Quantum Maps and Quantum States

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- There is a connection between positive maps and quantum states of bipartite systems
- Classification of positive maps

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ii)	Positivity	$\rho > 0$
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The set of density matrices $\mathcal{D}(\mathcal{H})$ is convex

$$\rho = s\rho_1 + (1-s)\rho_2$$

Pure states are extremal of $\mathcal{D}(\mathcal{H})$

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i) Preserves selfadjointness $\varphi(\rho)^\dagger = \varphi(\rho)$
ii) Preserves Positivity $\varphi(\rho) > 0$
iii) Preserves Normalization $\mathrm{Tr}\,\varphi(\rho) = 1$
iv) Preserves Linearity

$$\varphi(s\rho_1 + (1-s)\rho_2) = s\varphi(\rho_1) + (1-s)\varphi(\rho_2)$$

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Example: Unitary map

$\varphi(\rho) = U\rho U^\dagger$ (pure state)

The set of quantum maps $QMaps(\mathcal{D}(\mathcal{H}))$ is convex

Positive and Completely Positive Maps

1) Positive Map: $\varphi : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$

i) Selfadjointness $\varphi(a)^\dagger = \varphi(a^\dagger)$

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2) Completely Positive: $\varphi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$

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Example: Transposition map

$$\varphi_\tau(a) = a^T$$

Positive Maps in $\mathcal{L}(\mathbb{C}^n) = M_n$

The hermitian product of M_n

$$(a, b) = \mathrm{Tr}(a^\dagger\, b) \qquad \forall a, b \in M_n$$

Given a basis $\{f_\alpha; \alpha = 1, 2 \cdots n^2\}$ in M_n such that

$$(f_\alpha, f_\beta) = \delta_{\alpha\beta}$$

there is also a product in the space of maps

$$\langle \varphi_1, \varphi_2 \rangle = \sum_{\alpha=1}^{n^2} \mathrm{Tr}(\varphi_1(f_\alpha)^\dagger \varphi_2 f_\alpha)$$

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B Matrix :

$$\varphi(a) = \sum_{\alpha,\beta} B_{\alpha\beta}\ \varphi_{\alpha\beta}(a)$$

The matrix B is hermitian: $B^*_{\beta\alpha} = B_{\alpha\beta}$

Positive Maps in $\mathcal{L}(\mathbb{C}^n) = M_n$

[Sudarshan, Mathews and Rau '60]

If $|i>; i = 1, 2, \cdots n$ is a basis of $\mathbb{C}^n$,

$$f_\alpha = e_{ij} = |i><j|$$

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1. A Matrix :

$$A_{\alpha\alpha'} = A_{ij,i'j'}$$

$$\begin{cases} A^*_{ij,i'j'} = A_{ji,j'i'} \\ A_{ij,i'j'}\rho_{i'j'} > 0 \\ A_{ii,i'j'} = \delta_{i'j'} \end{cases}$$

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2. B Matrix :

$$B_{\alpha\alpha'} = B_{ii',jj'} \qquad \begin{cases} B^*_{ii',jj'} = B_{jj',ii'} \\ B_{ii',jj'}x_i^* y_{i'} x_j y_{j'}^* > 0 \\ B_{ii',ij'} = \delta_{i'j'} \end{cases}$$

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is positive for any k.

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3. φ is completely positive if there exist a family of operators $D_i; i = 1, 2 \cdots r$ in M_n such that φ can be decomposed as

$$\varphi(a) = \sum_{i=1}^{N} D_i \, a \, D_i^\dagger \qquad [\text{ If } N = 1, \varphi \;\text{ is extremal}]$$

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4. If $\rho = \rho_1 \otimes \rho_2 \in M_n \otimes M_m$ is a separable state the unitary evolution map

$$\mathrm{Tr}_2 U^\dagger \rho U = \varphi_U(\rho_1)$$

is completely positive.

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is not positive.

3. φ is not completely positive if there are two families of operators $D_i, ; i = 1, 2 \cdots N$ and $C_i; i = 1, 2 \cdots M$ in M_n such that φ can be decomposed as

$$\varphi(a) = \sum_{i=1}^{N} D_i\, a\, D_i^\dagger - \sum_{i=1}^{M} C_i\, a\, C_i^\dagger$$

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5. A map $\varphi : M_n \to M_n$ is decomposable if it is k-decomposable for any k.

Separable Positive Maps

1. A positive decomposable map φ can be expressed as the sum of positive and copositive maps

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2. In n=2 dimensions every positive map is decomposable.
3. In higher dimensions positive maps which are not decomposable are called atomic

$$\varphi_{\text{Choi}} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} + a_{22} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} + a_{33} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} + a_{11} \end{pmatrix}$$

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for any positive map φ [Horodecki]

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Negative modes of $\varphi \otimes \mathbb{I}_m$ are entangled states.

Separable States
PPT States
Entangled States

Completely Positive Maps
Decomposable Maps
Indecomposable Maps

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- This permits to stablish a correspondence between quantum maps φ and quantum states ω of the composite system $\mathbb{C}^n \otimes \mathbb{C}^m$

$$B_{\alpha\beta} \in M_n \otimes M_m \Longleftrightarrow \varphi \in \mathcal{L}(M_n)$$

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Composed States $\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$

Given a completely positive normalized map $\varphi : \mathcal{D}(\mathbb{C}^m) \to \mathcal{D}(\mathbb{C}^n);\ \varphi(\mathbb{I}_m) = \mathbb{I}_n$ it is possible to associate to any quantum state $\rho \in \mathcal{D}(\mathbb{C}^n)$ a composite state $\omega_\varphi^\rho \in \mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^m)$ satisfying

i) $\mathrm{Tr}\, \omega_\varphi^{\rho\dagger}\, a \otimes \mathbb{I}_m = \mathrm{Tr}\, \rho^\dagger\, a$

ii) $\mathrm{Tr}\, \omega_\varphi^{\rho\dagger}\, \mathbb{I}_n \otimes b = \mathrm{Tr}\, \varphi^\dagger(\rho)^\dagger\, b$

where $\varphi^\dagger : \mathcal{D}(\mathbb{C}^n) \to \mathcal{D}(\mathbb{C}^m)$ is the dual map of φ

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1. First method

$$\omega_\varphi^\rho = \sum_k \lambda_k\, m_k\, \rho_k \otimes \varphi^\dagger(\rho_k)$$

where we use the spectral decomposition of ρ

$$\rho = \sum \lambda_k\, m_k\, \rho_k; \quad \rho_k = \frac{1}{m_k}\mathbb{P}_k :\ m_k = \mathrm{Tr}\,\mathbb{P}_k \qquad [\mathit{Ohya}]$$

Composed States $\mathcal{D}(\mathbb{C}^n \otimes \mathbb{C}^n)$

2. <u>Second method.</u> Let $n = m$ and φ a CP map

$$\sigma_\varphi = \sum_{ij=1}^{n} \varphi(e_{ij}) \otimes e_{ij} = (\varphi \otimes \mathbb{I}_n) \sum_{ij=1}^{n} e_{ij} \otimes e_{ij} \in M_n \otimes M_n$$

Define a Quantum Conditional Probability Operator

$$\pi_\varphi(\sigma_\varphi) = (\sigma_1^{-1/2} \otimes \mathbb{I}_n)\, \sigma_\varphi\, (\sigma_1^{-1/2} \otimes \mathbb{I}_n), \qquad \sigma_1 = \mathrm{Tr}_2\, \sigma_\varphi > 0$$

which satisfies

$$\pi_\varphi(\sigma_\varphi) > 0 \quad \text{and} \quad \mathrm{Tr}_2\, \pi_\varphi(\sigma_\varphi) = \mathbb{I}_n$$

and finally the composed state

$$\omega_\varphi^\rho = (\rho^{1/2} \otimes \mathbb{I}_n)\, \pi_\varphi(\sigma_\varphi)\, (\rho^{1/2} \otimes \mathbb{I}_n)$$

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- Example:

$$\varphi(a) = \sum_{ij} c_{ij}\, e_{ij}\, a\, e_{ij}^* + \mu\, a$$

is completely positive iff

$$c_{ij} \geq 0, \quad i \neq j \qquad \text{and} \qquad |c_{ii}\delta_{ij} + \mu| \geq 0$$

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$$c_{ij} \geq 0, \quad i \neq j \qquad \text{and} \qquad |c_{ii}\delta_{ij} + \mu| \geq 0$$

and completely copositive if

$$c_{ii} + \mu \geq 0 : \qquad c_{ij} + c_{ji} > 2|\mu|, \quad i \neq j$$

$$c_{ij}c_{ji} \geq \mu^2, \quad i \neq j$$

SU(N) Examples

Gell-Mann basis of M_n: $\lambda_1, \lambda_2, \cdots, \lambda_{n^2-1}$

1. The map

$$\varphi(a) = \sum_{\alpha=1}^{n^2-1} x_\alpha \lambda_\alpha \operatorname{Tr} a \lambda_\alpha + \frac{1}{n} \mathbb{I} \operatorname{Tr} a$$

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2. Let ω be a mixed state in $\mathbb{C}^N$: $\omega \geq 0$ with $\operatorname{Tr} \omega = 1$

$$f_\alpha = \begin{cases} \lambda_\alpha & \text{for} \quad \alpha = 1, 2, \cdots, n^2 - 1 \\ \omega & \text{for} \quad \alpha = n^2 \end{cases}$$

$$\sum_{\alpha=1}^{n^2-1} |x_\alpha|^2 \leq 1$$

$$x_{n^2} = 1$$

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$$g_\alpha = \begin{cases} \lambda_\alpha - \mathbb{I}\,\mathrm{Tr}\,\omega^\dagger\,\lambda_\alpha; & \text{for} \quad \alpha = 1, 2, \cdots, n^2 - 1 \\ \mathbb{I}; & \text{for} \quad \alpha = n^2 \end{cases}$$

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$$g_\alpha = \begin{cases} \lambda_\alpha - \mathbb{I}\,\mathrm{Tr}\,\omega^\dagger \lambda_\alpha; & \text{for} \quad \alpha = 1, 2, \cdots, n^2 - 1 \\ \mathbb{I}; & \text{for} \quad \alpha = n^2 \end{cases}$$

The following maps are positive and copositive

$$\varphi(a) = \sum_{\alpha=1}^{n^2} x_\alpha f_\alpha \,\mathrm{Tr}\,(a g_\alpha)$$

$$\widetilde{\varphi}(a) = \sum_{\alpha=1}^{n^2} x_\alpha g_\alpha \,\mathrm{Tr}\,(a f_\alpha)$$

respectively.