Mechanics - Basic Physical Concepts

Math: Circle: $2\pi r$, πr^2; Sphere: $4\pi r^2$, $(4/3)\pi r^3$

Quadratic Eq.: $a x^2 + b x + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Cartesian and polar coordinates:

$$x = r\cos\theta,\ y = r\sin\theta,\quad r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$$

Trigonometry: $\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\alpha - \beta)$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

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$$\sin 2\theta = 2\sin\theta\cos\theta, \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2},\quad 1 + \cos\theta = 2\cos^2\frac{\theta}{2}$$

Vector algebra: $\vec{A} = (A_x, A_y) = A_x\hat{\imath} + A_y\hat{\jmath}$

Resultant: $\vec{R} = \vec{A} + \vec{B} = (A_x + B_x, A_y + B_y)$

Dot: $\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_x B_x + A_y B_y + A_z B_z$

Cross product: $\hat{\imath}\times\hat{\jmath} = \hat{k},\quad \hat{\jmath}\times\hat{k} = \hat{\imath},\quad \hat{k}\times\hat{\imath} = \hat{\jmath}$

$$\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$C = A\,B\sin\theta = A_\perp B = A\,B_\perp$, use right hand rule

Calculus: $\frac{d}{dx}x^n = n\,x^{n-1},\quad \frac{d}{dx}\ln x = \frac{1}{x},$

$$\frac{d}{d\theta}\sin\theta = \cos\theta,\quad \frac{d}{d\theta}\cos\theta = -\sin\theta,\quad \frac{d}{dx}\text{const} = 0$$

Measurements

Dimensional analysis: *e.g.*,

$F = m\,a \rightarrow [M][L][T]^{-2}$, or $F = m\,\frac{v^2}{r} \rightarrow [M][L][T]^{-2}$

Summation: $\sum_{i=1}^{N}(a\,x_i + b) = a\sum_{i=1}^{N}x_i + b\,N$

Motion

One dimensional motion: $v = \frac{ds}{dt},\quad a = \frac{dv}{dt}$

Average values: $\bar{v} = \frac{s_f - s_i}{t_f - t_i},\quad \bar{a} = \frac{v_f - v_i}{t_f - t_i}$

One dimensional motion (constant acceleration):

$v(t):\quad v = v_0 + a\,t$

$s(t):\quad s = \bar{v}\,t = v_0\,t + \frac{1}{2}a\,t^2,\quad \bar{v} = \frac{v_0 + v}{2}$

$v(s):\quad v^2 = v_0^2 + 2\,a\,s$

Nonuniform acceleration: $x = x_0 + v_0\,t + \frac{1}{2}a\,t^2 + \frac{1}{6}j\,t^3 + \frac{1}{24}s\,t^4 + \frac{1}{120}k\,t^5 + \frac{1}{720}p\,t^6 + \ldots$, (jerk, snap,...)

Projectile motion: $t_{rise} = t_{fall} = \frac{t_{trip}}{2} = \frac{v_{0y}}{g}$

$h = \frac{1}{2}g\,t_{fall}^2,\ R = v_{ox}\,t_{trip}$

Circular: $a_c = \frac{v^2}{r},\quad v = \frac{2\pi r}{T},\quad f = \frac{1}{T}$ (Hertz=s^{-1})

Curvilinear motion: $a = \sqrt{a_t^2 + a_r^2}$

Relative velocity: $\vec{v} = \vec{v}' + \vec{u}$

Law of Motion and applications

Force: $\vec{F} = m\,\vec{a},\ F_g = m\,g,\ \vec{F}_{12} = -\vec{F}_{21}$

Circular motion: $a_c = \frac{v^2}{r},\ v = \frac{2\pi r}{T} = 2\pi r f$

Friction: $F_{static} \le \mu_s N \qquad F_{kinetic} = \mu_k N$

Equilibrium (concurrent forces): $\sum_i \vec{F}_i = 0$

Energy

Work (for all F): $\Delta W = W_{A\rightarrow B} = W_B - W_A = F_\parallel\,s = F\,s\cos\theta = \vec{F}\cdot\vec{s} \rightarrow \int_A^B \vec{F}\cdot d\vec{s}$ (in Joules)

Effects due to work done: $F_{ext} = m\,a + F_c + f_{nc}$

$W_{ext}|_{A\rightarrow B} = K_B - K_A + U_B - U_A + W_{diss}|_{A\rightarrow B}$

Kinetic energy: $K_B - K_A = \int_A^B m\,\vec{a}\cdot d\vec{s},\quad K = \frac{1}{2}m\,v^2$

K (conservative $\vec{F}$): $U_B - U_A = -\int_A^B \vec{F}\cdot d\vec{s}$

$U_{gravity} = m\,g\,y,\qquad U_{spring} = \frac{1}{2}k\,x^2$

From U to $\vec{F}$: $F_x = -\frac{\partial U}{\partial x},\quad F_y = -\frac{\partial U}{\partial y},\quad F_z = -\frac{\partial U}{\partial z}$

$F_{gravity} = -\frac{\partial U}{\partial y} = -m\,g,\quad F_{spring} = -\frac{\partial U}{\partial x} = -k\,x$

Equilibrium: $\frac{\partial U}{\partial x} = 0,\quad \frac{\partial^2 U}{\partial x^2} > 0$ stable, < 0 unstable

Power: $P = \frac{dW}{dt} = F\,v_\parallel = F\,v\cos\theta = \vec{F}\cdot\vec{v}$ (Watts)

Collision

Impulse: $\vec{I} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i \rightarrow \int_{t_i}^{t_f}\vec{F}\,dt$

Momentum: $\vec{p} = m\,\vec{v}$

Two-body: $x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

$p_{cm} \equiv M\,v_{cm} = p_1 + p_2 = m_1 v_1 + m_2 v_2$

$F_{cm} \equiv F_1 + F_2 = m_1 a_1 + m_2 a_2 = M\,a_{cm}$

$K_1 + K_2 = K_1^* + K_2^* + K_{cm}$

Two-body collision: $\vec{p}_i = \vec{p}_f = (m_1 + m_2)\,\vec{v}_{cm}$

$v_i^* = v_i - v_{cm},\qquad v_i' = v_i^{*\prime} + v_{cm}$

Elastic: $v_1 - v_2 = -(v_1' - v_2'),$

$v_i^{*\prime} = -v_i^*,\quad v_i' = 2\,v_{cm} - v_i$

Many body center of mass: $\vec{r}_{cm} = \frac{\sum m_i\,\vec{r}_i}{\sum m_i} = \frac{\int \vec{r}\,dm}{\int m_i}$

Force on cm: $\vec{F}_{ext} = \frac{d\vec{p}}{dt} = M\vec{a}_{cm},\quad \vec{p} = \sum\vec{p}_i$

Rotation of Rigid-Body

Kinematics: $\theta = \frac{s}{r},\quad \omega = \frac{v}{r},\quad \alpha = \frac{a_t}{r}$

Moment of inertia: $I = \sum m_i\,r_i^2 = \int r^2 dm$

$I_{disk} = \frac{1}{2}M\,R^2,\quad I_{ring} = \frac{1}{2}M\,(R_1^2 + R_2^2)$

$I_{rod} = \frac{1}{12}M\,\ell^2,\quad I_{rectangle} = \frac{1}{12}M\,(a^2 + b^2)$

$I_{sphere} = \frac{2}{5}M\,R^2,\quad I_{spherical\ shell} = \frac{2}{3}M\,R^2$

$I = M\,(\text{Radius of gyration})^2,\quad I = I_{cm} + M\,D^2$

Kinetic energies: $K_{rot} = \frac{1}{2}I\,\omega^2,\quad K = K_{rot} + K_{cm}$

Angular momentum: $L = r\,m\,v = r\,m\,\omega\,r = I\,\omega$

Torque: $\tau = \frac{dL}{dt} = m\,\frac{dv}{dt}\,r = F\,r = I\,\frac{d\omega}{dt} = I\,\alpha$

$W_{ext} = \Delta K + \Delta U + W_f,\quad K = K_{rot} + \frac{1}{2}m\,v^2,\quad P = \tau\,\omega$

Rolling, angular momentum and torque

Rolling: $K = \frac{1}{2}\left(I_c + M\,R^2\right)\omega^2 = \frac{1}{2}\left(\frac{I_c}{R^2} + M\right)v^2$

Angular momentum: $\vec{L} = \vec{r}\times\vec{p},\ L = r_\perp\,p = I\,\omega$

Torque: $\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r}\times\frac{d\vec{p}}{dt} = \vec{r}\times\vec{F},\quad \tau = r_\perp\,F = I\,\alpha$

Gyroscope: $\omega_p = \frac{d\phi}{dt} = \frac{1}{L}\frac{dL}{dt} = \frac{\tau}{L} = \frac{m\,g\,h}{I\,\omega}$

Static equilibrium

$\sum\vec{F}_i = 0$, about any point $\sum\vec{\tau}_i = 0$

Subdivisions: $\vec{r}_{cm} = \frac{m_A\,\vec{r}_{Acm} + m_B\,\vec{r}_{Bcm}}{m_A + m_B}$

Elastic modulus = stress/strain

stress: F/A

strain: $\Delta L/L,\quad \theta \approx \Delta x/h,\quad -\Delta V/V$