Orbifolds

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Abstract

In this lecture we review the notion of an orbifold, the quotient of a manifold by the action of a discrete symmetry group, as it is used in string theory and conformal field theory. We discuss general features of orbifold CFT's and construct simple examples such as $S^1/\mathbb{Z}_2$. We also consider orbifolds as realistic models for string theory compactification spaces. In particular, we look at how supersymmetry is broken in such compactifications

1 Introduction

Orbifolds arose in string theory out of a desire to find realistic models of compactification that were still exactly solvable. The simplest model of compactification, namely toroidal, is not very realistic since it leads to $N = 4$ supersymmetry when six of the ten flat dimensions are compactified. More realistic compactifications on other manifolds tend to be extremely complicated. Orbifolds provide a compactification scheme that is both exactly solvable and far more realistic. Basically the idea is to start with a toroidal theory and "twist" the boundary conditions in such a way that there is no net charge corresponding to the broken symmetries on the worldsheet. We shall see that modular invariance puts strong constraints on the manner in which this can be done. Because orbifolds are exactly solvable free CFT's one can make explicit calculations of tree and loop level amplitudes in string theory.

Geometrically, an orbifold is a quotient space obtained by identify points in a manifold under some discrete symmetry group. That is, let M be a manifold and G be a discrete group with an action $G \times M \to M$. The quotient space, or orbifold, M/G is defined as the space of equivalence classes under the relation

$$x \equiv gx \quad \text{for all } g \in G \tag{1}$$

We say that G acts *freely* if for all $x \in M$, $gx = x$ implies that $g = 1$. Free actions have no *fixed points*, points for which $gx = x$, for any $g \in G$ where $g \neq 1$. If G acts freely on M then the quotient space M/G will be a manifold. If G does not act freely (i.e. some elements in G have fixed points) then M will fail to be a manifold at precisely the fixed points of G. Such points are called *orbifold singularities*. It is these singularities which distinguish orbifolds from ordinary manifolds (We consider manifolds as special cases of orbifolds).

As a simple example consider the real line $\mathbb{R}$ under the action of the $\mathbb{Z}_2$ group generated by

$$g\colon x \mapsto -x \tag{2}$$

The only fixed point of g is the point $x = 0$. The orbifold, $\mathbb{R}/\mathbb{Z}_2$ is topologically the half-line $[0, \infty)$. This space if locally homeomorphic to $\mathbb{R}$ everywhere expect the origin.

2 The Compactified Boson Revisited

The simplest example of an orbifold CFT is one with which we are already familiar: the free boson compactified on a circle of radius R. Let $M = \mathbb{R}$ and let $G = \mathbb{Z}$ be the group of translations generated by

$$x \to x + 2\pi R \tag{3}$$

If we identify points in $\mathbb{R}$ under the action of $\mathbb{Z}$ we get the circle

$$S^1 = \mathbb{R}/\mathbb{Z} \tag{4}$$

Note that in this case $\mathbb{Z}$ is freely acting and the orbifold S^1 is a smooth manifold. This is just the toroidal compactification of the real line. Considering $x(z, \bar{z})$ as a free scalar field we can construct the associated compactified CFT. Consider the vertex operator

$$V(z, \bar{z}) = \exp(ik \cdot x(z, \bar{z})) \tag{5}$$

which creates states of momentum k out of the vacuum in the free scalar field theory. Since we are now identifying $x \equiv x + 2\pi R$ we must have,

$$V = \exp(ik \cdot (x + 2\pi R)) = \exp(ik \cdot x) \tag{6}$$

which is only satisfied when

$$k = \frac{m}{R} \qquad m \in \mathbb{Z} \tag{7}$$

So we see that the momentum must be quantized in units of $1/R$ in the compact theory. Because of the geometry of the compact boson, we may consider fields that "wind" around the circle n times

$$x\left(e^{2\pi i} z, e^{-2\pi i} \bar{z}\right) = x(z, \bar{z}) + 2\pi n R \qquad n \in \mathbb{Z} \tag{8}$$

The need for the winding sectors can be seen from the point of view of modular invariance. The partition function for a free boson

$$Z_{\text{bos}}(\tau) = \frac{1}{\sqrt{\operatorname{Im} \tau}\, |\eta(\tau)|^2} \tag{9}$$

will not be modular invariant unless we consider all possible momentum and winding modes. The true partition function for the compactified boson is

$$Z_{\text{circ}}(R) = \frac{1}{\eta\bar{\eta}} \sum_{m,n=-\infty}^{\infty} q^{\frac{1}{2}p_L^2} q^{\frac{1}{2}p_R^2} \tag{10}$$

where

$$p_{L,R} = \frac{m}{R} \pm \frac{1}{2}nR \tag{11}$$

The Hilbert space of the compactified theory is then built on the highest weight states $|m,n\rangle$ of momentum m/R and winding $nR/2$. These states are created from the vacuum by the vertex operators

$$V_{m,n}(z,\bar{z}) = {:}\exp(ip_L x_L(z) + ip_R x_R(\bar{z})){:} \tag{12}$$

The states $|m,n\rangle$ have conformal weights given by

$$L_0 \,|m,n\rangle = \frac{1}{2}p_L^2 \,|m,n\rangle \tag{13a}$$

$$\bar{L}_0 \,|m,n\rangle = \frac{1}{2}p_R^2 \,|m,n\rangle \tag{13b}$$

where

$$L_0 = \sum_{m>0} \alpha_{-m}\alpha_m + \frac{1}{2}p_L^2 \tag{14a}$$

$$\bar{L}_0 = \sum_{m>0} \bar{\alpha}_{-m}\bar{\alpha}_m + \frac{1}{2}p_R^2 \tag{14b}$$

Arbitrary states built on $|m,n\rangle$ are of the form

$$\prod_{i=1}^{N}\prod_{j=1}^{\tilde{N}} \alpha_{-n_i}\bar{\alpha}_{-n_j} \,|m,n\rangle \tag{15}$$

We see that the Hilbert space of the compactified theory is much different from that of the original theory. In the original theory, we could create states of arbitrary momentum. Here momentum is quantized and modular invariance forces us to consider states of nonzero winding. It turns out that these are general features of all orbifold CFT's.

3 Orbifold CFT's

An orbifold CFT is built analogously to the geometric object. One starts with a given modular invariant theory $\mathcal{T}$ that admits a discrete group of symmetries G consistent with the operator algebra and then forms the "quotient" theory $\mathcal{T}/G$. As we discuss below, this will involve the projection onto G-invariant states as well as the introduction of twisted sectors into the Hilbert space. Frequently in string theory, an orbifold CFT will have a geometric interpretation as the quotient of a manifold by a discrete group of symmetries, but this is not always the case.

For an orbifold CFT to be well-defined the states and fields of the theory must be invariant under the group action. That is, elements of G must act like the identity in the Hilbert space of the orbifold. In order to ensure this, one must project the states and fields of the original theory $\mathcal{T}$ onto states that are invariant under G.

To construct the partition function $Z_{\mathcal{T}/G}$ of the orbifold theory it is therefore necessary to include the projection operator

$$P = \frac{1}{|G|} \sum_{g \in G} g \tag{16}$$

into the trace over states. Here $|G|$ means the order of the group G. In the Lagrangian formulism the insertion of an element g into the trace corresponds to taking the fields to be *twisted* by g in the time direction of the torus

$$\phi(z + \tau) = g\phi(z) \tag{17}$$

With the insertion of the entire operator (16) into the trace, the partition function in the Lagrangian formulism should then be given by

$$Z = \frac{1}{|G|} \sum_{g \in G} g \underset{1}{\square} \tag{18}$$

where by the box notation

$$g \underset{h}{\square} \tag{19}$$

we mean the path integral on the torus with fields satisfying boundary conditions twisted by g in "time" and h in "space"

$$\phi(z + \tau) = g\phi(z) \tag{20a}$$

$$\phi(z + 1) = h\phi(z) \tag{20b}$$

for any $g, h \in G$. It is not hard to see, however, that the partition function (18) is not modular invariant. Under the modular transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \tag{21}$$

fields twisted by g and h become twisted by $g^a h^b$ and $g^c h^d$ respectively (see Figure 1).

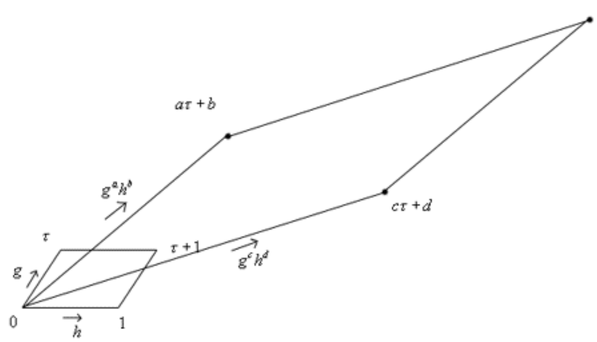

Figure 1: Spin structures on the torus under a modular transformation

That is, under the transformation (21)

$$g \underset{h}{\square} \to g^a h^b \underset{g^c h^d}{\square} \tag{22}$$

at least up to a overall phase.

To ensure modular invariance, it is then necessary to introduce sectors twisted in "space" by various elements of G.

$$\phi(z+1) = h\phi(z) \tag{23}$$

Such a field is not periodic in the original theory if ϕ and $h\phi$ are distinct, but is in the orbifold theory because of the identification. These additional sectors of the orbifold theory are called *twisted sectors*. They are necessary for modular invariance. Of course, in each twisted sector we must further project onto G-invariant states. The entire partition function of the orbifold theory is then given by

$$Z_{\mathcal{T}/G} = \frac{1}{|G|} \sum_{g,h\in G} g \underset{h}{\square} \tag{24}$$

Anomalous phases introduced by modular transformations can spoil the modular invariance of (24). We will return to this point later.

When G is nonabelian the path integral boundary conditions are inconsistent for $gh \neq hg$ and so the path integral vanishes. The sum in (24) can then be restricted to the case $gh = hg$. Note that if ϕ is in the sector twisted by g

$$\phi(z+1) = g\phi(z) \tag{25}$$

then under the action of h

$$h\phi(z+1) = hg\phi(z) = (hgh^{-1})h\phi(z) \tag{26}$$

and so $h\phi$ is in the sector twisted by hgh^{-1}. Thus, under the action of the group G twisted sectors in a given conjugacy class mix. For nonabelian orbifolds the independent twisted sectors are labeled not by the elements of G but by the conjugacy classes of G.

4 $S^1/\mathbb{Z}_2$ Orbifold

We will now consider another simple example which will illustrate in greater detail the procedure outlined above. We will take the compactified boson, topologically S^1, and identify under a $\mathbb{Z}_2$ group action generated by

$$g\colon x \to -x \tag{27}$$

There are two fixed points at $x = 0$ and $x = \pi R$. The orbifold $S^1/\mathbb{Z}_2$, depicted in Figure 2, is topologically the unit interval $[0,1]$ with the two fixed points corresponding to the endpoints of the interval.

The orbifold CFT will have both a untwisted sector and a sector twisted by g. States in the twisted sector are those for which

$$x(\sigma_1 + 2\pi) = -x(\sigma_1) \tag{28}$$

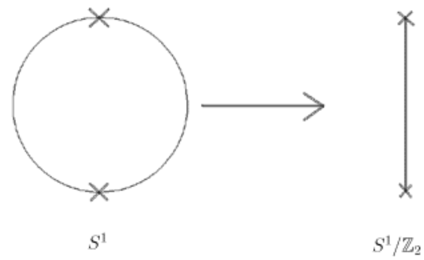

Figure 2: The orbifold $S^1/\mathbb{Z}_2$

and so can be thought of as antiperiodic bosons. However, in the orbifold theory the twisted boundary condition is actually a periodicity condition since the points x and $-x$ are identified.

In both the untwisted and twisted sectors we need to project onto g-invariant states. To do this we need to know how states transform under the action of g. Consider first the mode expansion of the U(1) current $j(z) = i\partial x(z)$

$$i\partial x(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{-n-1} \tag{29}$$

We see that under $x \to -x$ the mode operators transform as

$$\alpha_n \to -\alpha_n \tag{30a}$$

Similarly, from the mode expansion of $i\bar{\partial}x(\bar{z})$ we obtain

$$\bar{\alpha}_n \to -\bar{\alpha}_n \tag{30b}$$

To determine how $|m,n\rangle$ transforms under g we note that

$$|m,n\rangle = V_{m,n}\,|0,0\rangle \tag{31}$$

Equation (12) for the vertex operators shows that under $x \to -x$ both p_L and p_R go to minus themselves (this can also be seen by noting that $p_L = \alpha_0$ and $p_R = \bar{\alpha}_0$). From (11) then we see that under $x \to -x$

$$|m,n\rangle \to |-m,-n\rangle \tag{32}$$

We can explicitly verify the modular invariance of the $\mathbb{Z}_2$ orbifold by writing down the partition function. Since there are only two elements in the orbifold group we have

$$Z_{\mathrm{orb}}(R) = \frac{1}{2}\left(+\underset{+}{\square} + -\underset{+}{\square} + +\underset{-}{\square} + -\underset{-}{\square} \right) \tag{33}$$

where $(+)$ corresponds to the identity and $(-)$ to g. In the operator formulism the sum over boundary conditions in time corresponds to the insertion of a projection operator into the trace over states. In other words,

$$Z_{\mathrm{orb}}(R) = (q\bar{q})^{-1/24}\,\mathrm{tr}_{(+)}\,\frac{1}{2}(1+g)q^{L_0}\bar{q}^{\bar{L}_0} + (q\bar{q})^{-1/24}\,\mathrm{tr}_{(-)}\,\frac{1}{2}(1+g)q^{L_0}\bar{q}^{\bar{L}_0} \tag{34}$$

where $\mathrm{tr}_{(\pm)}$ is the trace in the untwisted and twisted sectors respectively. To compute these traces we must further decompose the Hilbert space into sectors with $g = \pm 1$ eigenvalues. We denote the untwisted sector by $H_{(+)}$ and the twisted sector by $H_{(-)}$. In the untwisted sector, from (30) and (32) we have

$$\begin{aligned} H^{+}_{(+)} &= \left\{\alpha_{-n_1}\cdots\alpha_{-n_\ell}\bar{\alpha}_{-n_{\ell+1}}\cdots\bar{\alpha}_{-n_{2k}}\left(|m,n\rangle + |-m,-n\rangle\right)\right\} \\ &\quad + \left\{\alpha_{-n_1}\cdots\alpha_{-n_\ell}\bar{\alpha}_{-n_{\ell+1}}\cdots\bar{\alpha}_{-n_{2k+1}}\left(|m,n\rangle - |-m,-n\rangle\right)\right\} \end{aligned} \tag{35a}$$

$$\begin{aligned} H^{-}_{(+)} &= \left\{\alpha_{-n_1}\cdots\alpha_{-n_\ell}\bar{\alpha}_{-n_{\ell+1}}\cdots\bar{\alpha}_{-n_{2k+1}}\left(|m,n\rangle + |-m,-n\rangle\right)\right\} \\ &\quad + \left\{\alpha_{-n_1}\cdots\alpha_{-n_\ell}\bar{\alpha}_{-n_{\ell+1}}\cdots\bar{\alpha}_{-n_{2k}}\left(|m,n\rangle - |-m,-n\rangle\right)\right\} \end{aligned} \tag{35b}$$

where $n_i \in \mathbb{Z}^+$.

In the twisted sector the antiperiodicity of x implies half-integer moding of the α's and $\bar{\alpha}$'s. Thus, to obtain states which are single-valued we introduce the twist operators $\sigma_{1,2}$. These twist operators have conformal dimension $\left(\frac{1}{16},\frac{1}{16}\right)$ and the following OPE's with the primary fields,

$$\partial x(z)\sigma_{1,2}(w,\bar{w}) \sim (z-w)^{-1/2}\tau_{1,2}(w,\bar{w}) \tag{36a}$$

$$\bar{\partial} x(\bar{z})\sigma_{1,2}(w,\bar{w}) \sim (\bar{z}-\bar{w})^{-1/2}\tilde{\tau}_{1,2}(w,\bar{w}) \tag{36b}$$

The $\tau_{1,2}$ and $\tilde{\tau}_{1,2}$ are excited twist operators with conformal dimensions

$$\tau_{1,2} \to \left(\frac{9}{16},\frac{1}{16}\right) \qquad \tilde{\tau}_{1,2} \to \left(\frac{1}{16},\frac{9}{16}\right) \tag{37}$$

The states identified with $\tau_{1,2}(0)\,|0\rangle$ and $\tilde{\tau}_{1,2}(0)\,|0\rangle$ can be written as $\alpha_{-1/2}\left|\frac{1}{16},\frac{1}{16}\right\rangle_{1,2}$ and $\bar{\alpha}_{-1/2}\left|\frac{1}{16},\frac{1}{16}\right\rangle_{1,2}$ respectively. The two twist operators correspond to the two ways in which the action $gx = -x$ can be realized. We can have $x \to -x$ or $x \to 2\pi - x$. Each realization is implemented by a different twist operator. In a general orbifold theory we will have twist fields localized near each of the fixed points of the orbifold. States in the twisted sector of an orbifold theory are built from the twisted ground states. These twisted ground states can be constructed by operating on the true vacuum with a twist operator as shown above. Thus, the twisted sector of the Hilbert space decomposes into the following subspaces

$$H^{+}_{(-)} = \left\{\alpha_{-n_1}\cdots\alpha_{-n_\ell}\bar{\alpha}_{-n_{\ell+1}}\cdots\bar{\alpha}_{-n_{2k}}\left|\tfrac{1}{16},\tfrac{1}{16}\right\rangle_{1,2}\right\} \tag{38a}$$

$$H^{-}_{(-)} = \left\{\alpha_{-n_1}\cdots\alpha_{-n_\ell}\bar{\alpha}_{-n_{\ell+1}}\cdots\bar{\alpha}_{-n_{2k+1}}\left|\tfrac{1}{16},\tfrac{1}{16}\right\rangle_{1,2}\right\} \tag{38b}$$

where now $n_i \in \left(\mathbb{Z}+\frac{1}{2}\right)^+$.

The partition function then can be written as the sum of the trace of states in $g = 1$ eigensubspaces of both the twisted and untwisted sectors:

$$Z_{\mathrm{orb}}(R) = (q\bar{q})^{-1/24}\,\mathrm{tr}_{H^{+}_{(+)}}\, q^{L_0}\bar{q}^{\bar{L}_0} + (q\bar{q})^{-1/24}\,\mathrm{tr}_{H^{+}_{(-)}}\, q^{L_0}\bar{q}^{\bar{L}_0} \tag{39}$$

We begin by computing the trace first in the untwisted sector:

$$\begin{aligned} (q\bar{q})^{-1/24}\,\mathrm{tr}_{H^{+}_{(+)}}\, q^{L_0}\bar{q}^{\bar{L}_0} &= (q\bar{q})^{-1/24}\,\mathrm{tr}_{(+)}\,\frac{1}{2}(1+g)q^{L_0}\bar{q}^{\bar{L}_0} \\ &= (q\bar{q})^{-1/24}\left[\mathrm{tr}_{(+)}\,\frac{1}{2}q^{L_0}\bar{q}^{\bar{L}_0} + \mathrm{tr}_{(+)}\,\frac{1}{2}gq^{L_0}\bar{q}^{\bar{L}_0}\right] \end{aligned} \tag{40}$$

Now, since the trace of a direct product of operators satisfies

$$\operatorname{tr} \bigotimes_i M_i = \prod_i \operatorname{tr} M_i \tag{41}$$

we can decouple the left and right moving modes. Thus we consider $\operatorname{tr}_{(+)} q^{L_0}$ and $\operatorname{tr}_{(+)} g q^{L_0}$ separately. The treatment for the antiholomorphic component is identical with the replacement $q \to \bar{q}$. Now

$$\begin{aligned} \operatorname{tr}_{(+)} q^{L_0} &= \operatorname{tr}_{(+)} q^{\sum_k \alpha_{-k}\alpha_k + \frac{1}{2}p_L^2} \\ &= \left(\prod_{k=1}^{\infty} \operatorname{tr}_{(+)} q^{\alpha_{-k}\alpha_k} \right) \operatorname{tr}_{(+)} q^{\frac{1}{2}p_L^2} \end{aligned} \tag{42}$$

where again we have used the fact that the traces factor for a direct product of matrices acting on different spaces. The first term in the factor we will evaluate is

$$\prod_{k=1}^{\infty} \operatorname{tr}_{(+)} q^{\alpha_{-k}\alpha_k} \tag{43}$$

Due to the commutation relations between mode operators,

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0} \tag{44}$$

only states of the form $(\alpha_{-k})^i |m, n\rangle$ contribute for each value of k in the infinite product. We then have

$$\prod_{n=1}^{\infty} \operatorname{tr}_{(+)} q^{\alpha_{-n}\alpha_n} = \prod_{n=1}^{\infty} \left(1 + q^n + q^{2n} + \cdots\right) = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \tag{45}$$

Comparison with the Dedekind η-function shows that

$$\prod_{n=1}^{\infty} \operatorname{tr}_{(+)} q^{\alpha_{-n}\alpha_n} = \frac{q^{1/24}}{\eta(q)} \tag{46}$$

Summing over all values of momentum and winding and then multiplying by the corresponding right-moving factors leads to the following

$$(q\bar{q})^{-1/24} \operatorname{tr}_{(+)} \frac{1}{2} q^{L_0} \bar{q}^{\bar{L}_0} = \frac{1}{2} \frac{1}{\eta\bar{\eta}} \sum_{m,n=-\infty}^{\infty} q^{\frac{1}{2}p_L^2(m,n)} \bar{q}^{\frac{1}{2}p_R^2(m,n)} \tag{47}$$

where as before

$$p_{L,R}^2(m,n) = \left(\frac{m}{R} \pm \frac{nR}{2}\right)^2 \tag{48}$$

Now we need to compute the trace with g inserted:

$$\operatorname{tr}_{(+)} \frac{1}{2} g q^{L_0} \bar{q}^{\bar{L}_0}$$

We follow a similar treatment here as in the trace without g inserted by considering the holomorphic piece

$$\operatorname{tr}_{(+)} g q^{L_0}$$

From the decomposition of $H_{(+)}$ into sectors with eigenvalues $g = \pm 1$ we see that at each level there are states built on

$$|m, n\rangle + |-m, -n\rangle$$

and

$$|m, n\rangle - |-m, -n\rangle$$

with the same L_0 eigenvalues but opposite eigenvalues under g. Thus they cancel and so only states built on the vacuum $|0, 0\rangle$ contribute. The sum over m, n collapses and since

$$p_{L,R}^2 \, |0, 0\rangle = 0 \tag{49}$$

we have

$$\mathrm{tr}_{(+)} \, g q^{\sum_n \alpha_{-n} \alpha_n} = \mathrm{tr}_{(+)} \, g \prod_{n=1}^{\infty} q^{\alpha_{-n} \alpha_n} = \prod_{n=1}^{\infty} \mathrm{tr}_{(+)} \, g q^{\alpha_{-n} \alpha_n} \tag{50}$$

Since

$$g \colon \alpha_{-n} \to -\alpha_{-n} \tag{51}$$

we have a factor of -1 for each contribution with states built with an odd number of mode operators and $+1$ for states built on an even number. Thus

$$\mathrm{tr}_{(+)} \, g q^{L_0} = \prod_{n=1}^{\infty} \left(1 - q^n + q^{2n} - q^{3n} + \cdots\right) = \prod_{n=1}^{\infty} \frac{1}{1 + q^n} \tag{52}$$

giving

$$(q\bar{q})^{-1/24} \, \mathrm{tr}_{(+)} \, \frac{1}{2} g q^{L_0} \bar{q}^{\bar{L}_0} = \frac{1}{2} \frac{(q\bar{q})^{-1/24}}{\prod_{n=1}^{\infty}(1 + q^n)(1 + \bar{q}^n)} = \left| \frac{\eta}{\vartheta_2} \right| \tag{53}$$

Finally, the total trace over states in the untwisted sector is

$$\begin{aligned}
(q\bar{q})^{-1/24} \, \mathrm{tr}_{H_{(+)}^+} \, q^{L_0} \bar{q}^{\bar{L}_0} &= (q\bar{q})^{-1/24} \, \mathrm{tr}_{(+)} \, \frac{1}{2}(1 + g) q^{L_0} \bar{q}^{\bar{L}_0} \\
&= \frac{1}{2} \frac{1}{\eta\bar{\eta}} \sum_{m,n=-\infty}^{\infty} q^{\frac{1}{2} p_L^2} \bar{q}^{\frac{1}{2} p_R^2} + \left| \frac{\eta}{\vartheta_2} \right| \\
&= \frac{1}{2} Z_{\mathrm{circ}}(R) + \left| \frac{\eta}{\vartheta_2} \right| \qquad (54)
\end{aligned}$$

where $Z_{\mathrm{circ}}(R)$ is simply the partition function for the compactified boson.

Now we need to calculate the trace over states in the twisted sector of the theory:

$$(q\bar{q})^{-1/24} \, \mathrm{tr}_{H_{(-)}^+} \, q^{L_0} \bar{q}^{\bar{L}_0} = (q\bar{q})^{-1/24} \, \mathrm{tr}_{(-)} \, \frac{1}{2}(1 + g) q^{L_0} \bar{q}^{\bar{L}_0} \tag{55}$$

We proceed as before, calculating separately $\mathrm{tr}_{(-)} \, q^{L_0}$ and $\mathrm{tr}_{(-)} \, g q^{L_0}$. We have

$$\begin{aligned}
\mathrm{tr}_{(-)} \, q^{L_0} &= \mathrm{tr}_{(-)} \, q^{\sum_{n>0} \alpha_{-n} \alpha_n + \frac{1}{48}} \\
&= q^{1/48} \prod_{n>0} \mathrm{tr}_{(-)} q^{\alpha_{-n} \alpha_n} \qquad (56)
\end{aligned}$$

where $n \in \left(\mathbb{Z}+\frac{1}{2}\right)^{+}$. Calculating the trace inside the product we find

$$\begin{aligned}
\operatorname{tr}_{(-)} q^{L_0} &= q^{1/48} \prod_{n\in\left(\mathbb{Z}+\frac{1}{2}\right)^{+}} \left(1 + q^{(1/2)n} + q^{(3/2)n} + \cdots\right) \\
&= q^{1/48} \prod_{n\in\left(\mathbb{Z}+\frac{1}{2}\right)^{+}} \frac{1}{(1-q^n)} = q^{1/48} \prod_{n=1}^{\infty} \frac{1}{(1-q^{n-1/2})}
\end{aligned} \tag{57}$$

This is the contribution from states built on $\left|\frac{1}{16},\frac{1}{16}\right\rangle_1$. There is an identical contribution from states built on $\left|\frac{1}{16},\frac{1}{16}\right\rangle_2$. Thus

$$\begin{aligned}
(q\bar{q})^{-1/24} \operatorname{tr}_{(-)} \frac{1}{2} q^{L_0} \bar{q}^{\bar{L}_0} &= 2\cdot\frac{1}{2}(q\bar{q})^{-1/48} \prod_{n=1}^{\infty} \frac{1}{(1-q^{n-1/2})(1-\bar{q}^{n-1/2})} \\
&= \left|\frac{\eta}{\vartheta_4}\right|
\end{aligned} \tag{58}$$

Performing a similar calculation for the trace with g inserted we obtain

$$\begin{aligned}
(q\bar{q})^{-1/24} \operatorname{tr}_{(-)} \frac{1}{2} g q^{L_0} \bar{q}^{\bar{L}_0} &= 2\cdot\frac{1}{2}(q\bar{q})^{-1/48} \prod_{n=1}^{\infty} \frac{1}{(1+q^{n-1/2})(1+\bar{q}^{n-1/2})} \\
&= \left|\frac{\eta}{\vartheta_3}\right|
\end{aligned} \tag{59}$$

where again the factor of two comes from the fact that we get two identical contributions to the trace from the two vacua $\left|\frac{1}{16},\frac{1}{16}\right\rangle_{1,2}$. The total trace over states in the twisted sector is then

$$\begin{aligned}
(q\bar{q})^{-1/24} \operatorname{tr}_{H^{+}_{(-)}} q^{L_0} \bar{q}^{\bar{L}_0} &= (q\bar{q})^{-1/24} \operatorname{tr}_{(-)} \frac{1}{2}(1+g) q^{L_0} \bar{q}^{\bar{L}_0} \\
&= \left|\frac{\eta}{\vartheta_4}\right| + \left|\frac{\eta}{\vartheta_3}\right|
\end{aligned} \tag{60}$$

Combining (54) and (60) we can write the entire partition function for the orbifold theory as

$$Z_{\text{orb}}(R) = \frac{1}{2} Z_{\text{circ}}(R) + \left|\frac{\eta}{\vartheta_2}\right| + \left|\frac{\eta}{\vartheta_4}\right| + \left|\frac{\eta}{\vartheta_3}\right| \tag{61}$$

Using the ϑ-function identity

$$\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3 \tag{62}$$

we can rewrite this as

$$Z_{\text{orb}}(R) = \frac{1}{2}\left(Z_{\text{circ}}(R) + \frac{|\vartheta_3\vartheta_4|}{\eta\bar{\eta}} + \frac{|\vartheta_2\vartheta_3|}{\eta\bar{\eta}} + \frac{|\vartheta_2\vartheta_4|}{\eta\bar{\eta}}\right) \tag{63}$$

It is straightforward to verify that this partition function is modular invariant using the modular transformation properties of the ϑ and η functions.

5 Orbifolding the String

We now consider the construction of a string theory by orbifolding ten dimensional flat spacetime. That is, we will consider the propagation of strings on a quotient of $\mathbb{R}^{9,1}$ by a discrete group of symmetries G. In general, G will not act freely and so the orbifold will contain singularities corresponding to the fixed points of G. One might think that string theory on such a space would not be well-defined. However it turns out that in most cases the correlation functions of the associated CFT are all finite and so strings actually propagate smoothly on these spaces.

5.1 Orbifold Groups

The most natural way in which to construct a spacetime orbifold is to choose for the symmetry group some discrete subgroup S of the full Poincaré group in $9+1$ dimensions. An element g of the Poincaré group

$$g = (A, a) \in \mathrm{O}(9,1) \ltimes \mathbb{R}^{9,1} \tag{64}$$

acts on $\mathbb{R}^{9,1}$ by first rotating by A then translating by a

$$gx = Ax + a \tag{65}$$

Multiplication and inversion in the Poincaré group are given by

$$(A, a)(A', a') = (AA', Aa' + a) \tag{66a}$$
$$(A, a)^{-1} = (A^{-1}, -A^{-1}a) \tag{66b}$$

It is easy to check that the subgroup of pure translations forms a normal subgroup:

$$(A', a')(1, a)(A', a')^{-1} = (1, A'a) \tag{67}$$

If we want a theory with d extended spacetime dimensions the orbifold group S should lie in the $\mathrm{O}(n) \ltimes \mathbb{R}^n$ subgroup of the full Poincaré group that acts trivially on the first $d = 10 - n$ coordinates. This gives

$$\mathbb{R}^{9,1}/S = \mathbb{R}^{d-1,1} \times K \tag{68}$$

where K is the compact orbifold

$$K = \mathbb{R}^n/S \tag{69}$$

formed by the action of S on the remaining n coordinates. Because S is discrete we can write it as

$$S = P \ltimes \Lambda \tag{70}$$

where P is a discrete subgroup of $\mathrm{O}(n)$ and Λ and discrete subgroup of $\mathbb{R}^n$. If K is to be compact Λ should be a n-dimensional lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. Note that P cannot be an arbitrary discrete subgroup of $\mathrm{O}(n)$, as it must act *crystallographically* on Λ. That is

$$Aa \in \Lambda \tag{71}$$

for all $A \in P$ and all $a \in \Lambda$. This is simply a statement that multiplication in S be closed. Borrowing terminology from crystallography we call S the *space group* and P the *point group*.

If the point group P is trivial then $S = \Lambda$ acts freely and the compact space K is simply an n-torus:

$$K = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n \tag{72}$$

Toroidal compactification is then included as a special case of an orbifold theory. Whatever the group P, the group of translations Λ is a normal subgroup of S and the quotient group

$$\bar{P} \equiv S/\Lambda \tag{73}$$

is isomorphic to P. Note that the group $\bar{P}$ has a well-defined action on the torus $\mathbb{T}^n$. That is, if two elements of S differ only by a lattice shift (so that they are equivalent in $\bar{P}$) then their action on two elements identified in $\mathbb{T}^n$ again differs only by a lattice shift and so leads to the same point in $\mathbb{T}^n$. This observation leads to two different ways of describing the orbifold K. One can either start with $\mathbb{R}^n$ and quotient by the full space group S or first form the torus $\mathbb{T}^n$ and then quotient by the group $\bar{P}$

$$K = \mathbb{R}^n/S \cong \frac{\mathbb{R}^n/\Lambda}{S/\Lambda} \cong \mathbb{T}^n/\bar{P} \tag{74}$$

The latter method is especially advantageous if the point group P is abelian. In that case one need only use the methods of abelian orbifolds (the translation group Λ is always abelian). If instead one wanted to quotient by the entire space group in one step, one would have to consider nonabelian orbifolds as the group $P \ltimes \Lambda$ is generally nonabelian even for abelian P and Λ. Thus we call the orbifold K abelian or nonabelian according to whether the point group P is abelian or nonabelian.

So far we have describe the action of the group S on the bosonic coordinates x^μ. For the superstring we must also describe the action of the group on the fermionic coordinates ψ^μ. In order to preserve worldsheet supersymmetry the twist must commute with the supercurrent

$$G = -\psi \cdot \partial x \tag{75}$$

This implies that for $g = (A, a) \in S$ must have

$$(g\psi)^\mu = A^\mu{}_\nu \psi^\nu \tag{76}$$

and likewise for the right movers. Note that the translations do not affect the spacetime fermions.

The space groups and point groups up to $n = 4$ have been completely classified. The 32 point groups and 230 space groups of the $n = 3$ case have been long studied in connection with crystallography. The case of real interest in string theory, however, is $n = 6$, for which the space groups have not been classified. We shall see later on, though, that if we are to preserve some supersymmetry and avoid tachyons in the theory we should restrict our choice of point group to those lying in a SU(3) subgroup of the full O(6).

5.2 The Hilbert Space

To construct the Hilbert space of the string orbifold theory we must project onto G-invariant states and introduce sectors twisted by various elements of G:

$$x(\sigma_1 + 2\pi) = gx(\sigma_1) \tag{77}$$

From the point of view of string theory the twisted sectors are necessary not only to preserve modular invariance but also because twisted strings can be produced in interactions of untwisted strings.

As discussed above we can construct the orbifold theory by first performing a toroidal compactification and then dividing by the action of the point group $\bar{P}$. The toroidal compactification is obtained by starting with $\mathbb{R}^n$ and identifying points under the discrete translation group $\Lambda = \mathbb{Z}^n$. The projection onto $\mathbb{Z}^n$-invariant states requires that the eigenvalues of the momentum operator take values in the dual lattice:

$$k \in \Lambda^* \tag{78}$$

Only for those values of k will the vertex operator $e^{ik\cdot x}$ be well-defined. The twisted sectors of this theory are just the usual winding sectors in which the string wraps around the various compact dimensions of the torus.

One then proceeds to construct the full orbifold theory by further projecting onto $\bar{P}$-invariant states and introducing sectors twisted by elements of $\bar{P}$. Since P is a discrete subgroup of $\mathrm{O}(n)$ each element $A \in P$ will have finite order N

$$A^N = 1 \tag{79}$$

Hence the eigenvalues of A will be Nth roots of unity. In the sector twisted by A the oscillators will not be moded as in the untwisted sector but rather will have their mode numbers shifted by some

$$\phi_i = \frac{r_i}{N} \qquad 0 \le r_i < N \tag{80}$$

Let us take the case of $n = 6$ for concreteness. For an arbitrary $A \in P$ we can always choose the axes so that A is of the form

$$A = \exp\left[2\pi i(\phi_2 J_{45} + \phi_3 J_{67} + \phi_4 J_{89})\right] \tag{81}$$

In considering the sector twisted by A it is convenient to introduce complex *spacetime* coordinates. We are already familiar with the notion of complex (Weyl) fermions. Complex bosons are defined in an analogous manner:

$$Z^i = \frac{1}{\sqrt{2}}(x^{2i} + ix^{2i+1}) \tag{82}$$

where i runs over $1, \ldots, 4$ covering the transverse modes of the string. We take the same basis for the complex fermionic coordinates. In the sector twisted by A the periodicity condition (77) takes the form

$$Z^i(\sigma + 2\pi) = e^{2\pi i\phi_i} Z^i(\sigma) \qquad i = 2, 3, 4 \tag{83}$$

In this sector the oscillators are moded as

$$\alpha^i_n : n \in \mathbb{Z} + \phi_i \qquad \bar{\alpha}^i_n : n \in \mathbb{Z} - \phi_i \tag{84a}$$

$$\psi^i_r : r \in \mathbb{Z} + \phi_i \qquad \bar{\psi}^i_r : r \in \mathbb{Z} - \phi_i \qquad \text{(R)} \tag{84b}$$

$$r \in \mathbb{Z} + \phi_i + \tfrac{1}{2} \qquad r \in \mathbb{Z} - \phi_i + \tfrac{1}{2} \qquad \text{(NS)} \tag{84c}$$

The quanta of the conjugate fields $(Z^i)^\dagger$ and $(\psi^i)^\dagger$ have the opposite mode numbers ($-\phi_i$ in place of ϕ_i and so on).

In order for a string in a sector twisted by A to have a massless state it must be capable of shrinking to zero size. The twisting condition (77) shows that this will only be possible if the string is localized around one of the fixed points of A. In other words the center-of-mass coordinate of the string can only lie at one of the fixed points. In each twisted sector there will be further subsectors built upon the separate fixed points. As these subsectors are related by obvious symmetries of the underlying torus they will have isomorphic spectrums. Twisted sectors that have no fixed points cannot have any massless states. For example, in toroidal compactification the nonzero winding sectors are free of fixed points and so there are no massless states with nonzero winding (at least for generic values of the torus radii).

We have seen that modular invariance in an orbifold theory requires the introduction of the twisted sectors as well as projection onto G-invariant states. The resulting sum over path integral sectors

$$Z_{\text{orb}} = \frac{1}{|G|} \sum_{g,h \in G} g \underset{h}{\square} \tag{85}$$

is naively modular invariant. However, this modular invariance can be spoiled by anomalous phases in the path integral under $\tau \to \tau + 1$. Such phases are determined by the level mismatch $L_0 - \bar{L}_0$ (mod 1). Only for left-right symmetric theories do the phases automatically cancel. For asymmetric theories, in particular for the heterotic string, one can derive a set of conditions to ensure the level matching of states in the various twisted sectors of the theory. As we will only be considering symmetric theories we will not derive these conditions here, but instead refer the interested reader to [2] or [7]. We should also mention that for abelian orbifolds it has been shown that level mismatch is the only potential obstruction to modular invariance.

5.3 Supersymmetry

To see how an orbifold breaks supersymmetry we need to look at the action of the orbifold group on the supercharges Q_α. We first consider the case where the point group is cyclic, $P = \mathbb{Z}_N$. Let A be a generator of P. We can choose the axes so that A is of the form (81). The point of taking the group P to be cyclic is so that any states invariant under the generator A will be invariant under the entire group. Under the rotation A the supercharges transform as

$$Q_\alpha \to D(A)_{\alpha\beta} Q_\beta \tag{86}$$

where $D(A)$ is the spinor representation of A. This can be written in the $\mathbf{s}$-basis as

$$Q_{\mathbf{s}} \to \exp(2\pi i\, \mathbf{s} \cdot \phi) Q_{\mathbf{s}} \tag{87}$$

where $\phi = (0, \phi_2, \phi_3, \phi_4)$ and the components of $\mathbf{s} = (s_1, s_2, s_3, s_4)$ run over all combinations of $\pm\frac{1}{2}$. Only those charges for which $\mathbf{s} \cdot \phi = 0$ will survive the projection. This condition will be satisfied only if

$$\pm\phi_2 \pm \phi_3 \pm \phi_4 = 0 \tag{88}$$

for some choice of signs. We can assume, without loss of generality, that the signs are all positive

$$\phi_2 + \phi_3 + \phi_4 = 0 \tag{89}$$

as other choices give equivalent physics. When (89) is satisfied there will be four unbroken supersymmetries corresponding to $s_1 = \pm\frac{1}{2}$ and $s_2 = s_3 = s_4 = \pm\frac{1}{2}$. This gives an unbroken $N = 1$ supersymmetry in $d = 4$. The stricter condition

$$\phi_2 + \phi_3 = \phi_4 = 0 \tag{90}$$

gives an unbroken $N = 2$ supersymmetry. Taking all ϕ_i's to be 0 is equivalent to toroidal compactification with unbroken $N = 4$ supersymmetry.

We note that because A takes the complex coordinates Z^i into linear combinations of themselves, the point group P lies in a U(3) subgroup of SO(6). Furthermore, condition (89) is equivalent to the condition that the generator of A (as an element of U(3)) be traceless, or in other words that A actually lie in an SU(3) subgroup. This suggests that the condition for an arbitrary point group P to preserve an $N = 1$ supersymmetry is that in lie in an SU(3) subgroup of SO(6). Indeed, under

$$\mathrm{SO}(9,1) \to \mathrm{SO}(3,1) \times \mathrm{SO}(6) \tag{91}$$

the spinor **16** of SO(9, 1) decomposes as

$$\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}) \tag{92}$$

Decomposing further

$$\mathrm{SO}(6) \to \mathrm{SU}(3) \tag{93}$$

we get

$$\mathbf{16} \to (\mathbf{2}, \mathbf{3}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{2}}, \mathbf{1}) \tag{94}$$

Hence, with $P \subseteq \mathrm{SU}(3)$ the generators $(\mathbf{2}, \mathbf{1})$ and $(\bar{\mathbf{2}}, \mathbf{1})$ survive the orbifold projection and the four dimensional theory will have an unbroken $N = 1$ supersymmetry. Likewise, condition (90) is equivalent to having P lie in an SU(2) subgroup and leaving an unbroken $N = 2$ supersymmetry.

The orbifold groups of real interest in string theory then are those with point groups in SU(3). The discrete subgroups of SU(3), unlike those of O(6), have in fact been completely classified. Restricting to those discrete subgroups which act crystallographically gives a finite number of possibilities. If one is not interested in preserving any supersymmetry one need not choose one of these point groups. However, almost all choices of groups not in SU(3) will give rise to tachyons in some twisted sector. There are a few exceptions, as noted in [2].

5.4 The Heterotic String

In the heterotic string we may choose to have orbifold group act of the gauge degrees of freedom as well. Unlike for the spacetime fermions, it is possible to have the translations Λ act on the gauge fermions. Choosing an action for Λ on the gauge degrees of freedom means associating a gauge transformation with a translation around a closed loop in $\mathbb{T}^n$. In other words, choosing a set of Wilson lines for the n generators of the fundamental group of the torus.

For the point group P the simplest thing to do is to simply embed P in an O(6) subgroup of a single E_8 of the full $\mathrm{E}_8 \times \mathrm{E}_8$ gauge group. This is referred to as *embedding the spin connection in the gauge connection.* For example, if we represent E_8 as 16 Majorana-Weyl fermions transforming as the **16** of O(16), then we can just take an O(6) subgroup of O(16) under which the **16** transforms as a **6** plus singlets. Or in the case that P actually lies in SU(3) subgroup of O(6) we represent E_8 as 8 Weyl fermions transforming as the **8** of U(8) and take an SU(3) subgroup under which the **8** transforms as a **3** plus singlets. One advantage of this standard choice is that the level matching conditions are automatically satisfied.

If the point group P lies in an SU(3) subgroup of O(6) it is useful to know how the adjoint representation of E_8 decomposes. Under

$$\mathrm{E}_8 \to \mathrm{SU}(3) \times \mathrm{E}_6 \tag{95}$$

we have

$$\mathbf{248} \to (\mathbf{8}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{78}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\bar{\mathbf{3}}, \overline{\mathbf{27}}) \tag{96}$$

This information is useful for construction the massless spectrum of the orbifold theory. Choices of than the standard one are possible, but there are associated complications. These are discussed in both [2] and [7].

5.5 Example: $\mathbb{T}^6/\mathbb{Z}_3$

We construct here a simple example of an $n = 6$ orbifold on which to compactify the string. Using the complex coordinates defined in (82) we define a $\mathbb{Z}_2$ lattice in each of the three complex planes via the generators

$$t_i \colon Z^i \to Z^i + R_i \tag{97a}$$

$$u_i \colon Z^i \to Z^i + e^{2\pi i/3} R_i \tag{97b}$$

Dividing by this lattice gives the 6-torus $\mathbb{T}^6 = \mathbb{C}^3/\mathbb{Z}^6 = (\mathbb{C}/\mathbb{Z}^2)^3$. One $\mathbb{T}^2$ component is shown in Figure 3.

We then take the orbifold point group to be the $\mathbb{Z}_3$ group generated by

$$\alpha \colon Z^i \to e^{2\pi i \phi_i} Z^i \tag{98}$$

where

$$\phi_2 = \phi_3 = \tfrac{1}{3} \quad \phi_4 = -\tfrac{2}{3} \tag{99}$$

It is not hard to verify that the rotation (98) is a symmetry of the tori defined in (97). This is most easily seen by tiling $\mathbb{C}$ with equilateral triangles of side length R_i or by observing that

$$e^{\pi i/3} = e^{2\pi i/3} + 1 \tag{100}$$

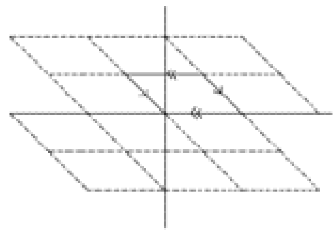

Figure 3: The torus $\mathbb{T}^2$

The action of α on a single $\mathbb{T}^2$ is shown in Figure 4. The fundamental region of this action can be taken to be the region shaded in grey. It is easy to see that for a given i there are three fixed points of the action:

$$Z^i = \frac{n_i}{\sqrt{3}} e^{i\pi/6} R_i \qquad n_i = 0, 1, 2 \tag{101}$$

By sewing together the edges of the fundamental region of $\mathbb{T}^2/\mathbb{Z}_3$ one sees that this orbifold is topologically a 2-sphere. It fails to be a manifold at the three conical singularities, around each of which the holonomy group is $\mathbb{Z}_3$. The action of α on the entire $\mathbb{T}^6$ will have $3^3 = 27$ fixed points corresponding to each Z^i satisfying (101). The orbifold $\mathbb{T}^6/\mathbb{Z}_3$ will have two twisted sectors (those twisted by α and α^{-1}), and in each twisted sector there will be 27 subsectors corresponding to expansions about the various fixed points. These subsectors have isomorphic spectra.

Note that the ϕ_i in (99) are chosen so as to satisfy (89). So the $\mathbb{Z}_3$ point group lies in an SU(3) rotation subgroup and leaves an unbroken $N = 1$ supersymmetry.

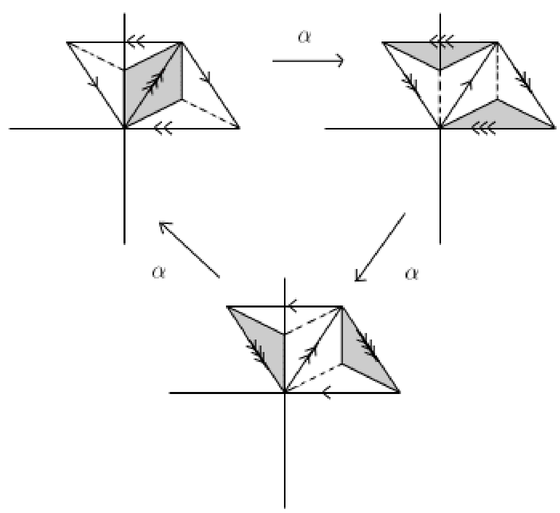

Figure 4: The action of α on $\mathbb{T}^2$

6 Summary

We have seen how to construct an orbifold theory from a free CFT via the action of a discrete group of symmetries. The basic procedure is to introduce sectors with twisted boundary conditions and then to project onto G-invariant states in all sectors. Orbifolds enlarge the set of exactly solvable free CFT's. We have also seen how we might use orbifolds as models for realistic compactifications of the superstring. Specifically, we have seen an example of compactification on a orbifold that preserves an $N = 1$ supersymmetry in four spacetime dimensions.

Besides the applications we have seen, orbifolds crop up in other places of string theory as well. The moduli space of certain smooth compactifications have special points that are described by orbifold CFT's. One can even think of the RNS formulation of the superstring as an orbifold theory: One starts by taking, in addition to the normal bosonic coordinates, fermionic coordinates ψ^μ whose transverse components transform as an $\mathbf{8_v}$ of SO(8). In the NS sector the ψ^μ obey antiperiodic boundary conditions. The spectrum contains a tachyon, massless vector and so on. One then considers the $\mathbb{Z}_2$ group generated by the operator $(-1)^F$ which anticommutes with the fermionic coordinates and commutes with the bosonic ones. The projection onto $(-1)^F$ invariant states, known as the GSO projection, removes the tachyon and other half-integer excitations and ensures a supersymmetric spectrum. Furthermore, one is forced to introduce the twisted, or Ramond, sector in which fermions are periodic.

References

[1] L. Dixon, J. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds*, Nucl. Phys. **B261** (1985), 678–686.

[2] L. Dixon, J. Harvey, C. Vafa, and E. Witten, *Strings on orbifolds (II)*, Nucl. Phys. **B274** (1986), 285–314.

[3] S. Hamidi and C. Vafa, *Interactions on orbifolds*, Nucl. Phys. **B279** (1987), 465–513.

[4] L. Dixon, D. Friedan, E. Martinec, and S. Shenker, *The conformal field theory of orbifolds*, Nucl. Phys. **B282** (1987), 13–73.

[5] P. Ginsparg, *Curiosities at $c = 1$*, Nucl. Phys. **B295** (1988), 153–170.

[6] P. Ginsparg, *Applied conformal field theory*, Les Houches, Session XLIX, Strings, Fields and Critical Phenomena (E. Brézin and J. Zinn-Justin, eds.), Elsevier Science Publishers, 1989.

[7] J. Polchinski, *String theory*, Cambridge University Press, Cambridge, 1998, §8.5 and Ch. 16.

[8] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal field theory*, Springer-Verlag, New York, 1997.