Landau and dynamical instabilities of the superflow of Bose-Einstein condensates in optical lattices

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The superfluidity of Bose-Einstein condensates (BECs) in optical lattices is investigated. Apart from the usual Landau instability, which occurs when a BEC flows faster than the speed of sound, the BEC can also suffer a dynamical instability, resulting in period doubling and other sorts of symmetry breaking of the system. Such an instability plays a crucial role in the dissipative motion of a trapped BEC in an optical lattice recently observed [Burger *et al.*, Phys. Rev. Lett. **86**, 4447 (2001)].

Optical lattices have long been used to manipulate ultracold atoms, with applications ranging from beam splitters [1] and accelerators [2,3] to lithography [4]. There are now growing interests in replacing the cold atoms with Bose-Einstein condensates (BECs) of alkali atoms [5–7] to explore the effects of coherence, atomic interaction, and superfluidity, with important applications in atom lasers [9] and high-precision interferometry [10].

In this paper, we investigate the superfluidity and instabilities of BECs in optical lattices. In free space, the superflow of a uniform BEC is represented by a plane wave, which has Landau instability when it travels faster than the sound. In an optical lattice, the natural objects of concern are the BEC Bloch waves, whose amplitudes are modulated with the same periodicity as the lattice. In the central region of the Brillouin zone of the lowest band, they are found to be local energy minima against all sorts of perturbations and thus represent the superflows of the BEC in optical lattices. For sufficiently strong repulsive interactions between the atoms, the superfluidity region can extend over the entire Brillouin zone; but for weaker interaction, Landau instability can occur in the outer regions of the zone, where the Bloch waves become energy saddle points. Many of the Bloch states with Landau instability can even be dynamically unstable in that small initial disturbances around them grow exponentially in time, resulting in period doubling and other forms of spontaneous breaking of the periodicity of the system. This dynamical instability is unique to BEC Bloch waves and is not present in BEC plane waves in free space. We map out the dangerous zones of the dynamical instability, characterize the growth rates, and discuss the experimental consequences.

We consider the situation of a one-dimensional optical lattice in which the motion in the perpendicular directions are confined [8] or can be disregarded [7]. We treat the atomic interaction with the mean-field theory, and obtain the grand-canonical Hamiltonian

$$H = \int_{-\infty}^{\infty} dx \left\{ \phi^* \left(-\frac{1}{2} \frac{\partial^2}{\partial x^2} + v \cos x \right) \phi + \frac{c}{2} |\phi|^4 - \mu |\phi|^2 \right\},$$

(1)

where all the variables are scaled to be dimensionless by the system's basic parameters, the atomic mass m, the wave number k_L of the two laser lights that generate the optical lattice, and the average density n_0 of the BEC. The chemical potential μ and the strength of the periodic potential v are in units of $4\hbar^2 k_L^2/m$, the wave function ϕ is in units of $\sqrt{n_0}$, x is in units of $1/2k_L$, and t is in units of $m/4\hbar k_L^2$. The coupling constant $c = \pi n_0 a_s/k_L^2$, where $a_s > 0$ is the s-wave scattering length.

Hamiltonian (1) is extremized by states in the form of Bloch waves, $\phi_k(x) = e^{ikx} \varphi_k(x)$, where $\varphi_k(x)$ is of the period of the optical lattice and can be expanded as a Fourier series. To find the numerical solution of a Bloch state in the lowest band, $\varphi_k(x)$, we truncate the series up to the Nth term $\varphi_k(x) = \sum_{-N}^{N} a_m e^{imx}$ (we used $N = 10$). The numerical solution is obtained by varying $\{a_m\}$ so that the wave function $\varphi_k(x)$ minimizes the system's total energy. The accuracy is checked by substituting the solutions into the Gross-Pitaevskii equation,

$$-\frac{1}{2} \frac{\partial^2}{\partial x^2} \phi + v \cos x \phi + c |\phi|^2 \phi = \mu \phi,$$

which is obtained by the variation of Hamiltonian (1).

To determine the superfluidity of these Bloch states, we need to find out if they remain energy minima against perturbations that break the periodicity. These perturbations can be decomposed into different modes labeled by q,

$$\delta \varphi_k(x, q) = u_k(x, q) e^{iqx} + v_k^*(x, q) e^{-iqx},$$

(2)

where q ranges between $-1/2$ and $1/2$ and the perturbation functions u_k and v_k are of periodicity of 2π in x. Since the system is periodic, the quadratic form of the energy deviation from the Bloch state ϕ_k is block diagonal in q, with each block given by

$$\delta E_k = \int_{-\infty}^{\infty} dx (u_k^*, v_k^*) M_k(q) \begin{pmatrix} u_k \\ v_k \end{pmatrix},$$

(3)

where

$$M_k(q) = \begin{pmatrix} \mathcal{L}(k+q) & c \varphi_k^2 \\ c \varphi_k^{*2} & \mathcal{L}(-k+q) \end{pmatrix},$$

(4)

with

$$\mathcal{L}(k) = -\frac{1}{2} \left(\frac{\partial}{\partial x} + ik \right)^2 + v \cos x - \mu + 2c |\varphi_k|^2.$$

(5)

If $M_k(q)$ is positive definite for all $-1/2 \leq q \leq 1/2$, the Bloch wave ϕ_k is a local minimum. Otherwise, δE_k can be negative for some q, and the Bloch wave is a saddle point.

We first consider the special case $v=0$, BEC in free space, where the Bloch state ϕ_k becomes a plane wave e^{ikx}. The operator $M_k(q)$ becomes a 2×2 matrix

$$M_k(q) = \begin{pmatrix} q^2/2 + kq + c & c \\ c & q^2/2 - kq + c \end{pmatrix}, \quad (6)$$

whose eigenvalues are found easily as

$$\lambda_\pm = \frac{q^2}{2} + c \pm \sqrt{k^2 q^2 + c^2}. \quad (7)$$

Since λ_+ is always positive, $M_k(q)$ fails to be positive definite only when $\lambda_- \leq 0$, or equivalently, $|k| \geq \sqrt{q^2/4 + c}$. It immediately follows that the BEC flow e^{ikx} becomes a saddle point when the flow speed exceeds the sound speed, $|k| > \sqrt{c}$. This is exactly the Landau condition for the breakdown of superfluidity [11], which has recently been confirmed experimentally on BEC [12].

The stability phase diagrams for BEC Bloch waves are shown in the panels of Fig. 1, where different values of v and c are considered. The results have reflection symmetry in k and q, so we only show the parameter region, $0 \leq k \leq 1/2$ and $0 \leq q \leq 1/2$. In the shaded area (light or dark) of each panel, the matrix $M_k(q)$ has negative eigenvalues, and the corresponding Bloch states ϕ_k are saddle points and have Landau instability. For those values of k outside the shaded area, the Bloch states are local energy minima and represent superflows. The superflow region expands with increasing atomic interaction c, and occupies the entire Brillouin zone for sufficiently large c. On the other hand, the lattice potential strength v does not affect the superflow region very much as we see in each row. The phase boundaries for $v \ll 1$ are well reproduced from the analytical expression $k = \sqrt{q^2/4 + c}$ for $v=0$, which is plotted as triangles in the first column.

A saddle-point Bloch state ϕ_k can still be dynamically stable in that small deviations from it remain small in the course of time evolution if no external persistent perturbations are present. This is the case for all Bloch states either in the absence of atomic interactions or periodic potentials. When both factors are present, many of the saddle-point Bloch states become dynamically unstable against certain perturbation modes q, shown as the dark-shaded regions in Fig. 1. These results are obtained from the linear stability analysis of the Gross-Pitaevskii equation [13],

$$i \frac{\partial}{\partial t} \phi = -\frac{1}{2} \frac{\partial^2 \phi}{\partial x^2} + v \cos(x) \phi + c |\phi|^2 \phi, \quad (8)$$

which governs the dynamics of the system. A Bloch state ϕ_k is a stationary solution of this equation, depending on time only through the phase factor $e^{-i\mu t}$. Writing the deviation in a form similar to Eq. (2), and expanding the above equation to first order, we find
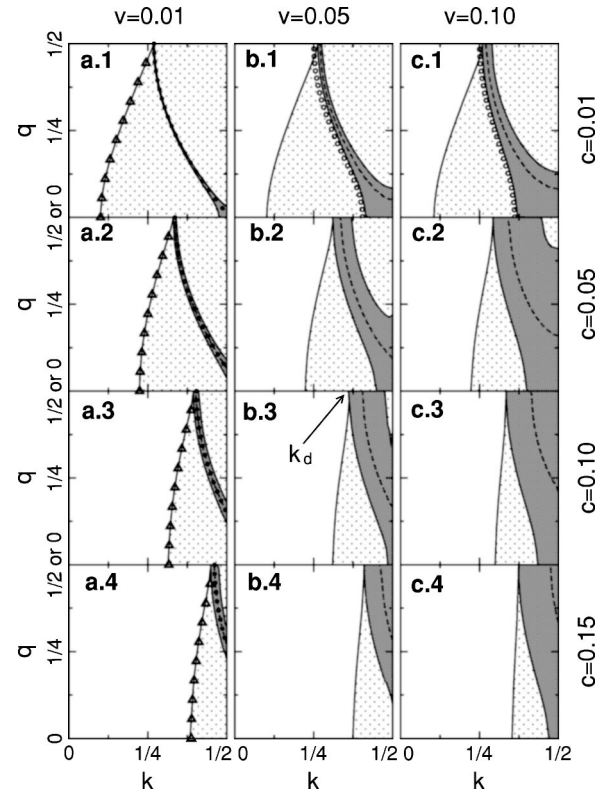

FIG. 1. Stability phase diagrams of BEC Bloch states in optical lattices. k is the wave number of BEC Bloch states, q denotes the wave number of perturbation modes. In the shaded (light or dark) area, the perturbation mode has negative excitation energy; in the dark shaded area, the mode grows or decays exponentially in time. The triangles in (a.1–a.4) represent the boundary, $q^2/4 + c = k^2$, of saddle point regions at $v=0$. The solid dots in the first column are from the analytical results of Eq. (12). The circles in (b.1) and (c.1) are based on the analytical expression (13). The dashed lines indicate the most unstable modes for each Bloch state k. k and q are in units of $2k_L$.

$$i \frac{\partial}{\partial t} \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \sigma M_k(q) \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad \sigma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (9)$$

The dynamical stability of the Bloch state ϕ_k is determined by the eigenvalues $\varepsilon_k(q)$ of the matrix $\sigma M_k(q)$. If they are real for all $-1/2 \leq q \leq 1/2$, the state is dynamically stable, otherwise it is dynamically unstable.

Before discussing our detailed results on the dynamical instability, we pause here to make some general remarks.

(i) When all the eigenvalues $\varepsilon_k(q)$ are real, the motions around the Bloch state ϕ_k are oscillations, which can be quantized to yield the phonon excitations [11,14]. The traditional Bogoliubov approach yields the same matrix $\sigma M_k(q)$ for the phonon spectrum. However, the bosonic commutation relation for the phonon operators imposes the skewed normalization condition $X^\dagger \sigma X = 1$, which selects only half of all the modes. The other half, satisfying $X^\dagger \sigma X = -1$, will be called antiphonon modes for ease of reference, but they really do not represent physical degrees of freedom independent of the phonons.

(ii) When the Bloch state is a local minimum [$M_k(q)$ positive definite], the dynamical eigenvalues $\varepsilon_k(q)$ of $\sigma M_k(q)$ are all real and the phonon branch of the spectrum is positive. The key to the proof is to notice that

$$\varepsilon_k(q)X^\dagger \sigma X = X^\dagger M_k(q)X \tag{10}$$

for an eigenvector X of $\sigma M_k(q)$. Because the right-hand side is positive and $X^\dagger \sigma X$ is real, $\varepsilon_k(q)$ is real and has the same sign as $X^\dagger \sigma X$. The physical meaning of this theorem is that, when it is a local minimum, the Bloch state ϕ_k is dynamically stable and its phonon excitations are not energetically favored.

(iii) Because the matrix $\sigma M_k(q)$ is non-Hermitian and real (when expressed in the momentum representation), complex eigenvalues can only appear in conjugate pairs, corresponding to modes growing or decaying exponentially at rates given by the imaginary part of the eigenvalues. Because both the quadratic forms in Eq. (10) are real, they must vanish when $\varepsilon_k(q)$ is complex. It is then impossible to enforce the normalization condition $X^\dagger \sigma X = 1$, corresponding to the fact that such modes cannot be quantized.

We now present our detailed results on the dynamical stability. Again, we first look at the case $v=0$, where the eigenvalues of $\sigma M_k(q)$ are

$$\varepsilon_\pm(q) = kq \pm \sqrt{q^2 c + q^4/4}. \tag{11}$$

These eigenvalues are always real; the BEC flows in free space are always dynamically stable. When $v \neq 0$, the situation is totally different: the eigenvalues $\varepsilon_k(q)$ of $\sigma M_k(q)$ can be complex and Bloch states can be dynamically unstable. The dark-shaded areas in Fig. 1 are the places where these $\varepsilon_k(q)$ are complex.

In the first column of Fig. 1, where $v \ll 1$, the dark-shaded areas are like broadened curves. These curves are the solutions of $\varepsilon_+(q-1) = \varepsilon_-(q)$,

$$k = \sqrt{q^2 c + q^4/4} + \sqrt{(q-1)^2 c + (q-1)^4/4}, \tag{12}$$

which are plotted as solid dots in Fig. 1. This is the resonant condition for a phonon mode to couple with an antiphonon mode by first-order Bragg scattering. The resonance is necessary because the complex eigenvalues can appear only in pairs, and they must come from a pair of real degenerated eigenvalues. Resonances within the phonon spectrum or within the antiphonon spectrum do not give rise to dynamical instability; they only generate gaps in the spectra. Somehow, in order to produce a mode with zero normalization, one must couple a pair of modes with opposite normalizations.

In the first row of Fig. 1, we have another extreme case $c \ll v$. The open circles along the left edges of these two dark-shaded areas are given by

$$E_1(k+q) - E_1(k) = E_1(k) - E_1(k-q), \tag{13}$$

where $E_1(k)$ is the lowest Bloch band of $H_0 = -1/2(\partial^2/\partial x^2) + v \cos x$. When $c=0$, this periodic system is linear; the excitation spectrum just corresponds to transitions from the condensate of energy $E_1(k)$ to other Bloch states of
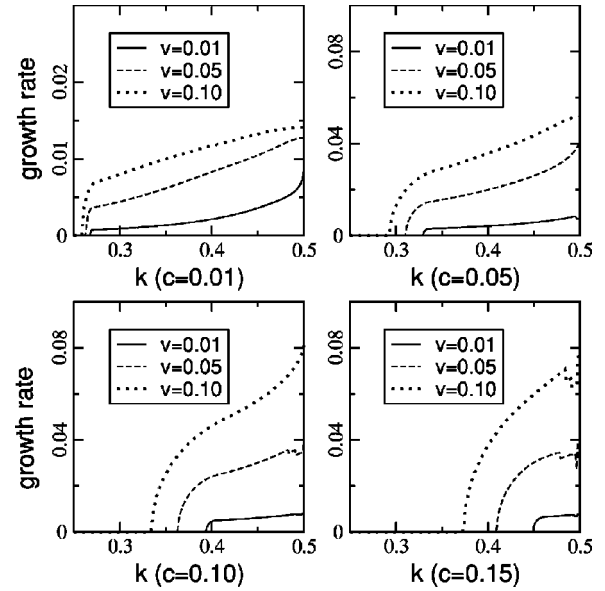

FIG. 2. Growth rates of the most unstable modes of each Bloch state ϕ_k. The erratic behavior of some curves around $k=1/2$ is due to the difficulty in finding the accurate Bloch waves ϕ_k in this region when $c>v$ [17]. k is in units of $2k_L$, the growth rate is in units of $4\hbar k_L^2/m$.

energy $E_n(k+q)$, or vice versa. The above equation is just the resonant condition between such excitations in the lowest band ($n=1$). Alternatively, this condition may be viewed as the energy and momentum conservation for two particles in the condensate to interact and decay into two different Bloch states $E_1(k+q)$ and $E_1(k-q)$.

One common feature of all the diagrams in Fig. 1 is that there is a critical Bloch wave number k_d beyond which the Bloch states ϕ_k are dynamically unstable. The onset instability at k_d always corresponds to $q=1/2$. In other words, if we drive the Bloch state ϕ_k from $k=0$ to $k=1/2$ the first unstable mode appearing is always $q = \pm 1/2$, which represents period doubling. Only for $k>k_d$ can longer wavelength instabilities occur. The growth of these unstable modes drives the system far away from the Bloch state and spontaneously breaks the translation symmetry of the system. The critical value of the Bloch wave number for the case of $v \ll 1$ is found to be $k_d = (c+1/16)^{1/2}$ by substituting $q = \frac{1}{2}$ into Eq. (12). In the other extreme case, $c \ll v$, the same substitution in Eq. (13) yields $k_d = 1/4$ with the help of periodicity of the band energy. Based on these results and the diagrams in Fig. 1, we find that $k_d \geq \frac{1}{4}$.

The dynamical instability discovered in this work should be observable in experiments. We have mapped out the dangerous zones of dynamical instability, which give us a good sense of where to look for unstable Bloch states and modes of instability. In Fig. 2, the rate of growth r for the most prominent mode (dashed lines in Fig. 1) of each Bloch state k is plotted in Fig. 2. The physical unit of the growth rate is $4\hbar k_L^2/m$, which is 4.0 μs^{-1} for sodium and 0.16 μs^{-1} for rubidium. Since the lifetime of BECs can be up to the order of seconds [15], these growth rates in Fig. 2 are significant. It is possible to directly observe the change of periodicity of

the BEC due to the dynamical instability, by monitoring the Bragg scattering of a probing laser light by the BEC cloud [16]. This dynamical instability can also cause the disruption of Bloch oscillations [17].

In a recent experiment, the superfluidity and instabilities of a BEC in an optical lattice was studied using a cigar-shaped (one-dimensional) magnetic trap [8]. After the BEC was prepared in the trap in the presence of the optical lattice, the trap was suddenly shifted by Δx along the longitudinal direction. This is equivalent to displacing the whole BEC off the center of the harmonic trap then releasing it. The subsequent oscillations of the BEC are nondamped if the initial displacement is small, but become dissipative if Δx is over a critical value Δx_c. This qualitatively agrees with our stability diagrams, because larger Δx implies larger velocity and therefore larger k. The dissipative behavior was explained as a manifestation of the Landau instability, but the dynamical

instability discussed here is likely to play a crucial role in our view. First, the experiment has $v \sim 0.2$ and $c \sim 0.02$, where the dynamical instability is rampant according to Fig. 2. Second, Δx_c increases with decreasing lattice potential v, which is in accordance with the trend of the growth rate as a function of v and k shown in the figure. Third, there is no dissipation when the BEC density is low, where Landau instability should be very strong but dynamical instability should be very weak according to Fig. 1. However, more detailed analyses are needed to take account of the effects of inhomogeneity and thermal cloud before a quantitative comparison with the experiment.

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