Resolution of Singular Spaces

A Project Report

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THESIS CERTIFICATE

This is to certify that the project report titled **Resolution of Singular Spaces**, submitted by **B. Aswin Kumar** (AE03B005), to the Indian Institute of Technology, Madras, for the award of the degree of **Bachelor of Technology**, is a bona fide record of the research work done by him under the supervision of **Dr. Suresh Govindarajan** during the academic year 2006-07. The contents of this thesis, in full or in parts, have not been submitted to any other Institute or University for the award of any degree or diploma.

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ABSTRACT

A complex manifold of real dimension $2n$ with a closed Kähler form and SU(n) holonomy is called a Calabi-Yau manifold. Yau proved a conjecture due to Calabi that Ricci flat metrics exist on these spaces. Apart from being of considerable mathematical interest, these manifolds are of special interest in string theory since they arise as possible geometries for the hidden spatial dimensions. Even non-compact Calabi-Yau manifolds are of interest for a variety of reasons, in spite of them being untenable as possible geometries for the hidden dimensions. Special cases of such non-compact manifolds are the crepant resolutions of singular spaces like orbifolds and conifolds. Explicit Ricci flat metrics for the case of resolved $\mathbb{C}^2/\mathbb{Z}_n$ are known to be the Gibbons-Hawking metrics. It would be of interest to compute the analogous Ricci-flat metrics for the case of $\mathbb{C}^3/\Gamma$. Making progress towards solving this problem is the central goal of the thesis. Initially, we review the background material and work out some simple examples using language of Guillemin (1994), Abreu (1997). We then summarize the work of Ray (1998). We also develop some new tools and indicate directions for future work.

KEYWORDS: Orbifolds, Resolutions of singularities, Kähler geometry, Symplectic geometry, Toric geometry

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NOTATION

$\mathbb{Z}$	Set of all integers
$\mathbb{Z}^+$	Set of all positive integers
$\mathbb{R}$	Set of all reals
$\mathbb{R}^+$	Set of all positive reals
$\mathbb{C}$	Set of complex numbers
$\mathbb{S}^n$	The n-(real) dimensional sphere.
$\mathbb{CP}^n$	The complex projective space of complex dimension n

CHAPTER 1

INTRODUCTION

This thesis will concentrate on certain differential geometric questions that arise in the context of string theory. String theory is an attempt to find an unified description of all known fundamental interactions. The interactions that are usually dominant at atomic scales are the electromagnetic, weak and strong forces. The Standard Model of particle physics furnishes an unified framework for treating these three interactions. Naive generalization of that formalism to the case of gravity leads to various technical problems. String theory, which is a theory of elementary one dimensional objects, removes some of those technical difficulties and is thus a candidate for a full theory of quantum gravity. However, some of the conditions required for string theory to be consistent seem to be in blatant disagreement with the world around us. One such requirement is that the space-time dimension should actually be ten (nine space + one time). This immediately falls in conflict with our understanding of the world as being four dimensional (three space + one time).

However, it is possible that we actually live in a 10 dimensional world with the other 6 spatial dimensions having properties that have made them undetectable at low energies. A simple way to build models of this kind would be have all the extra dimensions compactified *(curled up)* on some compact six-dimensional space. It turns out that there are some further conditions on the kind of manifolds that are allowed as candidates for compactification. Also, the effective four dimensional physics that is derived from such a model will depend quite critically on the actual manifold chosen for compactification.

It was realised in the 1980s that manifolds known as Calabi-Yau (CY) manifolds are suitable candidates such compactifications. These are a special class of complex manifolds which admit Ricci flat ($R_{\mu\nu} = 0$) metrics. A conjecture about the existence of such metrics was proposed by Calabi and it was later proved by Yau [1]. Though the existence of such metrics has been proved, it is extremely difficult to obtain the actual Ricci

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[1] Yau's proof was for the compact case. But, most results carry over to the non-compact cases.

flat metrics on these manifolds. Many indirect methods have been evolved to obtain these metrics for some special cases. Some of these cases are when the CYs are blown up orbifolds and conifolds. These geometries are obtained as crepant resolutions of a space with singularities. Since analysis with these spaces is relatively easier, they help us in constructing simple toy-models for compactifications. For example, in order to understand string theory in $M \times G$ where G is a compact manifold (of complex dimension 3) with a quotient singularity, we can instead analyze string theory in $\mathbb{C}^3/\Gamma$ where we choose Γ such that the local nature of the quotient singularity is captured. Such calculations have been done (for ex, Douglas and Moore (1996), Douglas *et al.* (1997), Douglas and Greene (1998) for $D0$ probes) and it was found that even in the case when a quotient singularity is present in the extra 6 dimensions, strings/other probes actually see a *resolved* space.

We will be interested in a mathematical question that is closely related to the above mentioned work. Apart from obtaining the topological properties of the space seen by the probes, can we obtain the Ricci flat metrics on these spaces? For the simple case of $\mathbb{C}^2/\Gamma$ where Γ is an abelian subgroup of $SU(2)$, it turns out that the required Ricci flat metrics are the well known Gibbons-Hawking family of *gravitational instantons*. These are self-dual solutions to the four dimensional, euclidean Einstein's equations. Historically, these were derived by methods analogous to those used in the construction of instanton solutions to Yang-Mills theories. The primary motivation came from the intention to formulate a theory of quantum gravity using euclidean path integrals. Since these solutions give absolute minima for the actions in their corresponding topological sectors, they will be crucial in any formulation of a theory using euclidean path integrals. But, our interest in these solutions is quite different.

Now, for the case of $\mathbb{C}^3/\Gamma$, the metrics obtained in Douglas and Greene (1998) are not Ricci flat. This is attributed to the fact that there is no hyper-Kähler structure in three complex dimensions and hence, when one does Kähler reduction, only the Kählerity of the resulting metric is ensured. This is unlike the case of two complex dimensions where one can directly obtain the Ricci flat metric by hyper-Kähler reduction. However, for the specific case of $\mathbb{C}^3/\mathbb{Z}_3$, one can obtain the Ricci flat metric by using a formalism due to Guillemin (1994), Abreu (1997). This was first pointed out in Ray (1998) where the actual Ricci flat metric is also derived. It would be of interest to carry out a similar construction of the Ricci flat metric on other orbifolds in the $\mathbb{C}^3$ family. Making progress towards solving this problem is the main concern of this thesis. We will be using Witten (1993) (and not Douglas and Moore (1996), Douglas *et al.* (1997), Douglas and Greene (1998)) as our starting point since the notion of symplectic quotient is easier to handle in this formalism.

The language introduced in Guillemin (1994), Abreu (1997) uses symplectic co-ordinates rather than the usual holomorphic/anti-holomorphic co-ordinates that one normally meets in Kähler geometry. It would be interesting to see if the next simplest case of $\mathbb{C}^3/\mathbb{Z}_5$ (unsolved, as of now) can be solved using this language. At a more simpler level, one would like to reproduce the entire Gibbons-Hawking family of metrics using the symplectic language. In fact, success in this construction could give crucial inputs as to how the problem for general $\mathbb{C}^3/\Gamma$ should be attacked in the symplectic language. With this mind, we set out explaining how the simplest metric in the multi-center Gibbons Hawking family, the Eguchi-Hanson metric, can be obtained in this formalism. But, one finds that there are no trivial generalizations of the method to the case of $\mathbb{C}^2/\mathbb{Z}_3$. The problems experienced in $\mathbb{C}^3/\mathbb{Z}_5$ are similar. After reviewing existing constructions, we will propose one possible to way of obtaining at least approximately Ricci flat metrics on these spaces.

The thesis is organized in the following fashion: We start by giving a brief introduction to these geometrical concepts in Chapter 2. We also introduce the language developed by Guillemin and Abreu and explain its relation to the usual complex language. Next, in Chapter 3, we explain the symplectic quotient construction and discuss how it arises in the context of supersymmetric gauge theories. We discuss some simple examples where familiar metrics are obtained as symplectic quotients using the language of Guillemin and Abreu. We then move on to working out the simple case of $\mathbb{C}^2/\mathbb{Z}_2$ in detail (Chapter 4). In Chapter 5, we review the results of Ray (1998) for the case of $\mathbb{C}^3/\mathbb{Z}_3$ and also make some observations which gives us greater insight into this case. In the final section, we make some comments about possible ways by which progress could be made in the case of $\mathbb{C}^3/\mathbb{Z}_5$ and other cases.

CHAPTER 2

Geometrical Background

2.1 Kähler Manifolds

We will introduce the geometrical concepts involved in a hierarchical manner with the goal of finally getting to what are called *toric Kähler manifolds*. Much of the material presented here can be found in any standard book on differential geometry. In particular, I have found the following sources useful: Nakahara (2003), Nash and Sen (1983), Eguchi *et al.* (1980), Greene (1997).

2.1.1 Topological Spaces

Most geometrical analysis is typically done using *continuous*[1] functions. However, there is a minimal amount of structure that should accompany a set of points for continuous functions to be described on them. This minimal requirement is what defines a topological space. More concretely,

A Topological space is a set X together with a collection $Y = \{X_\alpha\}$ (where X_α is a subset of X) satisfying the following conditions :

1. $\phi \in Y$ and $X \in Y$. That is, the empty set and the whole set should be part of the collection Y.
2. Any subcollection $\{Z_\alpha\}$ of X_α has the property that $\bigcup Z_\alpha \in Y$.
3. Any *finite* subcollection Z_α has the property that $\bigcap Z_\alpha \in Y$.

The collection Y is typically called a collection of open sets. In fact, the conditions above are motivated by the properties of open sets of $\mathbb{R}^n$, objects that are familiar from multi-variate calculus. The topological space is usually denoted by the double (X, Y). Now, once the notion of open sets has been established, we can proceed to define a continuous function as

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[1] I am appealing just to an intuitive understanding of continuity.

- A function $\phi : P \to Q$, where P, Q are topological spaces, is continuous if the inverse of this map, ϕ^{-1} maps open sets in Q to open sets in P.

Again, this just generalizes the usual notion of continuous functions which map $\mathbb{R}^n$ to $\mathbb{R}^n$. One can check that the conventional definition for continuous functions in calculus matches the above definition if P and Q are $\mathbb{R}^n$ with the usual topology (or *choice of open sets*). Both the domain and the range of ϕ should already possess the property of being topological spaces in order for us to be able to define continuous functions between them. As emphasized earlier, one may view the set of requirements listed for a topological space to be the minimal requirements on space so that continuous functions could be defined on them.

2.1.2 Differentiable Manifolds

Typically, the spaces encountered in physical problems are endowed with much more structure. For example, it could be possible to label the points in the space using co-ordinate charts. We usually ascribe a set of numbers to each point after setting up a co-ordinate system. This would correspond to prescribing local maps from the space to $\mathbb{R}^n$. Note that simple spaces like subsets of $\mathbb{R}^n$ can be covered by a single coordinate chart. But, the most general spaces may require many charts. We formalize these notions in the following definition

A topological space (X, Y) is a topological manifold if

- It can be covered with open sets $O_\alpha \in Y$ such that for each O_α, one has a map $\psi_\alpha : O_\alpha \to \mathbb{R}^n$ for a fixed $n \in \mathbb{Z}^+$ such that map ψ_α and its inverse ψ_α^{-1} are continuous.

The map ψ_α furnishes a local co-ordinate system for O_α. Now, as a special case, consider the continuous function $f : X \to \mathbb{R}$. Using this and the co-ordinate maps ψ_α, one can obtain a map from the co-ordinates themselves (alternatively, the image of ψ_α) to $\mathbb{R}^n$. This map is given by $f \circ \psi_\alpha^{-1} : \psi_\alpha(O_\alpha) \to \mathbb{R}$. This map is called the *co-ordinate representation* of f. Note that $\psi_\alpha(O_\alpha)$ are open sets in $\mathbb{R}^n$. So, the continuity of f is now mapped onto the problem of continuity of the map $f \circ \psi_\alpha^{-1}$ in multivariate calculus. One can check that the actual function is continuous in the sense of the above definition iff its co-ordinate representation is continuous in the calculus sense. Now, if some points were to fall under two open sets, we would want to be sure that the notion of continuity of a function does not end up being patch-dependent. Luckily, this is the case. It is worthwhile to note that any additional requirement beyond continuity will impose some new requirements. We will now see one such additional requirement.

A topological manifold is a *differentiable manifold* if the co-ordinate representations mentioned above are infinitely differential (C^∞) functions in the sense used in calculus. But, there is an additional subtlety that we encounter when the open sets overlap. Consider $f \circ \psi_1^{-1} : \psi_1(O_1) \to \mathbb{R}$ and $f \circ \psi_2^{-1} : \psi_2(O_2) \to \mathbb{R}$ and say we are interested in the behaviour of these maps in $\psi_1(O_1 \cap O_2)$ and $\psi_2(O_1 \cap O_2)$ (both are open sets in $\mathbb{R}^n$). Now, ideally, we would not expect that if the co-ordinate representation obtained using ψ_1 is C^∞ then so is the one obtained from ψ_2. Noting that $f \circ \psi_1^{-1} = f \circ \psi_2^{-1} \circ (\psi_2 \circ \psi_1^{-1})$, we see that the above requirement implies that the map $\psi_2 \circ \psi_1^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞. These maps are usually called transitions functions.

Now, with the structure available with a differential manifold, one can do calculus on the manifold. For starters, consider a smooth map $f : M \to R$ and a curve $p(t) : R \to M$. The variable t can be viewed as co-ordinate in the domain of p. Now, we want to find the derivate of $f \circ p : R \to R$. As it is immediately not clear what this means, rewrite the derivative in the following fashion

$$\frac{d(f \circ x^{-1} \circ x \circ p(t))}{dt} = \frac{\partial(f \circ x^{-1})}{\partial x^i} \frac{d(x^i(p(t))}{dt} , \tag{2.1}$$

where x^i are local co-ordinates on M. The above expression can now be viewed as a differential operator acting on $(f \circ x^{-1})$

$$\begin{aligned} \frac{d(f \circ p)}{dt} &= \frac{d(x^i(p(t))}{dt} \frac{\partial(f \circ x^{-1})}{\partial x^i} . \\ &= \mathbb{X}(f \circ x^{-1}) , \end{aligned} \tag{2.2}$$

where $\mathbb{X}$ is

$$\mathbb{X} = \frac{d(x^i[p(t)])}{dt} \frac{\partial}{\partial x^i} . \tag{2.3}$$

We can see that for the simple case of $\mathbb{X}$ acting on the co-ordinate x^i, we obtain the

’velocity’ vector (thus reproducing usual notions)

$$\mathbb{X}(x^i) \;=\; \frac{dx^j}{dt}\frac{\partial x^i}{\partial x^j} = \frac{dx^i}{dt}\ . \tag{2.4}$$

These differential operators are called *tangent vectors*. In the specific case above, we have found the tangent vector to the curve $p(t)$. Note that our evaluation of the derivatives are at a particular point. We have just avoided symbols like $|_{p_0}$ to keep the expressions simple. Now, any differential operator is of the form $\mathbb{X} = a^i \frac{\partial}{\partial x^i}$ and it is a tangent to some curve. It is easy to see that the set of all tangent vectors *at a point* have a vector space structure. The natural basis for that vector space (called the tangent space) is $\frac{\partial}{\partial x^i}$. This vector space is usually denoted as $T_p(M)$. The relevant field is $\mathbb{R}$ for real manifold. Its dimension is the same as the topological dimension n of the manifold. The tangent space, just like any other vector space, has a dual space given by linear maps from the tangent space to $\mathbb{R}$. This dual space is called the co-tangent space and denoted by $T_p^*(M)$. Any element of the co-tangent space is called a one-form. It turns out that differentials of functions ($'df'$) are naturally members of the co-tangent space. To see this, choose $df \in T_p^*(M)$ and *define* its action on $\mathbb{X}$ as

$$\langle df, \mathbb{X}\rangle = \mathbb{X}f\ . \tag{2.5}$$

For the case of f being the co-ordinates x^i and $\mathbb{X} = \frac{\partial}{\partial x^j}$, the above definition gives

$$\langle dx^i, \frac{\partial}{\partial x^j}\rangle \;=\; \frac{\partial x^i}{\partial x^j} = \delta^i_j\ . \tag{2.6}$$

The above relation is just what you would expect between the basis vectors in a vector space and its dual. So, dx^i is actually the basis dual to $\frac{\partial}{\partial x^j}$. For arbitrary $\omega = a_i dx^i$ and $\mathbb{X} = b^j \frac{\partial}{\partial x^j}$, (2.5) will give

$$\langle \omega, \mathbb{X}\rangle = a_i b^i\ . \tag{2.7}$$

So far, we have talked about vectors and co-vectors. We can generalize these notions to tensors in the following way

- A tensor of type (a, b) is an element of the following tensor product of vector spaces

$$T^a_b = T_p(M) \otimes \cdots \otimes T_p(M) \otimes T_p^*(M) \otimes \cdots \otimes T_p^*(M)\ , \tag{2.8}$$

where $T_p(M)$ is repeated a times and $T_p^*(M)$ is repeated b times.

There are a special class of co-tensors called differential forms. These are defined as co-tensors which are antisymmetric under exchange of any two of their arguments[2]. These differential forms are the basic building blocks for doing integrations on manifolds. We will just note that there is a natural operation which takes an n-form to an $(n+1)$ form. This is the exterior derivative d (something that we met in df).

$$d : \omega \to d\omega = \frac{\partial \omega_{i_1 \ldots i_n}}{\partial x^{i_{n+1}}} dx^{n+1} \wedge dx^{i_1} \cdots \wedge dx^{i_n} \ , \tag{2.9}$$

where the wedge product is defined as

$$dx^i \wedge dx^j = \frac{1}{2}(dx^i \otimes dx^j - dx^j \otimes dx^i) \ . \tag{2.10}$$

Now, we also note some further defintions which we will appeal to in later parts. First is the definition of the p-th DeRham cohomology group $H_d^p(X)$ for a real differentiable manifold X

$$H_d^p(X) = \frac{\{\omega | d\omega = 0\}}{\{\alpha | \alpha = d\beta\}} \ , \tag{2.11}$$

where we have just used the usual notation in group theory to obtain quotient groups. ω and α are p-forms.

Next is the notion of a Riemannian manifold. This is nothing but the double (X, g) where X is a differentiable manifold and g is a *symmetric* positive map

$$g : T_p \otimes T_p \to \mathbb{R} \ . \tag{2.12}$$

The symmetric aspect is crucial as this distinguishes it from 2-forms. In most physics literature, this map appears as the *metric tensor* g_{ij}. This is nothing but a component form of the map g

$$g = g_{ij} dx^i \otimes dx^j \ . \tag{2.13}$$

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[2] Recall that a co-vector $(0, n)$ takes n vectors as arguments.

2.1.3 Complex Manifolds

We now proceed to introduce more structure on our space. So far, we have always considered co-ordinate charts with a subset of $\mathbb{R}^n$ being the image where n is the topological dimension of the manifold. Alternatively, we could use maps with images in $\mathbb{C}^d$. It is easy to see that $d = n/2$ where n is the real dimension of the manifold. So, the first requirement for a manifold to admit charts with maps to $\mathbb{C}^d$ is that its real dimension should be an even number. Now, as a generalization of the C^∞ requirement on transition functions in real differential manifolds, the definition of a complex manifold requires that the transition functions be analytic.

In the case of complex manifolds, the relevant field for the tangent space is taken to be $\mathbb{C}$ (that is, one allows linear combinations with complex coefficients). This is usually called the complexified tangent space $T_p^{\mathbb{C}}$. A basis for this space can be written using linear combinations of the usual basis for the real tangent space

$$\left\{ (\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^{d+1}}), \cdots, (\frac{\partial}{\partial x^d} + i\frac{\partial}{\partial x^{2d}}), (\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^{d+1}}), \cdots, (\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^{d+1}}) \right\} . \tag{2.14}$$

Using complex co-ordinates, we can rewrite this as

$$\left\{ \frac{\partial}{\partial z^1}, \cdots, \frac{\partial}{\partial z^d}, \frac{\partial}{\partial \bar{z}^1}, \cdots, \frac{\partial}{\partial \bar{z}^d} \right\} . \tag{2.15}$$

Similarly, the basis for the dual space is

$$\left\{ dz^1, \cdots, dz^d, d\bar{z}^1, \cdots, d\bar{z}^d \right\} . \tag{2.16}$$

The above arrangement of the basis vectors is specifically tailored for the following re-arrangement of the tangent space at p

$$T_p^{\mathbb{C}} = T_p^{(1,0)} \oplus T_p^{(0,1)} , \tag{2.17}$$

where $T_p^{(1,0)}$ is the space obtained as span of $\{\partial/\partial z^i\}$ while $T_p^{(0,1)}$ is got from $\{\partial/\partial \bar{z}^i\}$. The former is called the holomorphic tangent space and the latter is called the anti-holomorphic tangent space. Similar extensions can be made for the co-tangent space.

Now the Riemannian metric introduced for real manifolds can be generalized to complex manifolds in the following way

$$g : T_p^{\mathbb{C}} \otimes T_p^{\mathbb{C}} \to \mathbb{C} \, . \tag{2.18}$$

Note that we have allowed for the map to take complex values. But, if we started with the a Riemannian metric on the real manifold and then complexified the tangent space and demand that the map still remains real and symmetric, we are led to the following restrictions on g

$$\begin{aligned} g_{ij} &= g_{ji} \, , \\ g_{i\bar{j}} &= g_{\bar{i}j} \, , \\ \overline{g_{ij}} &= g_{\bar{i}\bar{j}} \, , \\ \overline{g_{i\bar{j}}} &= g_{\bar{i}j} \, . \end{aligned} \tag{2.19}$$

where $g_{i\bar{j}}$ is defined as $g(\partial/\partial z^i, \partial/\partial \bar{z}^j)$ and so on.

Now, we further restrict ourselves to cases when only mixed components are non-zero. That is, $g_{ij} = g_{\bar{i}\bar{j}} = 0$. Such metrics are called *hermitian* metrics. So, in this special case, the metric reduces to

$$ds^2 = g_{i\bar{j}} dz^i \otimes d\bar{z}^j + g_{\bar{i}j} d\bar{z}^i \otimes dz^j \, . \tag{2.20}$$

The above condition of just the mixed components surviving can be obtained by demanding that there exists a map $J : T_p \to T_p$ with $J^2 = -I$ such that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$. This map is called the *complex structure*. Now, with this complex structure, we can always obtain the following two-form

$$\omega = i g_{i\bar{j}} dz^i d\bar{z}^j \, . \tag{2.21}$$

When $d\omega = 0$, it is called a Kähler form. Note that existence of a globally defined, closed two-form also implies that it is *symplectic* manifold. We shall use this feature later. Now, we will see the implications of demanding that $d\omega = 0$.

$$
\begin{aligned}
d\omega &= (\partial + \bar{\partial}) i g_{i\bar{j}} dz^i d\bar{z}^j = 0 \ . \\
\implies \frac{\partial g_{i\bar{j}}}{\partial z^k} &= \frac{\partial g_{k\bar{j}}}{\partial z^i} \ , \\
\frac{\partial g_{i\bar{j}}}{\partial \bar{z}^k} &= \frac{\partial g_{k\bar{j}}}{\partial \bar{z}^i} \ .
\end{aligned} \tag{2.22}
$$

We can recognize the above relations as an integrability condition indicating that the metric can be obtained from a single scalar function. We express $g_{i\bar{j}}$ as

$$
g_{i\bar{j}} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j} \ . \tag{2.23}
$$

This scalar function is called the Kähler potential and manifolds possessing such potentials are called Kähler manifolds. Kähler manifolds are fascinating mathematical objects and have attracted detailed analysis over many decades. Our interest is in a specific class of Kähler manifolds called toric Kähler manifolds. We will now proceed to introduce these concepts in the next section.

2.2 Toric Kähler Manifolds and the Symplectic viewpoint

In the previous section, we presented aspects of Kähler geometry. The complex language was used through out. However, there is an alternative way of looking at Kähler manifolds. It is using the language of symplectic geometry. M^{2n} is a symplectic manifold if there is a non-degenerate two-form ω which is closed ($d\omega = 0$). Now, by virtue of the condition $d\omega = 0$, every Kähler manifold is a symplectic manifold. So, Kähler manifolds are a special class of manifolds falling under the intersection of symplectic and complex geometry. But, most presentations and results of Kähler geometry make use of the complex language (like holomorphic co-ordinates) and not the symplectic language (like Darboux co-ordinates). A possible reason pointed out in Abreu (2000) is that all compatible [3] symplectic forms (for a fixed complex structure J_0) can be pa-

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[3] required condition is that $\omega(\cdot, J\cdot)$ is symmetric and positive definite.

rameterized by smooth functions on the manifold. That is, given J_0 and a ω_0, all other compatible symplectic forms in the same cohomology class are given by

$$\omega = \omega_0 + 2i\partial\bar{\partial}f, f \in C^\infty(M) \ . \tag{2.24}$$

Now, in principle, it is also possible to fix the *symplectic* form instead and vary the complex structure. But, this is not possible in practice as there is no effective parameterization of compatible complex structures for generic Kähler manifolds. But, Abreu (1997) has shown that for the special case of toric Kähler manifolds, a convenient parameterization of compatible complex structures is possible. In this set-up, Darboux co-ordinates are used and the symplectic form is fixed to

$$\omega = \sum dx_i \wedge dy_i \ . \tag{2.25}$$

Now, to proceed further, we need to introduce some aspects of toric Kähler manifolds. A Kähler toric manifold (M^{2n}) is a Kähler manifold which has an effective $\mathbb{T}^n$ action which leaves the Kähler form invariant. Futher, one demands that the torus action be integrable. This condition means that one has a moment map $\mu : M \to \mathbb{R}^n$. The image of this moment map is a convex subset of $\mathbb{R}^n$ ($\Delta \subset \mathbb{R}^n$), called the moment polytope. This image is of the form

$$\Delta = \{y \in \mathbb{R}^n | l_a(y) \geqslant 0, a = 1 \ldots d\} \ , \tag{2.26}$$

where the $l_a(y)$ are linear functions depending on the toric data.

Now, in usual complex co-ordinates, one can show that

$$\omega = 2i\partial\bar{\partial}K \ . \tag{2.27}$$

The complex structure in these co-ordinates takes the usual form

$$J = \begin{pmatrix} \mathbf{0} & \vdots & -\mathbf{I} \\ \cdots & \cdots & \cdots \\ \mathbf{I} & \vdots & \mathbf{0} \end{pmatrix} , \tag{2.28}$$

where **0** and **I** are $n \times n$ matrices.

We now pass over to log-complex co-ordinates

$$u_j = \log(z_j) \,, \tag{2.29}$$

and write $u_j = \rho + i\phi_j$. The toric nature of the manifold will mean that the metric would actually be independent of ϕ_j. Now, ω in these co-ordinates will be of the form

$$\omega_{mn} = \begin{pmatrix} 0 & \vdots & K_{ij} \\ \cdots & \cdots & \cdots \\ -K_{ij} & \vdots & 0 \end{pmatrix} \,, \tag{2.30}$$

where

$$K_{ij} = \frac{\partial^2 K}{\partial \rho^i \partial \rho^j} \,. \tag{2.31}$$

We can see that the Riemannian metric $g = \omega(\cdot, J\cdot)$ is of the form

$$g_{mn} = \begin{pmatrix} K_{ij} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & K_{ij} \end{pmatrix} \,. \tag{2.32}$$

That is, the metric is

$$ds^2 = K_{ij} d\rho^i d\rho^j + K_{ij} d\phi_i d\phi_j \,. \tag{2.33}$$

In the symplectic co-ordinates, these expressions are modified such that the symplectic form looks simpler while the complex structure carries derivatives of a potential. The symplectic co-ordinates are related to the log-complex co-ordinates in the following way.

- $$P^i = \frac{\partial K}{\partial \rho^i} \,. \tag{2.34}$$
- The symplectic potential G is given by the Legendre transform
$$G(P^i) = \rho_j(P) P^j - K(P^i) \,. \tag{2.35}$$

One can immediately see that

$$K_{ij}(\rho) = G^{ij}(P) \ , \tag{2.36}$$

and

$$K_{ij} d\rho^i d\rho^j = G_{ij} dP^i dP^j \ , \tag{2.37}$$

where

$$\begin{aligned} G_{ij} &= \frac{\partial^2 G}{\partial P^i \partial P^j} \ , \\ G^{ik} G_{kj} &= \delta^i_j \ . \end{aligned} \tag{2.38}$$

In these new co-ordinates, the metric g thus becomes

$$g_{mn} = \begin{pmatrix} G_{ij} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & G^{ij} \end{pmatrix} \ . \tag{2.39}$$

Alternatively,

$$ds^2 = G_{ij} dP^i dP^j + G^{ij} d\phi_i d\phi_j \ . \tag{2.40}$$

At this point, we also note some useful formulae for the Ricci scalar and Ricci tensor. These can be obtained directly from the above metric.

$$R = -\frac{\partial^2 G^{ij}}{\partial P^i \partial P^j} \ , \tag{2.41}$$

$$R_{ij} = -G^{il} \frac{\partial^2 G^{kj}}{\partial P^k \partial P^l} \ . \tag{2.42}$$

So far, we have not used any data specific to a particular Kähler toric manifold. The construction above is completely general. Now, Guillemin (1994) has shown that for general toric Kähler manifolds defined as per (2.26), there exists a canonical metric which can be obtained from the following simple symplectic potential[4]

$$G_{can} = \frac{1}{2}\Bigg[\sum_{i=1}^{n} l_i(P^j)\log(l_i(P^j))\Bigg] \, . \tag{2.43}$$

The canonical potential is smooth in the interior of the polytope defined by $l_i > 0$ and is singular on its boundary. We also note that the form of the canonical metric is very similar to that of the von Neumann/Shannon entropy. But it is not immediately clear if this corresponds to the entropy of some thermodynamic system. Now, to gain a better understanding of this passage from complex to the symplectic co-ordinates and the nature of the canonical metric, we will now consider the trivial example of flat space. The flat metric on $\mathbb{C}^n$ can be obtained from the following Kähler potential

$$K(z_j) = \frac{1}{2}\sum_{j=1}^{n} |z_j|^2 \, . \tag{2.44}$$

Now, write $z = \exp u_j$ where $u_j = \rho_j + i\eta_j$ so that $z_j\bar{z}_j = \exp(2\rho_j)$. The Legendre transform can now be obtained after defining

$$\begin{aligned} y_j &= \frac{\partial K}{\partial \rho_j} \, . \\ \implies y_j &= e^{2\rho_j} \, , \\ \implies \rho_j &= \frac{1}{2}\log y_j \, , \end{aligned}$$

and then obtaining

$$G(y_j) = y_j\rho_j(y_j) - K(y_j) \, . \tag{2.45}$$

Substituting,

$$G(y_j) = \frac{1}{2}\sum_{j=1}^{n} y_j \log(y_j) \, , \tag{2.46}$$

where we have ignored an irrelevant linear term in y_j. Now, comparing this with (2.43), we realize that in the special case of a flat metric in d-complex dimensions, the inequalities are $y_i > 0$ and the moment polytope is just the region in y_i space defined by $y_i > 0$. The striking similarity between the symplectic potential for flat space and the canonical metric for a toric Kähler manifold is not accidental. Guillemin's construction of the

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[4] Note that a very similar result was obtained implicitly in Witten (1993).

canonical metric was done by starting from a flat metric in a higher dimensional space and then obtaining the metric on the required toric manifold as a symplectic quotient. This is the reason for the identical structure. In the next chapter, we will be considering some specific symplectic quotients in this language. Those constructions should make things clearer.

CHAPTER 3

The symplectic quotient construction

Our interest in toric Kähler manifolds is partly motivated by their relation to certain supersymmetric gauge theories. We will first detail the mathematical way of looking at these quotient constructions and then briefly explain the connections to physics. Background material for this section can be found in Witten (1993), Aspinwall and Greene (1995) and Greene (1997).

A general toric variety can be obtained as a generalization of the usual $\mathbb{CP}^n$ in the following sense. We note that $\mathbb{CP}^n$ can be obtained by the following quotienting action

$$\mathbb{CP}^n = \frac{\mathbb{C}^{n+1} - \{0\}}{\mathbb{C}^*} \, , \tag{3.1}$$

where the $\mathbb{C}^*$ action is given by $(z_1, z_2, \ldots, z_n) \to (\lambda z_1, \ldots, \lambda z_n)$. This definition is akin to the definition of real projective spaces as the space of lines through origin. Now, a general toric variety is obtained by

$$V = \frac{\mathbb{C}^m - F_\Delta}{(\mathbb{C}^*)^p} \, , \tag{3.2}$$

where the F_Δ is a set of points. The action $\mathbb{C}^*$ and the actual set F_Δ will determine the variety. The above mentioned way of obtaining toric manifolds is called the holomorphic quotient. An alternative way of obtaining the same toric manifold is to split the $\mathbb{C}^*$ action into

$$\mathbb{C}^* \simeq \mathbb{R}^+ \times U(1) \, , \tag{3.3}$$

and then obtain the quotient in two steps. Such a procedure is called the symplectic quotient (For a detailed introduction, see Guillemin and Sternberg (1984)). This is done by first imposing a constraint $\mu : \mathbb{C}^n \to \mathbb{R}$ (called the *moment map*) and then a further quotient by a compact Lie group. The moment map and the Lie group are related in a specific way which is a generalization of the relation between the Hamiltonian function and the phase space flow in classical mechanics. To understand this better, consider the case of a harmonic oscillator. Noting that phase space plots for $H = H_0$ are circles, we can see that the Hamiltonian $H = (p^2 + q^2)/2$ generates a $U(1)$ action on the phase space. Similarly, a moment map $\mu = \sum_{i=1}^{n+1} |z_i|^2/2$ generates a $U(1)$ action[1] in $\mathbb{C}^{n+1}$. We can use this moment map to obtain the generalized Fubini-Study metric on $\mathbb{CP}^n$. First, consider the simple case of a quotienting from $\mathbb{C}^2$. The flat metric on $\mathbb{C}^2$ is given by the Kähler potential

$$K = \frac{1}{2}\sum_{i=1}^{2} |z_i|^2 \ . \tag{3.4}$$

The corresponding symplectic potential is given by

$$G = \frac{1}{2}\sum_{i=1}^{2} x_i \log x_i \ , \tag{3.5}$$

where we have taken $z_i = \sqrt{x_i}e^{i\theta}$ Now, the constraint (or *moment map*) takes the form

$$x_1 + x_2 = t \ . \tag{3.6}$$

Note that this single constraint forces the fields to lie on an $\mathbb{S}^3$. The further $U(1)$ quotient is taken just by forgetting θ_2 while writing out the metric. Thus, imposing these constraints will yield a space of real dimension 2. And knowing that $\mathbb{S}^3$ is a $U(1)$ fibration over $\mathbb{S}^2$, we realize that the space that has been obtained at the end is $\mathbb{S}^2$. And the induced metric on this manifold can be obtained by substituting g(3.6) into the symplectic potential in (3.5) and then finding

$$ds^2 = G_{ij}dP^i dP^j + G^{ij}d\theta_i d\theta_j \ , \tag{3.7}$$

where i, j would, for arbitrary quotients, run over $\{1, 2, \ldots, \text{dim}(M_{red})\}$ where dim(M_{red}) is the complex dimension of the manifold obtained after reduction. In the simple case that we are considering, dim$(M_{red}) = 1$. To evaluate the induced metric in this case, we set $t = 1$ and write x_1, x_2 as $x_1 = (1 - P), x_2 = P$. This is to ensure that the constraint is automatically satisfied. The polytope in this case is defined by the inequalities $P > 0$ and $1 - P > 0$. This is just a line segment of unit length (see fig (3.1)) Now, evaluate

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[1] $(z_1, z_2 ... z_n) \rightarrow (\lambda z_1, .. \lambda z_n)$ with $|\lambda| = 1$.

Figure 3.1: The moment polytope for $\mathbb{S}^2$

(3.7). We obtain,

$$ds^2 = \frac{dP^2}{1-P^2} + (1-P^2)d\phi^2 \; . \tag{3.8}$$

Now, by doing a simple co-ordinate transformation $(1-P^2) = \sin^2\theta$, we realize that the above metric is the usual metric on $\mathbb{S}^2$

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \; . \tag{3.9}$$

Setting t to some other value would have just changed the size of the sphere. Doing the analogous reduction with $\mathbb{C}^3$ replacing $\mathbb{C}^2$ would yield the usual Fubini-Study metric on $\mathbb{CP}^2$. Here the route will be $\mathbb{C}^3 \to \mathbb{S}^5 \to \mathbb{CP}^2$. It is worthwhile pointing out that an odd dimensional sphere $\mathbb{S}^{2n+1}$ can be written as a $U(1)$ fibration over $\mathbb{CP}^n$ and thus, it is easy to see that the route to generic $\mathbb{CP}^n$ in this formalism is $\mathbb{C}^{n+1} \to \mathbb{S}^{2n+1} \to \mathbb{CP}^n$. Now, we will get back to the specific case of $\mathbb{CP}^2$[See Abreu (1997, 2000)]. The constraint in this case is

$$x_1 + x_2 + x_3 = t \; , \tag{3.10}$$

and the canonical metric is, as before,

$$G = \frac{1}{2}\sum_{i=1}^{3} x_i \log x_i \; . \tag{3.11}$$

Now, choose $t = 3, x_1 = 1 + P^1, x_2 = 1 + P^2, x_3 = 1 - P^1 - P^2$. With this ,

$$G(P^1, P^2, P^3) = \frac{1}{2}\Big[(1+P^1)\log(1+P^1) + (1+P^2)\log(1+P^2) + (1-P^1-P^2)\log(1-P^1-P^2)\Big] \; . \tag{3.12}$$

Now, comparing with (2.43), we can see that the polytope is defined by the following

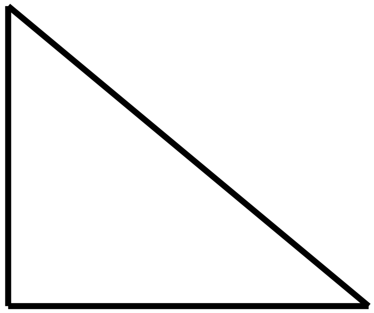

Figure 3.2: The moment polytope for $\mathbb{CP}^2$

inequalities

$$\begin{aligned} 1+P^1 &> 0\,, \\ 1+P^2 &> 0\,, \\ 1-P^1-P^2 &> 0\,. \end{aligned} \tag{3.13}$$

These inequalities will define a polytope which looks like fig (3.2) Now, we can find G_{ij}

$$G_{ij} = \frac{1}{2(1-P^1-P^2)} \begin{bmatrix} \frac{2-P^2}{1+P^1} & 1 \\ 1 & \frac{2-P^1}{1+P^2} \end{bmatrix}\,. \tag{3.14}$$

Inverting this matrix gives G^{ij}

$$G^{ij} = \frac{2}{3} \begin{bmatrix} (2-P^1)(1+P^1) & -(1+P^1)(1+P^2) \\ -(1+P^1)(1+P^2) & (2-P^2)(1+P^2) \end{bmatrix}\,. \tag{3.15}$$

We can check that it is indeed a constant curvature metric by calculating

$$R = -\frac{\partial^2 G^{ij}}{\partial P^i \partial P^j} = 4\,. \tag{3.16}$$

Now, we could move away from simple $\mathbb{C}^*$ actions and consider actions of the form $(z_1, z_2 ... z_{n+1}) \rightarrow (\lambda^{c_1} z_1, .. \lambda^{c_{n+1}} z_{n+1})$. These actions will be generated by a moment map $\mu = \sum_{i=1}^{n+1} c_i |z|_i^2/2$. Consider the special case of $\lambda_i = \lambda, i \neq n+1$ and $\lambda_{n+1} = \lambda^{-n}$. For the case of $n = 3$, the constraint will then reduce to

$$x_1 + x_2 - 2x_3 = t\,. \tag{3.17}$$

Now, consider the case where $t \gg 0$. Note that $x_i \geq 0$, strictly. When $t > 0$, x_1, x_2 cannot simultaneously vanish and setting $x_3 = 0$ gives $x_1 + x_2 = t$ which is exactly the constraint corresponding to a $\mathbb{CP}^1$. So, for large positive t, the manifold obtained by reduction is going to look like a $\mathbb{CP}^1 \simeq \mathbb{S}^2$. For generic positive values of t, the resulting manifold is a line bundle over $\mathbb{CP}^1$. Such metrics were constructed long ago by Calabi and in this case, it is known to be the same as the Eguchi-Hanson metric. Now, for large negative t, it is easy to see that x_3 has to be non-zero and using the $\mathbb{C}^*$ action, one can set x_{n+1} to be a particular non-zero value. Now, there is some freedom in choosing the rest of the x_i after fixing x_{n+1}. Looking at the $\mathbb{C}^*$ action, one can see that this corresponds to an overall multiplication by an n-th root of unity since only $\lambda^n = 1$ would leave x_{n+1} untouched. So, upon a $U(1)$ quotient, we will finally obtain an orbifold $\mathbb{C}^n/\mathbb{Z}_n$. One can see that the topological properties of the space obtained change as we go through $t = 0$. In the next section, we deal with this particular example in much greater detail showing explicitly that the manifold obtained for generic positive t is indeed the blow-up for $\mathbb{C}^2/\mathbb{Z}_2$. So far, we have not appealed to any physics in the above constructions of symplectic quotients. But, it turns out that these quotients arise naturally when one tries to find all possible vacuum configurations of fields in certain $N = 2$ superconformal theories. For our purpose, it is sufficient to note that the bosonic potential (U) for the theory considered in Witten (1993) contains many terms of which the Fayet-Iliopoulos D-term is one. In general, if we have a theory with n-complex fields, they could live in $\mathbb{C}^n$. But once we set $U = 0$, we are forced to set the D-term to zero and as a result, there is an additional constraint on the fields. If the constraint is of the form $\mu : \mathbb{C}^n \to \mathbb{R}$, it will bring down the real dimension of the space of possible vacua by one. A further quotienting by the gauge group would yield the true space of physically distinct vacua. In the special case of a $U(1)$ gauge group, this will again reduce the real dimension by a further count of one. Now, one can immediately identify the two-step process that we followed in the constructions of the symplectic quotients with the two physics motivated steps of a) setting the D-term to zero and then b) using gauge equivalence to retain only those vacua which could lead to distinct physics.

CHAPTER 4

Specific resolutions – $\mathbb{C}^2/\mathbb{Z}_2$

4.1 The Ricci flat metric from symplectic reduction

The D-term constraint in this case is

$$x_1 + x_2 - 2x_3 = t \; . \tag{4.1}$$

where x_i are the fields and t is the F-I parameter. Before imposing the constraint, let us consider the canonical metric on the space of the three fields

$$G = \frac{1}{2}\sum_{i=1}^{3} x_i \log[x_i] \; . \tag{4.2}$$

Now the region defined by $x_i > 0$ defines the moment polytope for $\mathbb{C}^3$. Here, it is just the first octant of the $x_1 - x_2 - x_3$ space. Now, the symplectic quotient is obtained by imposing the gauge equivalence and the above D-term constraint. After imposing (4.1), we will have two independent fields. We choose them in the following way

$$\begin{aligned} P^1 &= \frac{(t - x^2 + x^1)}{2} \; . \\ P^2 &= x^3 \; . \end{aligned} \tag{4.3}$$

The resulting space has complex dimension two and is in fact the resolution of $\mathbb{C}^2/\mathbb{Z}_2$. The moment polytope for this manifolds looks like Fig (4.1). Note that the blowing up of a $\mathbb{S}^2$ at the origin shows up nicely in the polytope diagram as a line segment(the polytope for $\mathbb{S}^2$!) chopping off a corner in the polytope for $\mathbb{C}^2$. The polytope is *open* since the space we are considering is non-compact. But the metric obtained from the

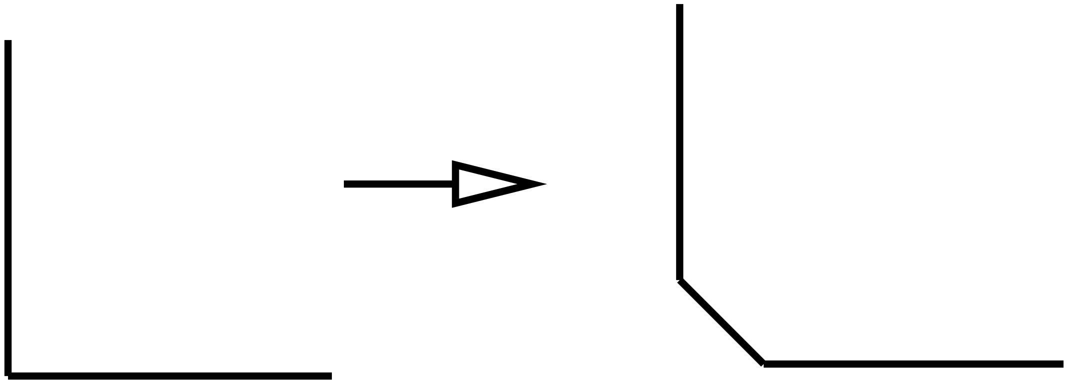

Figure 4.1: The moment polytope for resolved $\mathbb{C}^2/\mathbb{Z}_2$

canonical potential is not Ricci flat. To see this, substitute (4.3) into (4.1),

$$G(P^1,P^2) = \frac{1}{2}\Big[(P^1+P^2)\log(P^1+P^2)+P^2\log(P^2)+(t-P^1+P^2)\log(t-P^1+P^2)\Big] . \tag{4.4}$$

Computing the Ricci scalar from the above potential using $R = -\frac{\partial^2 G^{ij}}{\partial P^i \partial P^j}$ gives

$$R = \frac{3(t^2+12P^2)}{(t+6P^2)^3} . \tag{4.5}$$

Thus, we see that the metric is not even scalar flat for real, finite values of the blow-up parameter t ($t \to \infty$ does give $R \to 0$). Also note that R is just a function of P^2. For a general choice of P^1, P^2, this need not have been the case. The choice made in (4.3) was tailored to this aspect. This just simplifies certain calculations. Now, following Abreu (1997), we could try adding a general $f(P^1,P^2)$ to the canonical symplectic potential (4.4). But, noting that R is just a function of P^2, we try adding just $f(P^2)$. It turns out that this is sufficient in this case and the actual *Ricci flat* (not only scalar flat) metric can be found by adding a $f(P^2)$ to the canonical metric. The details of solving the resultant differential equations for $f(P^2)$ are presented in the Appendices. We just note the solution here

$$f(P^2) = \frac{1}{2}\Big[(t+P^2)\log(t+P^2) - (t+2P^2)\log(t+2P^2)\Big] . \tag{4.6}$$

It is important to note that the new singularities that have been added ($P^2 = -t, P^2 = -t/2$) lie outside the polytope for $\mathbb{C}^2/\mathbb{Z}_2$. Thus, we confirm that adding these pieces has not led to distortion of the original polytope.

4.2 The Ricci flat metric on $\mathbb{C}^2/\mathbb{Z}_2$ as a cone

In the previous section, we derived the Ricci flat metric on the blow-up of $\mathbb{C}^2/\mathbb{Z}_2$ by symplectic quotient using the language of Guillemin-Abreu. Here, we show that the metric thus obtained is indeed the well know Eguchi-Hanson metric or, alternatively, the two-center Gibbons-Hawking metric. (See Eguchi and Hanson (1979), Eguchi and Hanson (1978), Gibbons and Hawking (1978) for more details). Now, recall that the metric is given by

$$ds^2 = G_{ij} dP^i dP^j + G^{ij} d\theta_i d\theta_j \ , \tag{4.7}$$

where $G_{ij} = \partial^2 G/\partial P^i \partial P^j$ and the G in this case is given by

$$\begin{aligned} G(P^1, P^2) \;&=\; \frac{1}{2}\Big[(P^1 + P^2)\log(P^1 + P^2) + P^2\log(P^2) + (t - P^1 + P^2)\log(t - P^1 + P^2) \\ &\quad +(t + P^2)\log(t + P^2) - (t + 2P^2)\log(t + 2P^2)\Big] \ . \end{aligned} \tag{4.8}$$

Now, we make the following co-ordinate transformation

$$\begin{aligned} t &= 2a^2 \ , \\ P^2 &= r^2 - a^2 \ , \\ P^1 &= a^2 + r^2\cos(\theta) \ . \end{aligned} \tag{4.9}$$

The metric (4.7) reduces to the Eguchi-Hanson form

$$ds^2 = \left[1 - \left(\frac{a}{r}\right)^4\right]^{-1} dr^2 + r^2\left[1 - \left(\frac{a}{r}\right)^4\right]\sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2) \ . \tag{4.10}$$

where

$$\begin{aligned}
\sigma_x &= \frac{1}{2}[\sin\beta\; d\theta - \sin\theta\;\cos\beta\; d\alpha]\;, \\
\sigma_y &= \frac{1}{2}[-\cos\beta\; d\theta - \sin\theta\;\sin\beta\; d\alpha]\;, \\
\sigma_z &= \frac{1}{2}[d\beta + \cos\theta\; d\alpha]\;.
\end{aligned} \tag{4.11}$$

Here, we have relabelled co-ordinates (ϕ_1, ϕ_2) as (α, β). Note that what we have done is write the Ricci flat metric in a form which looks very similar to the cone form of the usual flat metric on $\mathbb{C}^2$

$$ds^2 = dr^2 + r^2(\sigma_x^2 + \sigma_y^2 + \sigma_z^2)\;.$$

The radial co-ordinate in (4.10) is P^2. The angle dual to this co-ordinate is ϕ_2 which is β as per our choice. This form is convenient to analyze some of the global properties of this metric. To do this, we carry out a further change of co-ordinates

$$u^2 = r^2\left[1 - \frac{a^4}{r^4}\right]\;. \tag{4.12}$$

The metric can now be written as

$$ds^2 = \left[1 + \frac{a^4}{r^4}\right]^{-2} du^2 + u^2\sigma_z^2 + r^2(\sigma_x^2 + \sigma_y^2)\;. \tag{4.13}$$

Now, we want to analyze the form of the metric near the point $r = a$, a point where the metric as written in (4.10) has a singularity. In the new co-ordinates, this corresponds to $u = 0$. So, in this co-ordinate system the metric near $r = a$ reduces to

$$ds^2 = \frac{du^2}{4} + u^2\sigma_z^2 + a^2(\sigma_x^2 + \sigma_y^2)\;. \tag{4.14}$$

Now, we just set (α, θ) to some constant value ($\implies d\alpha = d\theta = 0$). So, $\sigma_x = \sigma_y = 0$ and $\sigma_z = d\beta/2$. So the metric reduces to

$$ds^2 = \frac{1}{4}(du^2 + u^2 d\beta^2)\;. \tag{4.15}$$

Thus we see that the $u = 0$ singularity can be reduced to a mere co-ordinate singularity provided the period of β is 2π. But, this is exactly half of the usual period for β when the σs are used to write a metric on $\mathbb{S}^3$. So, to get a non-singular metric, we choose the period for β to be 2π. This explicitly tells us that there is a quotient by $\mathbb{Z}_2$. So, the asymptotic limit of this metric is not flat space but it is flat space with antipodal points identified (that is, $\mathbb{C}^2/\mathbb{Z}_2$).

4.3 Multi-center Gibbons-Hawkings metrics and the half-Legendre transform

Now, one would like to extend the above procedure to obtain the complete Gibbons-Hawking family of self-dual metrics in four real dimensions. But, this seems to present considerable algebraic difficulties. To see this, we take the work of Hitchin *et al.* (1987) where a solution based on an intermediate Legendre transform is presented for the general Gibbons-Hawking metric. For the n-center case, the solution is actually given by

$$F_n = \sum_{i=1}^{n} F_*(\bar{r} - \bar{\rho}_i) \ , \tag{4.16}$$

where $\bar{r} = (x, z, \bar{z})$ and $\bar{\rho}_i = (x_i, z_i, \bar{z}_i)$ are locations of n-centers in this mixed co-ordinate system. F_* itself is given by

$$F_*(\bar{r}) = r - x\log(x + r) + \frac{1}{2}x\log(4z\bar{z}) \ . \tag{4.17}$$

This is related to the Kähler potential by a Legendre transform

$$K(z, \bar{z}, w, \bar{w}) = F(x, z, \bar{z}) - (w + \bar{w})x \ , \tag{4.18}$$

with the following relation between $(w + \bar{w})$ and F

$$\frac{\partial F}{\partial x} = w + \bar{w} \ . \tag{4.19}$$

We can immediately see that this choice is midway between the usual mix of holomorphic, anti-holomorphic in the complex language and the two real co-ordinates of the symplectic picture. A further Legendre transform of F with respect to $(z+\bar{z})$ should yield a symplectic potential for resolved $\mathbb{C}^2/\mathbb{Z}_n$. This is indeed the case. We can check this explicitly for the first non-trivial case of the Eguchi-Hanson metric[1]. It is actually convenient to go to a new set of complex co-ordinates $(\rho,\bar{\rho})$ such that $z = e^{\rho}$ (So, $z\bar{z} = e^{\rho+\bar{\rho}}$). In this co-ordinate system, F_* is

$$\begin{aligned} F_*(\bar{r}) &= r - x\log(x+r) + \frac{1}{2}x\log(4e^{\rho+\bar{\rho}}) \ , \\ r &= \sqrt{x^2+4e^{\rho+\bar{\rho}}} \ . \end{aligned} \tag{4.20}$$

The F in this case is,

$$F = F_*(x+a) + F_*(x-a) \ . \tag{4.21}$$

and we obtain the symplectic potential as below

$$G(x,y) = x(\rho+\bar{\rho}) - F(x,y) \ , \tag{4.22}$$

where $(\rho+\bar{\rho})$ is expressed in terms of y using

$$\frac{\partial F}{\partial(\rho+\bar{\rho})} = y \ . \tag{4.23}$$

Now, inverting the above equation for the F given in (4.21) gives

$$\rho+\bar{\rho} = \log\left[\frac{(x+y)(y-x)(y+a)(y-a)}{4y^2}\right] . \tag{4.24}$$

Substituting this into (4.22), we get (ignoring linear terms which do not contribute to the metric)

$$\begin{aligned} G &= -(a-x)\,\log\left[\frac{(-a+y)\ (x+y)}{y}\right] + (a+x)\,\log\left[\frac{(a+y)\ (x+y)}{y}\right] \\ &- \ (x-y)\,\log\left[\frac{(-a+y)\ (a+y)\ (-x+y)\ (x+y)}{y^2}\right] . \end{aligned} \tag{4.25}$$

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[1] The one center case is just flat space and it works trivially.

Recollecting terms with the following relabelling of co-ordinates gives exactly the same G obtained in (4.8).

$$\begin{aligned} y &= \frac{t}{2} + P^2 \ , \\ a &= \frac{t}{2} \ , \\ x &= P^1 - \frac{t}{2} \ . \end{aligned} \tag{4.26}$$

We emphasize that the crucial step in going from F to G is the ability to invert (4.23). Our passage from z to ρ made the inversion simple in the two-center case. But, this advantage does not carry over to the general case of n centers. So, the problem of obtaining symplectic potentials for the entire Gibbons-Hawking family remains open.

The interesting feature in the solution using the intermediate Legendre transform is that there is a sense ((4.16)) in which the multi-center solution is a superposition of the single-centered ones. This is also the case with the original Gibbons-Hawking form

$$\begin{aligned} ds^2 &= V(d\tau + \omega \cdot d\mathbf{x})^2 + V^{-1} d\mathbf{x} \cdot d\mathbf{x} \ , \\ V &= \sum_{i=1}^{n} \frac{a_i}{|\mathbf{x} - \mathbf{x_i}|} \ , \\ \text{curl}\, \omega &= \text{grad}(V) \ . \end{aligned} \tag{4.27}$$

This linearity means that it is easy to write down the intermediate potential for the n-center case. But, even for the case of $\mathbb{C}^2/\mathbb{Z}_3$, knowing the intermediate Legendre transform does not amount to anything worthwhile since the inverting of (4.23) is not possible. To see this more explicitly, we write (4.23) for $\mathbb{C}^2/\mathbb{Z}_3$

$$\frac{\sqrt{4\, e^{\rho+\bar{\rho}} + (a-x)^2} + \sqrt{4\, e^{\rho+\bar{\rho}} + (b-x)^2} + \sqrt{4\, e^{\rho+\bar{\rho}} + (a+b+x)^2}}{2} = y \ , \tag{4.28}$$

where we have chosen $x = a, x = b, x = -a - b$ to be the locations of the centers. Inverting the above equation to express $(\rho + \bar{\rho})$ in terms of x and y would require us to find the roots of a polynomial of sixth degree. We have not been able to do this. Since there are no closed form solutions for generic polynomials with degree > 4, we believe that the required inversion may not be possible.

CHAPTER 5

Specific resolutions - $\mathbb{C}^3/\mathbb{Z}_3$

5.1 The Ricci flat metric from symplectic reduction

We will closely follow the work of Ray (1998) in this section. The D-term constraint for this case is

$$x_1 + x_2 + x_3 - 3x_4 = t \, . \tag{5.1}$$

There are a total of three independent fields(P^i) and we choose them in the following fashion

$$\begin{aligned} x_1 &= t - P^1 - P^2 + 3P^3 \, , \\ x_2 &= P^1 \, , \\ x_3 &= P^2 \, , \\ x_4 &= P^3 \, . \end{aligned} \tag{5.2}$$

As in the case of $\mathbb{C}^2/\mathbb{Z}_2$, we first write the symplectic potential corresponding to the canonical metric

$$G_{can} = \frac{1}{2}\left[P^1 \log P^1 + P^2 \log P^2 + P^3 \log P^3 + (t - P^1 - P^2 + 3P^3) \log(t - P^1 - P^2 + 3P^3)\right] . \tag{5.3}$$

The moment polytope (defined by $x_i(P^j) > 0$) in this case is given by Fig (5.1). The central triangular region corresponds to the $\mathbb{CP}^2$ that has been blow-up at the origin. Note that the figure given is actually a projection of the polytope into a two dimensional plane.

The Ricci scalar for the canonical metric is given by

$$R_{can} = \frac{6(t^2 + 72(P^3)^2)}{(t + 12P^3)^3} \, . \tag{5.4}$$

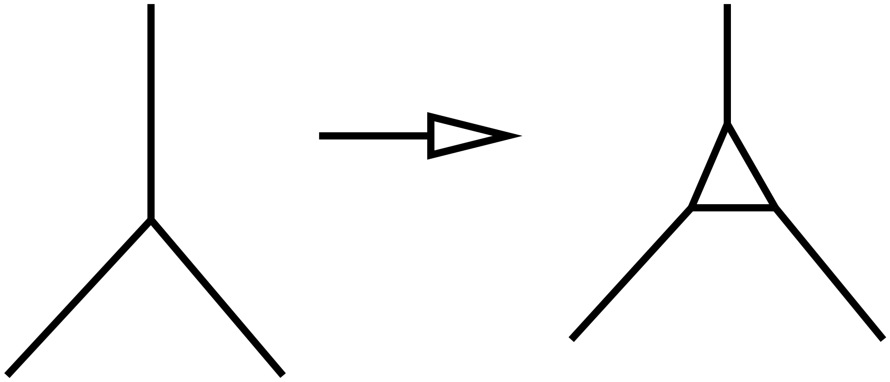

Figure 5.1: The moment polytope for resolved $\mathbb{C}^3/\mathbb{Z}_3$

Again, we note that the Ricci scalar is a function of just a single variable P^3. So, there is hope of finding a Ricci flat metric by just adding a function of a single variable to the canonical potential. So, we now add an $f(P^3)$ and try to obtain a Ricci flat metric on this manifold. The details of solving the resulting differential equations have already been worked out in detail in Ray (1998) and we will not reproduce them here. The process is identical to the one followed for $\mathbb{C}^2/\mathbb{Z}_2$ and the only subtlety involved (not sufficiently explained in Ray (1998)) is in choosing the value for the integration constant such that the singularity structure of the polytope is not modified. For the case of $\mathbb{C}^2/\mathbb{Z}_2$, we have provided the details (in the appendix) as to how this can be ensured. The present case of $\mathbb{C}^3/\mathbb{Z}_3$ is very similar. Now, we just note the final solution

$$f''(P^3) = \frac{-9}{2\,(t+3\,P^3)} + \frac{3\,(t+2\,P^3)}{2\,\left(t^2+3\,t\,P^3+3\,{P^3}^2\right)}\ . \tag{5.5}$$

5.2 The Ricci flat metric on $\mathbb{C}^3/\mathbb{Z}_3$ as a cone

Just like the case of $\mathbb{C}^2/\mathbb{Z}_2$, we can rewrite the Ricci flat metric on the blow-up of $\mathbb{C}^3/\mathbb{Z}_3$ in a cone form. To achieve this, make the following co-ordinate transformation

$$\begin{aligned} t &= \frac{a^2}{2}\ , \\ P^1 &= r^2\cos\theta_1\ , \\ P^2 &= r^2\cos\theta_2\ , \\ P^3 &= \frac{1}{6}(r^2-a^2)\ . \end{aligned} \tag{5.6}$$

This modifies the metric to the following cone-like form

$$\left[1-\frac{a^6}{r^6}\right]^{-1}dr^2+\frac{r^2}{9}\left[1-\frac{a^6}{r^6}\right](d\gamma+6d\alpha\cos\theta_1+6d\beta\cos\theta_2)^2+r^2d\Omega^2\ . \qquad (5.7)$$

where $d\Omega^2$ is the usual Fubini-Study metric on $\mathbb{CP}^2$. The algebra involved in the co-ordinate transformation is quite involved and some details are provided in the appendix to help the reader.

We can now proceed to check if the periodicities of the angles match the expectations. To do this, we set

$$u^2=\frac{1}{9}r^2\left[1-\frac{a^6}{r^6}\right]\ , \qquad (5.8)$$

and the metric reduces to

$$ds^2=\left[1+2\frac{a^6}{r^6}\right]^{-2}du^2+u^2(d\gamma+6d\alpha\cos\theta_1+6d\beta\cos\theta_2)^2+r^2d\Omega^2\ . \qquad (5.9)$$

So, near $r\to a(u\to 0)$, the metric is

$$ds^2\simeq\frac{du^2}{9}+u^2(d\gamma+6d\alpha\cos\theta_1+6d\beta\cos\theta_2)^2+r^2d\Omega^2\ . \qquad (5.10)$$

From this, it is clear that the angle γ should have a periodicity that is $1/3^{rd}$ the usual if $r\to 0$ were to be a mere co-ordinate singularity. This does confirm with our expectation considering the fact that this space has been obtained using a global $\mathbb{Z}_3$ quotient of $\mathbb{C}^3$.

Now, in order to understand better the passage to the Ricci flat metric, we want to track the deformation of the canonical metric to the Ricci flat one in the cone-like co-ordinates. For starters, we carry out the *same* co-ordinate transformation on the canonical metric. This gives us

$$ds^2=\left[\frac{3a^2-3r^2}{3a^2-4r^2}\right]^{-1}dr^2+\frac{r^2}{9}\left[\frac{3a^2-3r^2}{3a^2-4r^2}\right](d\gamma+6d\alpha\cos\theta_1+6d\beta\cos\theta_2)^2+r^2d\Omega^2\ . \qquad (5.11)$$

Note that the if we set $a=0$, *the coefficient of* dr^2 *does not reduce to one*[1]. So, the variable r is not the exact radial variable for the canonical metric. But, this does not

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[1] It is actually equal to $4/3$, which is presumably related to the fact we obtained a *three* dimensional space by carrying out a quotient on a *four* dimensional space.

unduly concern us as it is just a matter of a constant factor. The neat thing that we observe is that the base of the cone in the canonical metric is the same as that in the Ricci flat case. In fact, by writing both the metric in the following form

$$H(r)^{-1}dr^2 + \frac{r^2}{9}H(r)(d\gamma + 6d\alpha\cos\theta_1 + 6d\beta\cos\theta_2)^2 + r^2 d\Omega^2\ , \tag{5.12}$$

we see that the only change induced by the additive piece is the modification of $H(r)$. As a quick check of this assertion, we can immediately see how the coefficient of dr^2 is modified. Since we are adding just $f(P^3) \sim f(r)$, the only modification to dr^2 terms will be from

$$\Delta g_{rr} = f''(P^3(r))(dP^3)^2\ , \tag{5.13}$$

where we have used Δg_{rr} to denote the change. Also, $dP^3 = rdr/3$. So, we get

$$\Delta g_{rr} = \frac{-(3a^4 + 2a^2r^2 + r^4)dr^2}{3(a^4 + a^2r^2 + r^4)}\ , \tag{5.14}$$

and finally,

$$\left[\frac{3a^2 - 3r^2}{3a^2 - 4r^2}\right]^{-1} dr^2 + \Delta g_{rr} = \left[1 - \frac{a^6}{r^6}\right]^{-1} dr^2\ . \tag{5.15}$$

where the RHS is nothing but the $H(r)^{-1}$ in the Ricci flat case. The result obtained here is in confirmation with the expectations that $1 - H(r) \sim 1/r^6$ when four cycles[2] are blown up (Benvenuti *et al.* (2005)). This arises in the context of the AdS-CFT correspondence (Maldacena (1998), Klebanov and Witten (1998)) which relates the exponent of r (six in this case) to the dimension of the operator corresponding to the blow-up in the CFT.

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[2] The corresponding result for two cycles in $1 - H(r) \sim 1/r^2$.

CHAPTER 6

Ricci-flat metrics on $\mathbb{C}^3/\Gamma$ and other cases

As we stated earlier, the primary goal of this work is to make progress towards obtaining Ricci flat metrics for some of the cases in the $\mathbb{C}^3/\Gamma$ family. The next simplest in this family is blown up $\mathbb{C}^3/\mathbb{Z}_5$ (unsolved till now). So, we now consider the possibility of obtaining a Ricci flat metric on this space by similar methods. At the outset, we clarify that we have not yet been able to achieve the final goal. But, we have been able to make some progress by developing new tools. We also indicate possible directions for future work.

The D-terms in this case are given by

$$\begin{aligned} x_1 + x_2 + x_5 - 3x_4 &= t_1 \,, \\ x_3 + x_4 - 2x_5 &= t_2 \,. \end{aligned} \tag{6.1}$$

Note the existence of multiple blow-up parameters. One of them corresponds to blowing up a $\mathbb{CP}^2$(it has a triangle as its moment polytope) and the other corresponds to the Hirzebruch surface $\mathbb{F}_3$ (it has a rectangle as the moment polytope). Five fields and two constraints implies that we have three independent fields and we choose independent the fields P^i as

$$\begin{aligned} x_1 &= t_1 + 3P^3 - P^1 - P^2 \,, \\ x_2 &= P^1 \,, \\ x_3 &= t_2 + 2P^2 - P^3 \,, \\ x_4 &= P^3 \,, \\ x_5 &= P^2 \,. \end{aligned} \tag{6.2}$$

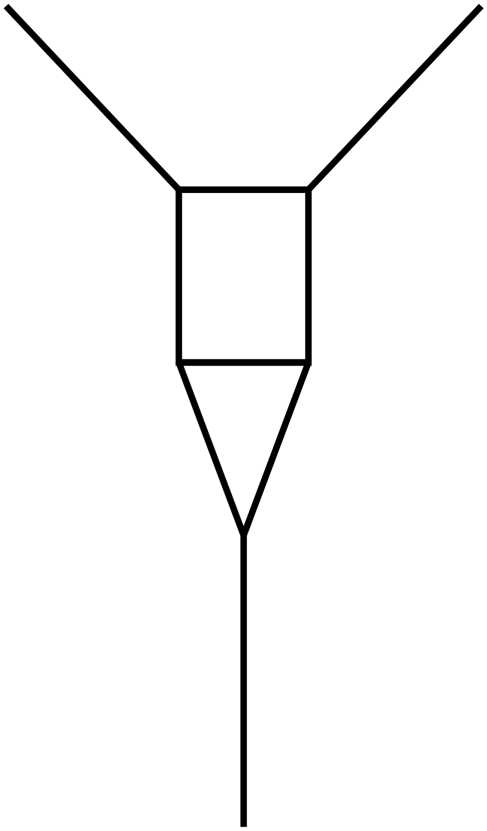

Figure 6.1: The moment polytope for resolved $\mathbb{C}^3/\mathbb{Z}_5$

The canonical metric for this case is

$$\begin{aligned} G &= \frac{1}{2}\Big[P^1 \log P^1 + P^2 \log P^2 + P^3 \log P^3 + (t_1 - P^1 - P^2 + 3P^3)\log(t_1 - P^1 - P^2 + 3P^3) \\ &+ (t_2 + 2P^2 + P^3)\log(t_2 + 2P^2 + P^3)\Big]\,. \end{aligned} \tag{6.3}$$

The polytope inequalities are $x_i(P^j) > 0$ and in P^j space, they define a region which looks like Fig (6.1). As before, we look at the Ricci scalar for hints on how to obtain the Ricci flat metric.

$$R_{can} = \frac{-2\,\left(6\,P^2 + t_2\right)}{4\,{P^2}^2 + 9\,{P^3}^2 - 6\,P^2\,\left(10\,P^3 + t_1\right) - 12\,P^3\,t_2 - t_1\,t_2}\,. \tag{6.4}$$

In a break of pattern, the Ricci scalar is no longer a function of a single variable. Any attempt to directly add $F(P^2, P^3)$ to the canonical metric runs into trouble as the equations for Ricci flatness are now PDEs and they do not seem to have any straightforward solutions. So, one probably needs to look at other ways to figure out the required deformation.

Another new feature in the case of $\mathbb{C}^3/\mathbb{Z}_5$ is regarding the behaviour of R for large t_i. In the limit of $t_1 \to \infty$, we can see that $R_{can} \to 0$. This mimics the behaviour that is seen in the case of R_{can} for $\mathbb{C}^2/\mathbb{Z}_2$ and $\mathbb{C}^3/\mathbb{Z}_3$. But, interestingly, the $t_2 \to \infty$ limit *does not* lead to such a behaviour. Taking $t_2 \to \infty$ gives

$$R_{t_2\to\infty} = \frac{2}{12P^3 + t_1} \,. \tag{6.5}$$

To rectify this, we could first try correcting G_{can} such that we get a Ricci flat metric for $t_2 \to \infty$. It turns out that this is indeed possible. Noting that the $R_{t_2\to\infty}$ is a function of just P^3, we try adding an $f(P^3)$ to the canonical metric. Solving for the resulting equations for Ricci flatness gives

$$f''(P^3) = \frac{-9}{2\,(3\,P^3 + t_1)} + \frac{3\,(2\,P^3 + t_1)}{2\,\left(3\,{P^3}^2 + 3\,P^3\,t_1 + {t_1}^2\right)} \,. \tag{6.6}$$

This additive piece is identical to the one added to the canonical metric for $\mathbb{C}^3/\mathbb{Z}_3$ in order to make it Ricci flat! But, in this case, we just have a metric which becomes Ricci flat in the limit $t_2 \to \infty$. For future work, one could take the new G as a starting point and then look at ways to add pieces so that the Ricci flat metric for finite t_1, t_2 is obtained. Since we do have the Ricci flat metric for the large t_i case, the logical thing would be to work out $1/t_i$ corrections. It could be possible that the cone-like form of the metric could be the ideal one to work with. But, the problem of finding the required co-ordinate transformation for $\mathbb{C}^3/\mathbb{Z}_3$ (and generic $\mathbb{C}^3/\Gamma$) remains. Some preliminary calculations indicate that the required transformations are not as simple as the ones we have met ($\mathbb{C}^2/\mathbb{Z}_2$, $\mathbb{C}^3/\mathbb{Z}_3$).

The summary from this brief discussion of $\mathbb{C}^3/\mathbb{Z}_5$ is that the case of multiple blow-ups is tough to handle in this formalism. We saw this coming even in the case of $\mathbb{C}^2/\mathbb{Z}_3$ (the simplest multiple blow-up case in two dimensions). Obtaining the symplectic potential from the half-Legendre transform was not possible. One must also mention that it is not clear if there is any analogous *linearity* in an intermediate Legendre transform for the case of three complex dimensions. Future work could also involve this aspect.

Now, one can wonder if it is possible to extend this formalism beyond the orbifolds. It can be checked that the case of the conifold does work. In this context, we note that the work of Martelli *et al.* (2006) (MSY) could play an important role. In their work, MSY have given a procedure to determine the correct radial variable for arbitrary Einstein-Sasakian manifolds. The examples that we have been considering do fall under this category since $\mathbb{C}^3/\Gamma$ can be written as real cone over $\mathbb{S}^5/\Gamma$. If writing the canonical metric as a cone is going to ultimately help us in obtaining the Ricci flat metrics, then the work of MSY could be directly used for generalizations of the conifold. Note that unlike the cases of the conifold and more general $Y^{p,q}$ spaces that they consider, it is easier to obtain the radial variable for the case of the orbifolds that we are considering. It can be done by taking all blow-up parameters to be zero and then comparing with the flat metric.

CHAPTER 7

Summary and future directions

To summarize, we have reviewed the symplectic reduction way of obtaining Ricci flat metrics on orbifolds using the language developed by Guillemin and Abreu. For the case of $\mathbb{C}^3/\mathbb{Z}_3$, one is able to go beyond what has been done in complex co-ordinates (Douglas and Greene (1998)) and obtain the actual Ricci flat metric. We have presented a careful account of deriving the Ricci flat metric in this new language, filling many potholes in existing literature. This derivation gives us hope that the more general cases could be dealt within this setup. But, the existence of multiple blow-up parameters in all these cases makes the problem much more difficult to handle. In fact, one doesn't even know if closed form expressions exist for the Ricci flat metric on these spaces. Having lost hope on analytic methods, many researchers have resorted to numerical work for similar problems (for ex, see Doran *et al.* (2007)).

However, we have developed some new tools that give hope for an analytical solution to the problem. Chief amongst them is the technique of writing the metrics obtained from symplectic reduction as cones. One major by-product of this is the explicit confirmation of the AdS-CFT prediction for blow-ups of four cycles in the case of $\mathbb{C}^3/\mathbb{Z}_3$ (Benvenuti *et al.* (2005)). We believe that the cone-form of the canonical metric would be an ideal starting point for any further attempt to obtain the exact/approximate Ricci flat metrics or for checking predictions of AdS-CFT. It is in this direction that we propose to carry out our future work.

APPENDIX A

Deriving the Ricci flat metric for $\mathbb{C}^2/\mathbb{Z}_2$

As explained earlier, we try to obtain the Ricci flat metric by adding a $f(P^2)$ to the canonical metric ((4.4)).

$$G(P^1,P^2)=\frac{1}{2}\Big[(P^1{+}P^2)\log(P^1{+}P^2){+}P^2\log(P^2){+}(t{-}P^1{+}P^2)\log(t{-}P^1{+}P^2){+}F(P^2)\Big]\,,$$

and then set

$$R_{ij}=-G^{il}\frac{\partial^2 G^{kj}}{\partial P^k\partial P^l}=0\,. \tag{A.1}$$

Note that that g_{ij} after addition of $f(P^2)$ is

$$g_{ij}=\frac{1}{2}\left[\begin{array}{cc} \frac{1}{t-P^1+P^2}+\frac{1}{P^1+P^2} & \frac{1}{P^1+P^2}-\frac{1}{t-P^1+P^2} \\ \frac{1}{P^1+P^2}-\left(\frac{1}{t-P^1+P^2}\right) & \frac{1}{P^2}+\frac{1}{t-P^1+P^2}+\frac{1}{P^1+P^2}+2\,F''(P^2) \end{array}\right]. \tag{A.2}$$

Now, we just redefine $f=F''(P^2)$ and write out the various components of R_{ij}

$$\begin{aligned}
R_{11} &= -G^{11}\left\{\frac{\partial^2 G^{11}}{\partial P^1\partial P^1}+\frac{\partial^2 G^{21}}{\partial P^2\partial P^1}\right\}-G^{12}\left\{\frac{\partial^2 G^{11}}{\partial P^1\partial P^2}+\frac{\partial^2 G^{21}}{\partial P^2\partial P^2}\right\},\\
R_{12} &= -G^{11}\left\{\frac{\partial^2 G^{12}}{\partial P^1\partial P^1}+\frac{\partial^2 G^{22}}{\partial P^2\partial P^1}\right\}-G^{12}\left\{\frac{\partial^2 G^{12}}{\partial P^1\partial P^2}+\frac{\partial^2 G^{22}}{\partial P^2\partial P^2}\right\},\\
R_{21} &= -G^{21}\left\{\frac{\partial^2 G^{11}}{\partial P^1\partial P^1}+\frac{\partial^2 G^{21}}{\partial P^2\partial P^1}\right\}-G^{22}\left\{\frac{\partial^2 G^{11}}{\partial P^1\partial P^2}+\frac{\partial^2 G^{21}}{\partial P^2\partial P^2}\right\},\\
R_{22} &= -G^{21}\left\{\frac{\partial^2 G^{12}}{\partial P^1\partial P^1}+\frac{\partial^2 G^{22}}{\partial P^2\partial P^1}\right\}-G^{22}\left\{\frac{\partial^2 G^{12}}{\partial P^1\partial P^2}+\frac{\partial^2 G^{22}}{\partial P^2\partial P^2}\right\}.
\end{aligned}$$

So, $R_{ij}=0$ will be satisfied if the following equations hold

$$\begin{aligned}
\frac{\partial^2 G^{11}}{\partial P^1\partial P^1}+\frac{\partial^2 G^{21}}{\partial P^2\partial P^1} &= 0\,,\\
\frac{\partial^2 G^{11}}{\partial P^1\partial P^2}+\frac{\partial^2 G^{21}}{\partial P^2\partial P^2} &= 0\,,\\
\frac{\partial^2 G^{12}}{\partial P^1\partial P^1}+\frac{\partial^2 G^{22}}{\partial P^2\partial P^1} &= 0\,,\\
\frac{\partial^2 G^{12}}{\partial P^1\partial P^2}+\frac{\partial^2 G^{22}}{\partial P^2\partial P^2} &= 0\,.
\end{aligned}$$

This set leads to three non-trivial equations out of which two are second order and one is first order. The first order equation comes from first equation above and is given by

$$3 + 2\, f^2\, P^2\, \left(t + 2\, P^2\right) + 2\, f\, \left(t + 5\, P^2\right) + P^2(t + 2\, P^2) f' \;\; = 0\ . \tag{A.3}$$

The general solution for the above equation can be obtained as

$$f = \frac{-1}{2\, P^2} - \frac{2}{t + 2\, P^2} + \frac{8\, \left(t\, c + 2\, c\, P^2\right)}{-1 + 4\, t^2\, c + 16\, t\, c\, P^2 + 16\, c\, {P_2}^2}\ . \tag{A.4}$$

Now, we use the freedom in choosing the integration constant c to ensure that the added piece $f(P^2)$ does not remove any of the existing l_i. This is to ensure that the polytope structure is not modified. Further, we also need to check that if any new singularities are introduced, they should be outside the polytope. To choose the constant, note that for the specific the specific value of $c = 1/4t^2$, the $-1/2P^2$ term gets cancelled. This is exactly what is required to ensure that there is no change in singularity structure. The presence of this term would have meant the cancellation of the $\frac{1}{2}P^2 \log P^2$ term in the canonical metric. Such a cancellation is a trivial way to obtain Ricci flatness and it is not what we seek. We emphasize that fixing the constant in (A.4) is an absolutely critical step since, for all other values of c, we *do not* get the metric on blown up $\mathbb{C}^3/\mathbb{Z}_5$. Now, after fixing c to the required value, we finally get

$$f = \frac{1}{2(t + P^2)} - \frac{2}{t + 2P^2}\ . \tag{A.5}$$

Quite remarkably, it turns out that this solution to the first order equation automatically satisfies the other second order equations. The method outlined here can be carried over in an identical manner to the case of $\mathbb{C}^3/\mathbb{Z}_3$. The choosing of the integration constant can also be done in a similar fashion.

APPENDIX B

Cone form for the $\mathbb{C}^3/\mathbb{Z}_3$

In (5.6), we presented the co-ordinate transform that gives us the cone form of the Ricci flat metric on $\mathbb{C}^3/\mathbb{Z}_3$. The tricky part in that transformation is realizing that the base part is indeed a metric on $\mathbb{CP}^2$. To aid the reader in following the calculations, we present some details. The base part actually looks like this

$$\begin{aligned}\bar{g} =& \frac{1}{18}\Bigg[36 d\alpha \cos\theta_1\{d\alpha - 4 d\beta\cos\theta_2\} - 36\{d\alpha^2 + d\beta^2 + d\alpha^2\cos 2\theta_1 + d\beta^2(\cos 2\theta_2 - \cos\theta_2)\} \\ &+ \frac{-9\sin\theta_1 d\theta_1\{4\sin\theta_2 d\theta_2 + (1 - 2\cos\theta_2)d\theta_1\tan\theta_1\} + 9(2\cos\theta_1 - 1)d\theta_2^2\tan\theta_2}{2\cos\theta_1 + 2\cos\theta_2 - 1}\Bigg] .\end{aligned}$$

Now, we apply the following co-ordinate transformation

$$\theta_1 = \cos^{-1}\left(\frac{1+P^1}{6}\right), \tag{B.1}$$

$$\theta_1 = \cos^{-1}\left(\frac{1+P^1}{6}\right). \tag{B.2}$$

With this, the metric reduces exactly to the form

$$\bar{g} = \bar{g}_{ij} dP^i dP^j + \bar{g}^{ij} d\phi_i d\phi_j \ . \tag{B.3}$$

where $\bar{g}_{ij}$ and $\bar{g}^{ij}$ are exactly like (3.14) and (3.15) respectively (upto some overall multiplication factors). ϕ_i are (α, β). Thus, we are convinced that the angle part does indeed reduce to a metric on $\mathbb{CP}^2$.

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