FROM NONCLASSICAL OPTICS TO QUANTUM INFORMATION SCIENCE: THE LEGACY OF SUDARSHAN'S DIAGONAL REPRESENTATION

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INTRODUCTION

Commenting on information theory which came into being, in one stroke, through the remarkable 1948 work of Shannon, A. I. Khinchin writes in his Mathematical Foundations of Information Theory (Dover, 1957):

Rarely does it happen in mathematics that a new discipline achieves the character of a mature and developed scientific theory in the first investigation devoted to it. Such in its time was the case with the theory of integral equations, after the fundamental work of Fredholm; so it was with information theory after the work of Shannon.

A similar description seems to fit perfectly Sudarshan's seminal work of 1963 which founded the field of Quantum Theory of

Optical Coherence. And, together with the theorems on general quantum dynamical maps he formulated two years earlier (more popularly known under the brand name of Kraus Representation, it has formed the robust framework for quantum optics of open systems. In the past 40 plus years Sudarshan's theory has been elaborated in many ways, has been extended in enormous number of dimensions, and has been applied to several walks of life beyond optics; as we shall see, the emerging Quantum Information Science is no exception.

While referring to the Born and Wolf classic, the unusual book of J.W. Simmons and M.J. Guttmann, States, Waves and Photons: A Modern Introduction to Light (Addison-Wesley, 1970) takes the liberty to add the subtitle The Bible of Non-quantum

Optics, to emphasize the fact that on any issue of optics, which is not specifically quantum, this work has some authoritative insight to offer. The point is that non-quantum subsumes classical. Borrowing this terminology, Sudarshan's paper may be called the bible of non-classical optics, the fact that this paper did not occupy anything much more than two printed pages notwithstanding. After all, the poems of Thiruvalluvar occupied not even two full lines; yet they remain ranked among the greatest works mankind has ever seen!

BEFORE SUDARSHAN

The pre-quantum or semiclassical theory of optical coherence and photoelectric detection were formulated by the Rochester School in the fifties and early sixties. The key element emphasised in these formulations was the role of the positive frequency part of the real optical field; this is the analytical signal. It was Glauber who showed in his 1963 PRL paper that quantum correlations could be, and had to be, formulated by simply substituting the positive frequency part of the hermitian field operators in place of the analytic signals in the corresponding classical expressions. He thus brought the coherent states, the eigenstates of these positive frequency operators, into the centre stage. He observed that the density operator of the thermal state is diagonal in the coherent state basis. Finally, Glauber related the photoionization probability of a pair of atoms, atom 1 at t_1 and atom 2 at t_2, to the quantum correlation function of order four in the field operators.

It seems Glauber had little reason to suspect that this diagonal form for thermal states could be part of a more general result or, equivalently, that the coherent state matrix elements of a density operator [his Eq.(3)] are not unique. In any case, there is nothing in the text of the paper, or in the list of references cited by him, to suggest that Glauber was, in any serious sense, aware of the overcompleteness of the coherent states, at the time of writing this paper of his.

Glauber's PRL paper appeared in print on February 01, 1963. It was received on December 27, 1962.

ENTER SUDARSHAN

Sudarshan's paper appeared in the April 01, 1963 issue of PRL. It was received on March 01, 1963.

Referring to Glauber's PRL paper, Sudarshan notes: statistical states of a quantized electromagnetic field have been considered recently, and a quantum mechanical definition of coherence function of arbitrary order presented. He has no interest in hair-splitting on the fact that Glauber had considered only the correlation function of order four, as noted above.

Sudarshan refers to the papers of Bargmann, Segal, Klauder, and Schweber on the analytic function representation of operators. None of these had been referred to by Glauber.

Sudarshan takes note of the non-orthogonality and over-completeness of the coherent states: We can make use of the overcompleteness of the states to represent every density matrix in the "diagonal" form $\cdots$

To complete the diagonal representation

$$\rho = \int d^2z\, \phi(z)\, |z\rangle\langle z|,$$

he gives the inversion formula expressing the phase-space ‘density’, or quasi-probability, $\phi(z)$ in terms of the density operator ρ. Note that the diagonal representation becomes a representation only with the presentation of the inversion formula!

Sudarshan notes that $\phi(z)$, the density function in the diagonal representation, is real in view of the hermiticity of ρ, but not necessarily positive definite. He shows that, in taking expectation values, every operator should be written in the normally-ordered form to go hand in hand with the diagonal representation, so that the expression for all correlation functions will have the same form as their classical counterparts, with $\phi(\{z_k\})$ playing the role of the probability density of the classical ensemble. This is the essence of the Optical Equivalence

Theorem. All nonclassicalities, if any, of a given state ρ are fully captured in the departure of the corresponding $\phi(\{z_k\})$ from a genuine classical probability, Sudarshan notes.

This departure continues to remain the definition of nonclassicality or quantumness and nonclassical optics. And this status is sure to continue. Finally, in Eq. (7) Sudarshan shows how the density operator can be determined from the measured correlation functions, if all the correlation functions are known (Quantum Tomography).

Note: At the time when Sudarshan's paper is sent to PRL (March 1, 1963), and in fact even at the time when it appeared in print on April 1, 1963, it seems that there is no statement or hint of the diagonal representation as a general theorem from anyone else, even as a preprint.

INTERLUDE

Glauber's next paper appeared in the Physical Review of June 15, 1963. It was received on February 11, 1963. It says: The present paper, which is the first of a series of fundamental papers on optics, is devoted largely to defining the concept of coherence. There is also given some hint as to what is to be expected in the next instalment of this series. For, after introducing the coherent states in Eq.(4.9), he states: We shall discuss the properties of such states at length in the paper to follow.

SECOND Phys. Rev. PAPER or GLAUBER IN HIS OWN WORDS

This paper appeared in the Physical Review of September 15, 1963. It was received on April 29, 1963, four weeks after Sudarshan's paper appeared in print. Though Glauber promised a series in his first Phys. Rev. paper, the series seems to have terminated here.

Towards the end of the abstract Glauber notes: A particular form is exhibited for the density operator which makes it possible to carry out many quantum mechanical calculations by methods resembling those of classical theory. This representation presents clear insights into the essential distinctions between the quantum and classical descriptions of the field. Be alerted and warned that this is part Glauber's abstract of his work, and is not to be misread as Glauber's eloquent summary of Sudarshan's work.

Again, while summarizing the contents of the paper, Glauber states: The application of the formalism to physical problems is begun in section VII, where we introduce a particular form for the density operator which seems especially suited to the treatment of radiation by macroscopic sources. Glauber makes an important point, namely that no useful application of the formalism will emerge without this particular form or representation. At this point Glauber makes no reference to Sudarshan, thereby tempting the reader to approach Section VII with great reverence, in anticipation of a genuinely new representation, which would form the basis for all applications.

Writing Section VII, particularly page 2776, appears to have been a painful task, even for Glauber who is one of the all time great expositors. First, he should avert the risk of this Section making the reader feel that Sections V and VI are simply remnants from an earlier draft before Sudarshan's paper appeared in print. Thus Glauber begins by presenting his view of why the unnecessary discussion in those two Sections had to be gone through: In the preceding section we have demonstrated the generality of the use of coherent states as a basis. Not all fields require for their description density operators of quite so general a form. Indeed, for a broad class of radiation fields which includes, as we shall see, virtually all of those studied in optics, it becomes possible to reduce the density operator to a considerably simpler form. This form is one which brings to light many similarities between quantum electro-dynamical calculations and the corresponding classical ones. And he goes on to add: Its use offers deep insights into the reasons why some of the fundamental laws of optics, such as those for the superposition of fields and calculation of the resulting intensities, are the same in classical theory, even when very few quanta are involved. This is precisely what Sudarshan had asserted in respect of his diagonal representation, but Glauber does not seem to care, for he has chosen not to cite Sudarshan as yet!

The promised particular form or representation is presented in Eq.(7.6):

$$\rho = \int P(\alpha)\,|\alpha\rangle\langle\alpha|\,d^2\alpha.$$

Glauber anticipates that the reader will wonder at this point: But is this not simply Sudarshan's diagonal representation, with ϕ replaced by P and z by α? So this time he adds a footnote: The existence of this form for the density operator has also been observed by E.C.G. Sudarshan, Phys. Rev. Lett. 10, 277 (1963). His note is discussed briefly at the end of Sec. X. Note that the phrase used is 'observed', and not even 'independently discovered', whereas the correct thing to say would have been 'discovered earlier by E.C.G. Sudarshan'.

In Glauber's opinion the bible of quantum optics is just a note, and not a paper. I wonder what will be his characterization of the writings of Thiruvalluvar.

So the curious reader moves on to this 'brief discussion' in Section X. Indeed there is a second and last reference to the work of Sudarshan: During the completion of the present paper a note by Sudarshan has appeared which deals with some of the problems of photon statistics that have been treated here. Sudarshan has observed what we have called the P representation of the density operator and has stated its connection with the representation based on the n-quantum states. To that extent, his work agrees with ours in Section VII to IX.

The above way of citing an earlier published paper is extraordinary, to say the least. Note that the same phrase 'observed' strikes again. How does the poor reader know that Glauber expects 'observed' to be read as 'discovered earlier'? Note also that it is not "To that extent, our work of Section VII to IX is along the same lines as implied in Sudarshan's work", but it is "To that extent, his work agrees with ours in Section VII to IX". How did Glauber know for certain that Sudarshan's gift of prophesy was so strong as to ensure that what he writes in February would agree with what Glauber will choose to claim in April?

Back to p.2776. Glauber goes on to specify the class of states for which his P-representation exists (or, rather, states he would be willing to allow). On this one he is categorical: The function $P(\alpha)$ need not be subject to any regularity conditions, but its singularities must be integrable ones. It is convenient to allow $P(\alpha)$ to have delta function singularities so that we may think of a pure coherent state as represented by a special case of equation (7.3). There is no question of Glauber allowing $P(\alpha)$ to assume negative values: It is the kind of operator we might be naturally led to if we were given knowledge that the oscillator is in a coherent state, but one which corresponds to an unknown eigenvalue α. That is, only incoherent mixtures or convex sums of coherent states are acceptable for his P-representation.

This lands Glauber on his next difficulty: All the states Glauber has just allowed are precisely the ones which are classical according to Sudarshan's diagonal representation theorem. In what appears to be a half-hearted attempt to overcome this difficulty, Glauber resorts towards the end of the page to a restricted or fine-tuned definition of classical states; this one is original and genuinely independent of Sudarshan. He alerts the reader to his observation earlier in the paper that coherent states $|\alpha\rangle$ and $|\alpha'\rangle$ are approximately orthogonal to one another only when $|\alpha - \alpha'| >> 1$, and goes on to add: When the function $P(\alpha)$ tends to vary little over such large ranges of the parameter α, the non-orthogonality of the coherent states will make little difference, and $P(\alpha)$ will then be interpretable approximately as a probability density. The functions $P(\alpha)$ which vary this slowly will, in general, be associated with strong fields, ones which may be described approximately in classical terms. Glauber does not seem to care that this class which he is willing to allow is nearly empty. Nor does he seem to care that if this decree of his is accepted, the set of all his classical states will not even form a convex set. In any case, I am yet to come across someone who has taken Glauber's attempt to modify Sudarshan's definition of classicality seriously.

As for his final difficulty in respect of writing this page, Glauber should have been acutely aware of the fact that he has no definition for his representation. To a beginning student it may appear that Eq.(7.6) is 'derived' through Eq.(7.3). This is not true, appearance notwithstanding; indeed, the latter is just the coherent state matrix element of the former.

Glauber knows that no integral representation is defined until the inversion formula is established. That is why in the case of his $R(\alpha^*, \beta)$ representation wherein he knows the inversion formula he presents it not just once, but four times: Eqs.(5.4), (5.7), (6.1), and (6.2). That he is not able to define his P-representation in the first place, however, does not seem to deter him from referring in Section IX to a vague 'generalized P distribution'.

There is little in this paper to suggest that Glauber has appreciated in any significant manner the depth, breadth, or universality of Sudarshan's paper, which forms the basis for all quantum optics. It is likely that Glauber got deceived by the length of this note. Interestingly, Sudarshan's diagonal representation theorem seems to have a deep inbuilt robustness or quantum security in the form of this no-go theorem: It is impossible to formulate a diagonal representation only for a subset of the state-space without appealing to Sudarshan's original theorem for the inversion formula. You take either all or none!

I have always admired Glauber for the clarity of his writings. Thus, I wonder how the half-hearted and self-contradicting incoherent hotchpotch arrangements in the second half of this paper could ever have flown out of Glauber's own pen.

WHAT IS THE P-REPRESENTATION ANYWAY?

DIAGONAL REPRESENTATION AND QUANTUM INFORMATION

As a final item I wish to show you how Sudarshan's diagonal representation has become the key to our understanding of quantum information of canonical or continuous variable systems. This area has become an active domain of research ever since the demonstration of teleportation by Professor Kimble's group in the late nineties.

The vacuum state is un-entangled or separable. And so are also the coherent states, being phase-space translates of the vacuum state. Since convex sums of separable states are separable, by definition, it follows from the diagonal representation theorem that all classical states are separable. We conclude that nonclassicality is a necessary condition for inseparability.

Local unitary transformations take separable states into separable states, and entangled states into entangled states. It thus follows that nonclassicality which can be transformed away by local unitaries cannot result in entanglement. Thus nonlocal nonclassicality is a necessary condition for entanglement. This much is immediate from the diagonal representation. But is it a sufficient condition?

One form of nonclassicality is squeezing, and it goes most naturally paired with Gaussian states. Thanks to the diagonal representation, it has turned out to be rather easy to prove that nonlocal nonclassicality is the necessary and sufficient condition for entanglement in respect of all (pure or mixed) Gaussian states. It does not matter how many parties are there, and it does not matter how many modes are there with each parties.

The importance of this result, and through it the power of the diagonal representation should be appreciated. To this end, it should be pointed out that in the case of a 3×3 bipartite system consisting of 3-dimensional Hilbert space on either side, we do not yet have an effective procedure for testing if a given mixed state is entangled or not. This is because Sudarshan has not given us yet a diagonal representation in the finite-dimensional case!

While squeezing is an example of phase-sensitive nonclassicality, it is not the only nonclassicality. There can be a whole range of phase-insensitive nonclassicalities associated with the photon number distribution (PND) of state ρ. Antibunching is just one of them. Let

$$p(n) \equiv \langle n|\rho|n\rangle, \quad n = 0, 1, 2, \cdots \tag{1}$$

define the PND, $|n\rangle$ being the Fock state with n quanta. It is convenient to define an associated distribution q_n through

$$q_n \equiv n! p(n)$$

Using the diagonal representation, one can then show that the PND is classical if and only if it satisfies the infinite set of inequalities

$$L^{(N)}, \quad \tilde{L}^{(N)} \geq 0, \quad N = 1, 2, 3, \cdots$$

where

$$L^{(N)} = \begin{pmatrix} q_0 & q_1 & q_2 & \cdot & \cdot & q_N \\ q_1 & q_2 & q_3 & \cdot & \cdot & q_{N+1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ q_N & q_{N+1} & q_{N+2} & \cdot & \cdot & q_{2N} \end{pmatrix}, \quad (2)$$

$$\tilde{L}^{(N)} = \begin{pmatrix} q_1 & q_2 & q_3 & \cdot & \cdot & q_{N+1} \\ q_2 & q_3 & q_4 & \cdot & \cdot & q_{N+2} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ q_{N+1} & q_{N+2} & q_{N+3} & \cdot & \cdot & q_{2N+1} \end{pmatrix}. \quad (3)$$

Considering the 2×2 diagonal blocks of the above matrices, we conclude that even if one of the following infinite set of conditions given by

$$\begin{pmatrix} q_{n-1} & q_n \\ q_n & q_{n+1} \end{pmatrix} \geq 0, \qquad (4)$$

$$i.e,\ \ q_{n-1}q_{n+1} - {q_n}^2 \geq 0$$

is violated, the state is nonclassical.

For an antibunched state, one of the three-term conditions

$$q_{n-1}q_{n+1} - {q_n}^2 \geq 0, \quad n = 1,\, 2,\, 3,\, \cdots$$

for some value of n will definitely be violated; but these conditions capture a whole range of nonclassicalities even in a state which is not antibunched!

Let us pass through one port of a beam splitter a radiation mode with PND $p(n)$ with vacuum entering the other port. Let $p(n)$ be a nonclassical PND, so that the input is nonclassical, but the nonclassicality is local. Now the beam splitter is not capable of producing or destroying nonclassicality, but it can transform local nonclassicality into a nonlocal nonclassicality. An analysis of this configuration shows that the two output modes are entangled whenever the input PND is nonclassical. Moreover, this entanglement is provably distillable!

To conclude: Shannon's work set the agenda for the classical information theory community. The agenda has not changed in spite of the enormous progress. An immediate consequence of the diagonal representation is that nonlocal nonclassicality is the only potential source of entanglement. How much of this potential can be harnessed as useful (distillable) entanglement will remain, it seems, the agenda for the quantum information community for a long time to come.

And that is the legacy of Sudarshan's diagonal representation!