care of the short distance effects in a better way than the nonlinear Schrödinger Lagrangian.

The usual nonlinear Schrödinger Lagrangian is recovered if we take $|0\rangle$ as the simple product state of zero momentum. In this case, the density $\rho(\mathbf{r})$ is simply $\exp(2\alpha(\mathbf{r}))$. In the general case, Eq.(49) still resembles the usual nonlinear Schrödinger Lagrangian by having identical terms involving the phase field S. The term involving α is generally a nonlocal and nonlinear functional of ρ, but is independent of S. A full scale study of the implementation and implications of the nonlocal nonlinear Schrödinger theory will be presented in separate publications.

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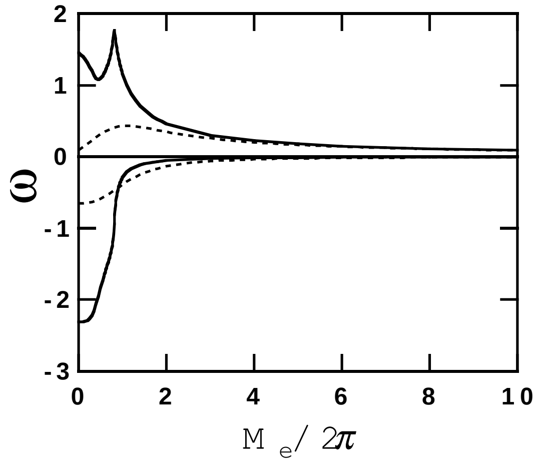
FIGURES

FIG. 1. Plot of the real (upper curves) and imaginary parts (lower curves) of the cyclotron frequency as a function of the external mass, calculated by using the analytical solutions (solid) and by using Eq.(33) (dashed). The frequency ω is in units of τ^{-1}, and the external mass $M_e/2\pi$ is in units of $\xi^2\bar{\rho}m$.



2
1
0
ω
-1
-2
-3
0
2
4
6
8
10
M e/ 2π