String Theory

PHY 396P/Q

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Lecture notes from Jacques Distler's string theory course taken by Jeff Olson during the 2001/2002 academic year. *These notes are very incomplete and probably widely inaccurate. Use at your own risk.*

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1 Introduction

String theory began as a theory of strong interactions prior to QCD. In 1968 Veneziano postulated a beta function form for the scattering amplitude of the strong interaction. Generalizations of this formula were developed and went by the name of *dual resonance models*. Later these models were recognized as describing the dynamics of a relativistic string. Two different kinds of strings were studied: open strings, which had two freely moving ends, and closed strings, which formed closed loops.

open string

lowest mode	pion	$m^2 < 0$
first excited mode	vector boson (spin-1)	$m = 0$

closed string

lowest mode	pomeron	$m^2 < 0$
first excited mode	spin-2	$m = 0$

If strings were to describe the strong interaction their tension had to be of the order of the energy scale squared.

$$T = \text{ Tension} \sim (1\text{ GeV})^2$$

It was found that these dual resonance models worked correctly only in 26 or 10 spacetime dimensions. This and other problems led researches to all but abandon the theory when QCD came along.

In 1974 and 1975, however, some noted that the presence of a massless spin-2 mode of the closed string together with the extra dimensions suggested the theory could make sense as a unified theory of quantum gravity. In this context the string tension would be on the order of the Planck energy squared.

$$T \sim (19\text{ GeV})^2$$

2 The Free Particle

Before studying string theory itself we will review the "simplest" of all possible physical systems: the free particle. The action associated to the worldline of a massive particle in D dimensional Minkowski spacetime can be chosen to be proportional to its invariant length:

$$S = m \int d\tau \sqrt{-\dot{x}_\mu \dot{x}^\mu} \tag{1}$$

where

$$\dot{x}^\mu = \frac{dx^\mu}{d\tau}$$

and τ is some parameterization of the worldline. Note that we are using the metric signature $(-,+,+,\cdots,+)$.

The problem with this action is that it is nonlinear and hard to quantize.

$$p_\mu = \frac{\delta \mathcal{L}}{\delta \dot{x}^\mu} = \frac{-m\dot{x}_\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}}$$

The difficulties with this are the facts that we are

- unable to solve for the velocities, $\dot{x}^\mu$
- restricted by the mass shell constraint

$$p_\mu p^\mu = -m^2$$

The origins of these difficulties is a gauge-invariance in the action

$$\tau \mapsto f(\tau)$$

In other words, the length of a worldline is independent of the parameterization. The classical equation of motion

$$\dot{p}_\mu = 0 \qquad p_\mu = \text{const}$$

has the solution

$$x_\mu(\tau) = x_\mu(0) + \frac{p_\mu}{m} f(\tau)$$

for arbitrary $f(\tau)$.

There are two primary ways of coping with these difficulties: the covariant method and the non-covariant method.

2.1 The Covariant Method

The basic idea here is to

1. Ignore the constraint and use standard Poisson brackets

$$\{p_\nu, x^\mu\} = -\delta_\nu{}^\mu$$

$$H_0 = p_\mu \dot{x}^\mu - \mathcal{L} = 0$$

2. Impose the constraint

$$p^2 + m^2 \sim 0$$

on physical states

H_0 is classically equivalent to

$$H = H_0 + \frac{e(\tau)}{2}(p^2 + m^2)$$

where $e(\tau)$ is a Lagrange multiplier.

The Lagrangian in the first order formulism is

$$\mathcal{L}_1 = p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 + m^2)$$
$$S_1 = \int d\tau \left[p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 + m^2) \right]$$

Note that the dependence on p is Gaussian so we can integrate it out. This gives the second order action

$$S_2 = \int d\tau \frac{1}{2} \left(\frac{1}{e} \dot{x}_\mu \dot{x}^\mu - m^2 e \right) \tag{2}$$

This is classically equivalent to our original action (1). This can be seen by plugging in the equation of motion for e

$$e = \frac{1}{m} \sqrt{-\dot{x}_\mu \dot{x}^\mu}$$

However, the action given by (2) is a bit more general since we can now let $m \to 0$. Gauge invariance is now

$$\tau \mapsto f(\tau)$$
$$e \mapsto \frac{e}{\dot{f}(\tau)}$$

2.1.1 Constrained Mechanics

Consider a phase space $\mathcal{M}$ with constraints $\phi_i = 0$ for $i = 1, \ldots, k$. We wish to obtain the constrained dynamics. Consider the subspace of $\mathcal{M}$,

$$\mathcal{M}' = \{\phi_i = 0\}$$

Observables are defined module ϕ_i

$$A \sim A + f_i\, \phi_i \qquad \text{on } \mathcal{M}'$$

Note that in general, $\mathcal{M}'$ is not a symplectic manifold (it is not even necessarily even dimensional). Also, we cannot just set $\phi_i = 0$ and forget about them.

$$\{H, \phi_i\} \neq 0$$
$$\{\phi_i, \cdot\} \text{ nontrivial}$$

That is to say, the equivalence relation is not necessarily preserved by the Poisson brackets. There are three primary subcases to consider

1. *First class constraint.*

$$\{\phi_i, \phi_j\} \sim 0 \qquad \text{(vanishes on } \mathcal{M}'\text{)}$$
$$\{\phi_i, H\} \sim 0$$

   In other words, time evolution keeps you on $\mathcal{M}'$. We now identify points in $\mathcal{M}'$ which lie along the same flow line.

$$\widetilde{\mathcal{M}} = \mathcal{M}' / \{\phi_i, \cdot\}$$

   $\widetilde{\mathcal{M}}$ is called the space of leaves of foliation generated by $\{\phi_i, \cdot\}$. Note

$$\dim \mathcal{M}' = \dim \mathcal{M} - k$$
$$\dim \widetilde{\mathcal{M}} = \dim \mathcal{M} - 2k$$

   In particular, $\widetilde{\mathcal{M}}$ is even dimensional. Now, if A and B are constant along flows,

$$\{\phi_i, A\} \sim \{\phi_i, B\} \sim 0$$

   then we have

$$\{A + f_i\ \phi_i, A + g_j\ \phi_j\} \sim \{A, B\}$$

   and $\{\ ,\}$ descends to a Poisson bracket on $\widetilde{\mathcal{M}}$.

2. *Second class constraint.*

$$\{\phi_i, \phi_j\} \sim S_{ij}$$

   where S_{ij} is a nondegenerate antisymmetric matrix-valued function of $\mathcal{M}'$. Here, $\{\phi_i, \cdot\}$ does not leave you on $\mathcal{M}'$. We need to modify the symplectic structure to project back onto $\mathcal{M}'$. Let us define the *Dirac bracket*,

$$\{A, B\}_D \equiv \{A, B\} - \{A, \phi_i\} (S^{-1})^{ij} \{\phi_j, B\}$$

   Now the flow generated by $\{\phi_i, \cdot\}_D$ leaves you on $\mathcal{M}'$.

$$\{\phi_i, \phi_j\}_D = 0$$

   If

$$\{\phi_i, H\}_D = \psi_i \not\sim 0$$

   then add these ψ_i as secondary constraints. We thus obtain a new symplectic manifold

$$\left(\widetilde{\mathcal{M}}, \{\cdot, \cdot\}_D\right) = \mathcal{M}' / \{\phi_i, \cdot\}_D$$

3. *Mixed first and second class constraints.*

   Here one can attempt to add degrees of freedom to make it totally first or second class or failing that, just give up.

2.1.2 Covariant Quantization

Returning to the action (2)

$$S = \int d\tau \frac{1}{2} \left(\frac{1}{e} \dot{x}_\mu \dot{x}^\mu - m^2 e \right)$$

The canonical momentum are

$$\begin{aligned} p^\mu &= \frac{1}{e} \dot{x}^\mu \\ \pi &= 0 \end{aligned}$$

where π is the momentum conjugate to e. $\pi = 0$ is then a primary constraint.

$$H = \frac{1}{2} e(p^2 + m^2) \qquad \text{module constraint}$$

So we have the secondary constraint

$$\begin{aligned} \{H, \pi\} &= \frac{1}{2}(p^2 + m^2) \\ &= 0 \end{aligned}$$

The complete constraints are then all first class:

$$\begin{aligned} \left\{ \pi, \frac{1}{2}(p^2 + m^2) \right\} &= 0 \\ \{\pi, H\} &= 0 \\ \left\{ \frac{1}{2}(p^2 + m^2), H \right\} &= 0 \end{aligned}$$

Now partial gauge-fix on the flow $\{\pi, \cdot\}$ by setting $e = 1$. We can then forget about e and π so that the dynamical variables are

$$x^\mu \quad \text{and} \quad p_\mu = \dot{x}_\mu$$

with the single first class constraint

$$p^2 + m^2 = 0$$

In the quantum theory we replace p_μ with the operator

$$p_\mu = -i \frac{\partial}{\partial x^\mu}$$

A state is a function $\Phi(x)$ and the constraint is

$$(-\Box + m^2)\Phi = 0$$

which is just the familiar Klein-Gordon equation.

2.2 The Noncovariant Method

The most natural gauge would be

$$x^0 = \tau$$
$$p^0 = \sqrt{p_i p^i + m^2}$$

however this is somewhat inconvenient because of the square root. Instead we choose *light-cone gauge*. Let,

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$$

Then break up A^μ into A^+, A^- and the transverse $d-2$ vector $\mathbf{A}$. The inner product is then

$$A \cdot B = -A^+B^- - A^-B^+ + \mathbf{A} \cdot \mathbf{B}$$

This corresponds to the metric

$$\eta_{+-} = \eta_{-+} = -1, \quad \eta_{ij} = \delta_{ij}$$

Our light cone gauge choice is then

$$x^+(\tau) = p^+\tau$$

The constraints are

$$\phi_1 = \frac{x^+}{p^+} - \tau \sim 0$$
$$\phi_2 = \frac{1}{2}(p^2 + m^2)$$
$$= \frac{1}{2}(-2p^+p^- + \mathbf{p}^2 + m^2) \sim 0$$

We have

$$\left\{x^+, p^-\right\} = \eta^{+-} = -1$$
$$\{\phi_1, \phi_2\} = 1$$

and so these constraints are second class,

$$S_{ij} = \{\phi_i, \phi_j\} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Recall that the Dirac bracket is defined as

$$\{A, B\}_D = \{A, B\} - \{A, \phi_i\} (S^{-1})^{ij} \{\phi_j, B\}$$

Explicitly,

$$\begin{aligned}
\left\{x^i, p^j\right\}_D &= \left\{x^i, p^j\right\} = \delta^{ij} \\
\left\{x^-, p^+\right\}_D &= \left\{x^-, p^+\right\} = -1 \\
\left\{x^-, p^-\right\}_D &= -\left\{x^-, \phi_2\right\}\left\{\phi_1, p^-\right\} = \frac{p^-}{p^+} \\
\left\{x^i, p^-\right\}_D &= \frac{p^i}{p^+} \\
\left\{x^-, x^i\right\}_D &= -\left\{x^-, \phi_1\right\}(S^{-1})^{12}\left\{\phi_2, x^i\right\} \\
&= \frac{x^+}{(p^+)^2}(-p^i) \\
&= -\frac{p^i}{p^+}\frac{x^+}{p^+} \\
&= -\frac{p^i}{p^+}(\phi_1 + \tau) \sim -\frac{p^i}{p^+}\tau
\end{aligned}$$

Note that if we define

$$x^- = x_0^- + \tau p^-$$

then

$$\left\{x_0^-, x^i\right\}_D = 0$$

Now solve the constraint $\phi_2 = 0$,

$$p^- = \frac{1}{2p^+}(\mathbf{p}^2 + m^2)$$

The dynamical variables are

$$x^-, p^+, x^i, p^i$$

The Hamiltonian generates τ translations,

$$\frac{\partial}{\partial\tau} \sim p^+\frac{\partial}{\partial x^+}$$

So that

$$H = p^+p^- + \alpha\phi_2$$

The second term has been added to keep us on the constraint surface (this is a result of switching to Dirac brackets). To determine the value of α we set

$$0 = \frac{d\phi_1}{d\tau} = \{\phi_1, H\} + \frac{\partial\phi_1}{\partial\tau} = (-1+\alpha) - 1$$

giving $\alpha = 2$. The Hamiltonian is then[1]

$$H = \frac{1}{2}(\mathbf{p}^2 + m^2)$$

[1]Something is wrong here. $\alpha = 2$ does not give the right Hamiltonian. If we had $\alpha = 1$ it would work out.

3 The Bosonic String

The Nambu-Goto action is

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det h}$$

where

$$h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu$$
$$X^\mu(\sigma, \tau)$$

If we introduce the notation

$$\dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau}$$
$$X'^\mu = \frac{\partial X^\mu}{\partial \sigma}$$

we can write the Nambu-Goto action as

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2} \tag{3}$$

Let's introduce an auxiliary field $g_{\alpha\beta}$

$$S = -\frac{1}{4\pi\alpha'} \int d\tau d\sigma \sqrt{-\det g}\; g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu \tag{4}$$

This action is called the Polyakov action. It is classically equivalent to the Nambu-Goto action. The equation of motion for $g_{\alpha\beta}$ is

$$T_{\alpha\beta} = \frac{1}{4\pi\alpha'} \left[\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} g_{\alpha\beta} \left(g^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu \right) \right]$$
$$= 0$$

$$\sqrt{-\det h} = \sqrt{-\det g}\, \frac{1}{2} g^{\gamma\delta} \partial_\gamma X^\mu \partial_\delta X_\mu$$

Exercise 3.1. *What happens when you add a "cosmological constant" term to* (4)*?*

$$\left(\sqrt{-\det g}\right) \sim \text{ mass term for free particle}$$

closed string

$$0 \le \sigma \le \pi \qquad \text{periodic b.c.}$$

open string

$$0 \le \sigma \le \pi \qquad \text{b.c. ?}$$

$$\delta S = 0 \text{ for } \delta X^\mu(\tau_i) = \delta X^\mu(\tau_+) = 0$$

4 The Superstring

4.1 Majorana Fermions in $d = 2$ Dimensions

Instead of γ's we'll use ρ's

$$\{\rho^A, \rho^B\} = -2\eta^{AB}$$

We choose the following specific representation for the ρ's

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$\rho^{01} = \frac{1}{2}\left[\rho^0, \rho^1\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

ρ^0 and ρ^1 are pure imaginary, so the Dirac operator

$$i\not{D} = ie^\alpha_A \rho^A D_\alpha$$

is real. Here e^A_α is the zweibein, and ρ^α_A the inverse. The Majorana condition is

$$\bar{\psi} = \psi^{\mathrm{T}}\rho^0$$

$$\overline{\rho^A\psi} = \bar{\psi}\rho^A$$

$$\rho_A\rho^B\rho^A = 0$$

$$\begin{aligned} \bar{\psi}\chi &= i(\psi_2\chi_1 - \psi_1\chi_2) \\ &= -i(\chi_1\psi_2 - \chi_2\psi_1) \\ &= \bar{\chi}\psi \end{aligned}$$

$$\begin{aligned} D_\alpha\psi &= \partial_\alpha\psi + \frac{1}{4}\omega_{\alpha AB}\rho^{AB}\psi \\ &= \partial_\alpha\psi + \frac{1}{2}\omega_{\alpha 01}\rho^{01}\psi \end{aligned}$$

where $\omega_{\alpha AB}$ is the spin connection. But

$$\rho^0\rho^A\rho^{01} = \rho^A\rho^1$$

so

$$\bar{\psi}\rho^A\rho^{01}\psi = \psi^{\mathrm{T}}\rho^A\rho^1\psi = 0$$

and hence we can drop the spin connection term in the action.

$$i\bar{\psi}\not{D}\psi = i\rho^\alpha_A\bar{\psi}\rho^A\partial_\alpha\psi$$

In conformal gauge

$$\rho^\alpha_A = e^{-\phi/2}\delta^\alpha_A$$

Let

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \qquad e = \det \rho^\alpha_A = \sqrt{g}$$

$$ei\bar{\psi}D\psi = 2e^{1/2}(\psi_1\partial_+\psi_1 + \psi_2\partial_-\psi_2)$$

4.2 The Action

The action is

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\; e\Big[g^{\alpha\beta}\partial_\alpha X_\mu \partial_\beta X^\mu - i e^\alpha_A \bar{\psi}_\mu \rho^A \partial_\alpha \psi^\mu + 2\bar{\chi}_\alpha e^\alpha_A e^\beta_B \rho^B \rho^A \psi_\mu \partial_\beta \psi^\mu + \frac{1}{2}\rho^\alpha_A \rho^\beta_B \bar{\chi}_\alpha \rho^B \rho^A \chi_\beta \bar{\psi}_\mu \psi^\mu\Big]$$

The symmetries of the action S are

1. Diffeomorphisms
2. Local supersymmetry

$$\begin{aligned}
\delta X^\mu &= \bar{\epsilon}\psi^\mu \\
\delta \psi^\mu &= -i\rho^A \epsilon\; e^\alpha_A \left(\partial_\alpha X^\mu - \bar{\psi}^\mu \chi_\alpha\right) \\
\delta e^A_\alpha &= -2i\bar{\epsilon}\rho^A \chi_\alpha \\
\delta \chi_\alpha &= \nabla_\alpha \epsilon
\end{aligned}$$

   where

$$\nabla_\alpha \epsilon = D_\alpha \epsilon - 2i\left(\bar{\chi}_\alpha \rho^{01}\rho^A \chi_\beta e^\beta_A\right)\epsilon$$

3. Weyl rescaling

$$\begin{aligned}
e^A_\alpha &\to f e^A_\alpha \\
\psi^\mu &\to f^{-1/2}\psi^\mu \\
\chi_\alpha &\to f^{1/2}\chi_\alpha \\
X^\mu &\to X^\mu
\end{aligned}$$

4. Local Lorentz

$$\begin{aligned}
e^A_\alpha &\to \left(\delta^A_B \cosh\lambda + \epsilon^A_B \sinh\lambda\right) e^B_\alpha \\
\psi_\mu &\to e^{\frac{1}{2}\lambda\rho^{01}}\psi^\mu \\
&= \left(\cosh\frac{\lambda}{2} + \rho^{01}\sinh\frac{\lambda}{2}\right)\psi^\mu \\
\chi_\alpha &\to e^{\frac{1}{2}}\lambda\rho^{01}\chi_\alpha
\end{aligned}$$

5. Gravitino shift

$$\delta\chi_\alpha = i e^A_\alpha \rho_A \eta$$

   where η is a Majorana fermion.

$$\begin{aligned}
e^\alpha_A \rho^A \rho^B \delta\chi_\alpha &= e^\alpha_A \rho^A \rho^B e^C_\alpha \rho_C \eta \\
&= \rho^A \rho^B \rho_A \eta = 0
\end{aligned}$$

5 Conformal Field Theory

Consider a world-sheet parameterized by σ and τ, where now

$$-\infty < \tau < \infty$$
$$0 \leq \sigma \leq 2\pi$$

for closed strings. The light cone coordinates on the string are

$$\tau \pm \sigma$$

Now analytically continue to Euclidean coordinates

$$\tau \pm \sigma \to -i(\tau \pm i\sigma)$$

and define

$$w = \tau + i\sigma, \qquad \bar{w} = \tau - i\sigma$$

We can then conformally map the world-sheet to the punctured complex plane by defining

$$z = e^{w}, \qquad \bar{z} = e^{\bar{w}}$$

Under this transformation left(right)-moving fields become analytic (antianalytic) fields, and time ordering becomes radial ordering. Equal τ slices conformally map to concentric circles in the complex plane.

We assume that we can construct a classically scale-invariant field theory. That is, we have a dilatation symmetry

$$X^{\alpha} \to X^{\alpha} + \delta\lambda X^{\alpha}$$

The dilatation current is

$$D_{\alpha} = T_{\alpha\beta} X^{\beta}$$

The conservation of this current implies

$$\partial_{\alpha} D^{\alpha} = 0 \implies T^{\alpha}_{\alpha} = 0$$

In 2D take the flat plane so

$$g_{z\bar{z}} = \frac{1}{2}, \qquad g_{zz} = g_{\bar{z}\bar{z}} = 0$$

and

$$T_{z\bar{z}} = 0$$

Of course, energy-momentum is conserved (this is already assumed) so

$$\partial_{\bar{z}} T_{zz} + \partial_{z} T_{\bar{z}z}$$

so

$$T_{\bar{z}z} = 0 \implies \partial_{\bar{z}} T_{zz} = 0$$

That is

$$T(z) \equiv T_{zz}(z)$$

is analytic. And similarly

$$\bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z})$$

is antianalytic.

We have conformal symmetry if

$$T_\alpha{}^\alpha = 0$$

is invariant under not just dilatations $X^\alpha \to (1+\delta\lambda)X^\alpha$ but also under conformal transformations

$$X^\alpha \to Y^\alpha(X)$$

where

$$g_{\alpha\beta} \to \frac{\partial Y^\gamma}{\partial X^\alpha}\frac{\partial Y^\delta}{\partial X^\beta}\, g_{\gamma\delta}(Y(X)) \equiv \rho(x) g_{\alpha\beta}$$

For $d > 2$

$$\underset{\text{Poincaré symmetry}}{\mathrm{ISO}(d-1,1)} \to \underset{\substack{\text{Conformal group}\\ \text{or}\\ \text{AdS group in } d+1 \text{ dim}}}{\mathrm{SO}(d,2)}$$

We're doing Euclidean so

$$\mathrm{ISO}(d) \to \mathrm{SO}(d+1,1)$$

These amount to different real sections of the complex group $\mathrm{SO}(d+2,\mathbb{C})$.

The new conserved currents are

$$D_\alpha = T_{\alpha\beta}X^\beta$$
$$K^{\alpha\beta} = X^2 T^{\alpha\beta} - 2X^\alpha D^\beta$$

For $d = 2$ we get and infinite dimensional symmetry algebra

$$\partial_{\bar{z}}(z^{n+1}T(z)) = 0$$
$$\partial_z(\bar{z}^{m+1}\bar{T}(\bar{z})) = 0$$

under the infinitesimal conformal transformation

$$z \to z + \epsilon(z)$$
$$\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$$

whose Lie algebra is $\mathrm{Vir} \oplus \mathrm{Vir}$

$$\epsilon(z) = \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}$$

The generators are

$$\ell_n = -z^{n+1}\frac{d}{dz}, \qquad \bar{\ell}_n = -\bar{z}^{n+1}\frac{d}{d\bar{z}}$$

with the Lie bracket

$$[\ell_n, \ell_m] = (n-m)\ell_{m+n}$$

Take the complexified Lie Algebra $\text{Vir}_{\mathbb{C}} \otimes \text{Vir}_{\mathbb{C}}$ whose different real sections are Euclidean (conformal group) and Minkowskian (pseudoconformal group).

The conserved charge is

$$L_\epsilon = \oint_{\log|z|=\tau} \frac{dz}{2\pi i} \epsilon(z) T(z)$$

Let us pretend that we have some correlation function

$$\langle X \rangle = \langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle$$

Make an arbitrary diffeomorphism

$$z \to z + \epsilon(z, \bar{z})$$

then

$$\begin{aligned} 0 = \sum_{k=1}^{n} &\langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \delta_\epsilon \mathcal{O}_k(z_k, \bar{z}_k) \cdots \mathcal{O}_n(z_n, \bar{z}_n) \rangle \\ &+ \frac{1}{\pi} \int d^2 z \left\{ \partial_{\bar{z}} (\epsilon \langle T(z) X \rangle) + \partial_z (\bar{\epsilon} \langle \bar{T}(\bar{z}) X \rangle) \right\} \end{aligned}$$

Consider, for example, a scalar field theory

$$\begin{aligned} \delta \mathcal{L} &= \frac{\delta \mathcal{L}}{\delta \phi} \xi \cdot \partial \phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\mu (\xi \cdot \partial \phi) \\ &= \left(\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \right) \xi \cdot \partial \phi + \partial_\mu \left(\xi^\nu \underbrace{\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \partial_\nu \phi}_{-\frac{1}{\pi} T^\mu{}_\nu} \right) \end{aligned}$$

Note that

$$\begin{aligned} \partial_{\bar{z}} \langle T(z) X \rangle &= 0 \\ \partial_z \langle \bar{T}(\bar{z}) X \rangle &= 0 \end{aligned}$$

everywhere except at the points $z = z_i$ and $\bar{z} = \bar{z}_i$. Also note that we can use the divergence theorem for complex coordinates:

$$\begin{aligned} \int d^2 z \left(\partial_{\bar{z}} A + \partial_z \bar{A} \right) &= \frac{i}{2} \int dz \wedge d\bar{z} \left(\partial_{\bar{z}} A + \partial_z \bar{A} \right) \\ &= -\frac{i}{2} \int_S d(A dz - \bar{A} d\bar{z}) \\ &= -\frac{i}{2} \oint_{\partial S} (A dz - \bar{A} d\bar{z}) \end{aligned}$$

Thus we can write

$$0 = \langle \delta_\epsilon X \rangle - \oint_C \frac{dz}{2\pi i} \epsilon(z) \langle T(z) X \rangle + \oint_C \frac{d\bar{z}}{2\pi i} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle$$

where C is a big circle containing all the points z_i.

$$\langle \delta_\epsilon X \rangle = \oint_C \frac{dz}{2\pi i} \epsilon(z) \langle T(z) X \rangle - \oint_C \frac{d\bar{z}}{2\pi i} \bar{\epsilon}(\bar{z}) \langle \bar{T}(\bar{z}) X \rangle$$

Now contract the contour to a sum of small circles around each of the points z_i.

$$\delta_\epsilon \mathcal{O}_i(z_i, \bar{z}_i) = \oint_C \frac{dz}{2\pi i} \epsilon(z) T(z) \mathcal{O}_i(z_i, \bar{z}_i) - \oint_C \frac{d\bar{z}}{2\pi i} \bar{\epsilon}(\bar{z}) \bar{T}(\bar{z}) \mathcal{O}_i(z_i, \bar{z}_i)$$

But this is just the definition (in radial ordering) of

$$\begin{aligned} [L_\epsilon, \mathcal{O}(w, \bar{w})] &= \left(\oint_{\substack{\log|z|> \\ \log|w|}} - \oint_{\substack{\log|z|< \\ \log|w|}} \right) \frac{dz}{2\pi i} \epsilon(z) T(z) \mathcal{O}(w, \bar{w}) \\ &= \oint_w \frac{dz}{2\pi i} \epsilon(z) T(z) \mathcal{O}(w, \bar{w}) \end{aligned}$$

From now on we will just drop the antiholomorphic term in order to save writing. Its presence is always implicit.

$$\begin{aligned} \delta_\epsilon \mathcal{O}(w, \bar{w}) &= [L_\epsilon, \mathcal{O}(w, \bar{w})] \\ &= \oint_w \frac{dz}{2\pi i} \epsilon(z) T(z) \mathcal{O}(w, \bar{w}) \end{aligned}$$

The variation of the energy-momentum tensor is

$$\delta_\epsilon = \epsilon(w) \partial T(w) + 2\partial\epsilon(w) T(w) + \underbrace{\frac{c}{12} \partial^3 \epsilon(w)}_{\text{Schwinger term}}$$

The factor of two in the second term is the conformal weight of T. The above variation follows from the operator product expansion (OPE)

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \cdots \tag{5}$$

To see this consider a contour integral around the point w

$$\begin{aligned} &\oint_w \frac{dz}{2\pi i} \epsilon(z) T(z) T(w) \\ &\quad = \oint_w \frac{dz}{2\pi i} (\epsilon(w) + (z-w)\partial\epsilon(w) + \frac{1}{2}(z-w)^2 \partial^2 \epsilon(w) + \frac{1}{6}(z-w)^3 \partial^3 \epsilon(w)) \\ &\qquad \times \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \right) \\ &\quad = \epsilon \partial T + 2\partial\epsilon T + \frac{c}{12} \partial^3 \epsilon \end{aligned}$$

Exercise 5.1 (The Virasoro algebra). *Using the Laurent expansion of* $T(z)$

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z^{n+2}} L_n \tag{6a}$$

where

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z) \tag{6b}$$

show that equation (5) *implies the Virasoro algebra:*

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n,-m} \tag{7}$$

Exercise 5.2. *Consider*

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c}{(z-w)} + \cdots$$

and let

$$J^a_n = \oint \frac{dz}{2\pi i} z^n J^a(z)$$

Compute $\left[J^a_n, J^b_m\right]$.

As defined $T(z)$ does not transform as a conformal tensor (of weight 2). Instead, under $z \to y(z)$,

$$T(z) \to \left(\frac{dy}{dz}\right)^2 T(y) + \frac{c}{12}\{y, z\}$$

where the *Schwarzian derivative* $\{y, z\}$ is defined as

$$\{y, z\} \equiv \frac{d^3y/dz^3}{dy/dz} - \frac{3}{2}\left(\frac{d^2y/dz^2}{dy/dz}\right)^2$$

Exercise 5.3. *Verify that under successive transformations* $z \to y \to w$ *the Schwarzian derivative satisfies*

$$\{w, z\} = \left(\frac{dy}{dz}\right)^2 \{w, y\} + \{y, z\}$$

Note that the Schwarzian vanishes for $\mathrm{PSL}(2, \mathbb{C})$ transformations:

$$y = \frac{az+b}{cz+d}, \qquad ad - bc = 1$$

Theorem 5.1. *If none of the points* z_i *are at infinity, then* $\langle T(z)X\rangle$ *is regular as* $z \to \infty$. *In fact,*

$$\langle T(z)X\rangle \sim z^{-4} \quad \text{as } z \to \infty$$

Proof. Consider integrating on a large circle C enclosing all of the points z_i

$$\oint_C \frac{dz}{2\pi i} z^{n+1} \langle T(z) X \rangle$$

Conformal invariance implies that the limit radius of $C \to \infty$ should exist. To study it, change variables of integration

$$y = \frac{1}{z}$$

Then

$$\begin{aligned} \oint_C \frac{dz}{2\pi i} z^{n+1} T(z) &\to - \oint_C \frac{dy}{2\pi i}\, y^{-(n+3)} (y^4 T(y)) \\ &= - \oint_C \frac{dy}{2\pi i}\, y^{1-n} T(y) \end{aligned}$$

We can now contract C to the origin in the y-plane (∞ in the z-plane). If $n \leq 1$ the integrand is analytic at $y = 0$ and so

$$\oint_C \frac{dy}{2\pi i}\, y^{1-n} T(y) = 0, \qquad n \leq 1$$

Translating back to the z-plane, we can conclude that as $C \to \infty$

$$\oint_C \frac{dz}{2\pi i} z^{n+1} \langle T(z) X \rangle \to 0, \qquad n \leq 1$$

and so $\langle T(z) X \rangle$ itself must vanish like z^{-4}. □

The subalgebra of the Virasoro algebra spanned by

$$L_{-1}, L_0, L_1 \qquad \text{and} \qquad \bar{L}_{-1}, \bar{L}_0, \bar{L}_1$$

has no central extension. Together these generate $\mathrm{SL}(2, \mathbb{C})$.

$$\begin{aligned} [L_0, L_{\pm 1}] &= \pm L_{\pm 1} \\ [L_1, L_{-1}] &= 2 L_0 \end{aligned}$$

They also generate the conformal transformation

$$z \longrightarrow z + \epsilon_{-1} + \epsilon_0 z + \epsilon_1 z^2$$

More specifically

$L_{-1}, \bar{L}_{-1}$	generate translations
$L_0 + \bar{L}_0 = z\partial_z + \bar{z}\partial_{\bar{z}} = r\frac{\partial}{\partial r}$	generates dilatations (evolutions in τ)
$i(L_0 - \bar{L}_0) = i(z\partial_z - \bar{z}\partial_{\bar{z}}) = \frac{\partial}{\partial \theta}$	generates rotations
$L_1, \bar{L}_1$	generate special conformal transformations

Let us assume that the operators $\mathcal{O}_i$ scale homogeneously under dilatations and rotations.

$$[L_0, \mathcal{O}(0)] = h\mathcal{O}(0)$$
$$\left[\bar{L}_0, \mathcal{O}(0)\right] = \bar{h}\mathcal{O}(0)$$

If you do not like the point 0, define

$$L_n^{(z)} = \oint_z \frac{d\phi}{2\pi i}(\phi - z)^{n+1}T(\phi)$$

and

$$\left[L_0^{(z)}, \mathcal{O}(z,\bar{z})\right] = h\mathcal{O}(z,\bar{z}), \text{ etc.}$$

The quantity $h + \bar{h}$ is called the scaling dimension of $\mathcal{O}$ and $h - \bar{h}$ is called the spin of $\mathcal{O}$. Under translations we have

$$[L_{-1}, \mathcal{O}(0)] = \partial\mathcal{O}(0)$$
$$\left[\bar{L}_{-1}, \mathcal{O}(0)\right] = \bar{\partial}\mathcal{O}(0)$$

and so part of the OPE of T with $\mathcal{O}$ is determined:

$$T(z)\mathcal{O}(w,\bar{w}) = \cdots + \frac{h\mathcal{O}(w,\bar{w})}{(z-w)^2} + \frac{\partial_w\mathcal{O}(w,\bar{w})}{(z-w)} + \cdots$$

Definiton. If the most singular term in the OPE of T with $\mathcal{O}$ is a double-pole then we call $\mathcal{O}$ a *primary conformal field.*

If $\mathcal{O}$ is primary then

$$[L_n, \mathcal{O}(0)] = 0 \qquad n \geq 1$$

Exercise 5.4. *Show that a primary field* $\mathcal{O}(z,\bar{z})$ *satisfies*

$$[L_n, \mathcal{O}(z,\bar{z})] = z^{n+1}\partial_z\mathcal{O}(z,\bar{z}) + h(n+1)z^n\mathcal{O}(z,\bar{z})$$

The vacuum satisfies

$$L_n|0\rangle = 0 \qquad n \geq -1$$

This is equivalent to $SL(2,\mathbb{C})$ invariance. In the analytic continuation to Minkowskian signature (imaginary τ), T and $\bar{T}$ are Hermitian. It follows that we must have

$$L_n^\dagger = L_{-n}$$

and so

$$0 = \langle 0|L_n^\dagger \qquad n \geq -1$$

or

$$0 = \langle 0|L_n \qquad n \leq 1$$

Given a primary field, $\mathcal{O}$ we can generate an infinite number of *descendant fields*:

$$\mathcal{O}^{(n_1,n_2,\ldots)}(z,\bar{z}) = \cdots (L_{-2}^{(z)})^{n_2}(L_{-1}^{(z)})^{n_1}\mathcal{O}(z,\bar{z})$$

The collection of the operators will be denoted by $[\mathcal{O}]$. Together they furnish a representation of the Virasoro algebra. In particular, since

$$L_{-1}^{(z)}\mathcal{O}(z,\bar{z}) = \partial\mathcal{O}(z,\bar{z})$$

$\mathcal{O}$ and all of its derivatives are in $[\mathcal{O}]$.

5.1 Correlation Functions

We saw that a primary field, $\mathcal{O}$ satisfies

$$\partial_\epsilon\mathcal{O}(z,\bar{z}) = \epsilon(z)\partial_z\mathcal{O}(z,\bar{z}) + h\mathcal{O}(z,\bar{z})\partial_z\epsilon(z)$$

Comparing with the RHS,

$$\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \sum_{i=1}^{n}\left(\frac{h_i}{(z-z_i)^2} + \frac{1}{(z-z_i)}\frac{\partial}{\partial z_i}\right)\langle\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle \quad (8)$$

Consider the contour integral

$$I = \oint_C \frac{dz}{2\pi i}(z-z_1)(z-z_2)\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle$$

where C is a contour surrounding both z_1 and z_2. Contracting,

$$I = (h_1 - h_2)(z_1 - z_2)\langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle$$

We have assumed that neither of the points z_1 or z_2 are at infinity so by Theorem 5.1 $\langle T(z)X\rangle \sim z^{-4}$ as $z \to \infty$. So taking the contour C to be arbitrary large we see that the integral must vanish

$$I = 0$$

So either $h_1 = h_2$ (and $\bar{h}_1 = \bar{h}_2$) or

$$\langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle = 0$$

By a similar argument

$$0 = \oint_C \frac{dz}{2\pi i}(z-z_2)\langle T(z)\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle$$

From a circle around z_2 the residue is

$$h\langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\rangle$$

From a circle around z_1 the residue is (use $z - z_2 = z - z_1 + (z_1 - z_2)$)

$$\left(h + (z_1 - z_2)\frac{\partial}{\partial z_1}\right)\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\rangle$$

so

$$0 = \left(2h + (z_1 - z_2)\frac{\partial}{\partial z_1}\right)\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\rangle$$

which implies

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\rangle = \text{ const } (z_1 - z_2)^{-2h}(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}}$$

Correlation functions, in general, must be invariant under conformal transformations. Specifically they should be invariant under the action of $PSL(2, \mathbb{C})$. By considering the effect of a rescaling and rotation its not hard to show that the one point correlation function must vanish:

$$\langle \mathcal{O}(z, \bar{z})\rangle = 0$$

The two point function we now know

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\rangle = \begin{cases} \text{const } (z_1 - z_2)^{-2h}(\bar{z}_1 - \bar{z}_2)^{-2\bar{h}} & {}^{h_1=h_2=h}_{\bar{h}_1=\bar{h}_2=\bar{h}} \\ 0 & \text{otherwise} \end{cases}$$

The next logical question is what is the three point function

$$\langle \mathcal{O}_1(z_1, \bar{z}_1)\mathcal{O}_2(z_2, \bar{z}_2)\mathcal{O}_3(z_3, \bar{z}_3)\rangle = ?$$

Before we calculate this we should comment on the insertion of multiple energy-momentum tensors into the correlation function. Equation (8) above gives the expression for the insertion of one energy-momentum tensor. The corresponding equation for multiple tensors is

$$\begin{aligned} &\langle T(z)T(y_1)\cdots T(y_m)\mathcal{O}_1(z_1, \bar{z}_1)\cdots \mathcal{O}_n(z_n, \bar{z}_n)\rangle \\ &\quad = \left\{\sum_{i=1}^{n}\left(\frac{h_i}{(z - z_i)^2} + \frac{1}{z - z_i}\frac{\partial}{\partial z_i}\right) + \sum_{j=1}^{m}\left(\frac{2}{(z - y_j)^2} + \frac{1}{z - y_j}\frac{\partial}{\partial y_j}\right)\right\} \\ &\qquad\qquad \times \langle T(y_1)\cdots T(y_m)\mathcal{O}_1(z_1, \bar{z}_1)\cdots \mathcal{O}_n(z_n, \bar{z}_n)\rangle \\ &\qquad + \sum_{j=1}^{m}\frac{c/2}{(z - y_j)^4}\langle T(y_1)\cdots T(y_{j-1})T(y_{j+1})\cdots T(y_m)\mathcal{O}_1(z_1, \bar{z}_1)\cdots \mathcal{O}_n(z_n, \bar{z}_n)\rangle \end{aligned}$$

This expression is used to compute correlation functions of descendant fields while equation (8) is used with primary fields.

It is convenient to choose an orthonormal basis of primary fields such that

$$\langle \mathcal{O}_n(z_1, \bar{z}_1)\mathcal{O}_m(z_2, \bar{z}_2)\rangle = \frac{\delta_{nm}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}}$$

5.1.1 States and Operators

We can define a primary state as

$$|\mathcal{O}\rangle = \lim_{z\to 0} \mathcal{O}(z,\bar{z})|0\rangle$$

A primary state satisfies

$$L_n|\mathcal{O}\rangle = 0 \qquad n \geq 1$$

For a ket $|\mathcal{O}\rangle$ the corresponding bra is

$$\langle\mathcal{O}| = \lim_{z\to\infty} \langle 0|\mathcal{O}(z,\bar{z}) z^{2L_0} \bar{z}^{2\bar{L}_0}$$

5.1.2 The 3-point Function

The three point function

$$\langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3,\bar{z}_3)\rangle$$

can be most easily computed by taking $z_1 \to \infty$ and $z_3 \to 0$. One can then show that the dependence on z and $\bar{z}$ (or z_2 and $\bar{z}_2$) is given by

$$\langle\mathcal{O}_i|\mathcal{O}_j(z,\bar{z})|\mathcal{O}_k\rangle = c_{ijk} z^{h_i-h_j-h_k} \bar{z}^{\bar{h}_i-\bar{h}_j-\bar{h}_k}$$

where c_{ijk} is a constant.

Exercise 5.5. *Prove this by considering*

$$\langle\mathcal{O}_i|T(w)\mathcal{O}_j(z,\bar{z})|\mathcal{O}_k\rangle$$

and taking the appropriate contour integral.

More generally, by $\mathrm{PSL}(2,\mathbb{C})$ transformations, we have

$$\begin{aligned}&\langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3,\bar{z}_3)\rangle \\ &\qquad = c_{123}\left(z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}\right)^{-1}\left(\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_1+\bar{h}_3-\bar{h}_2}\right)^{-1}\end{aligned}$$

where the c_{ijk} are OPE coefficients.

5.1.3 The 4-point Function

Note that the one, two, and three point functions are completely determined up to constant coefficients by conformal invariance. Life is not so easily for the four point function:

$$G^{(4)} = \langle\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\mathcal{O}_3(z_3,\bar{z}_3)\mathcal{O}_4(z_4,\bar{z}_4)\rangle$$

An element of $\mathrm{PSL}(2,\mathbb{C})$ is uniquely determined by its action on any three points. Hence we could fix

$$(z_1,z_2,z_4) \to (\infty,1,0)$$

by the transformation

$$y(z) = \frac{(z_1 - z_2)(z - z_4)}{(z_1 - z)(z_2 - z_4)}$$

Under this transformation z_3 is sent to the point

$$z_3 \to x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)} = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

and the four point function becomes

$$G^{(4)} = \prod_{i=1}^{4} \left(\frac{dy}{dz}\right)^{h_i}_{z=z_i} \left(\frac{d\bar{y}}{d\bar{z}}\right)^{\bar{h}_i}_{\bar{z}=\bar{z}_i}$$
$$\times \lim_{\zeta \to z_1} \left(\frac{dy}{dz}\right)^{h_1}_{z=\zeta} \left(\frac{d\bar{y}}{d\bar{z}}\right)^{\bar{h}_1}_{\bar{z}=\bar{\zeta}} (y(\zeta))^{-2h_1} \left(\bar{y}(\bar{\zeta})\right)^{-2\bar{h}_1} \langle \mathcal{O}_1 | \mathcal{O}_2(1,1)\mathcal{O}_3(x,\bar{x}) | \mathcal{O}_4 \rangle$$

or

$$G^{(4)} = \left[z_{12}^{h_3+h_4-h_1-h_2} z_{13}^{-2h_3} z_{14}^{h_2+h_3-h_1-h_4} z_{24}^{h_1-h_2-h_3-h_4}\right] \cdot \overline{[\quad]} \langle \mathcal{O}_1 | \mathcal{O}_2(1,1)\mathcal{O}_3(x,\bar{x}) | \mathcal{O}_4 \rangle$$

$$\sum_{i=1}^{4} \epsilon(z_i) \frac{\partial}{\partial z_i} x = 0$$

for $\epsilon(z) = a + bz + cz^2$ as we have been using, the dependence on x is completely undetermined.

5.2 Examples

5.2.1 Free Massless Scalar Field

The action is given

$$S = \frac{1}{2\pi} \int d^2 z \; \partial X \bar{\partial} X$$

with

$$\langle X(z_1, \bar{z}_1) X(z_2, \bar{z}_2) \rangle = -\log |z_1 - z_2|^2$$
$$T(z) = -\frac{1}{2} {:}\partial X \partial X{:}$$
$${:}\partial X \partial X{:}(z) = \lim_{\zeta \to z} \left(\partial X(\zeta) \partial X(z) + \frac{1}{(\zeta - z)^2} \right)$$

$$\begin{aligned} T(z)T(w) &= \frac{1/2}{(z-w)^4} - \frac{{:}\partial X(z) \partial X(w){:}}{(z-w)^2} \\ &= \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \end{aligned}$$

so the central charge is

$$c = 1$$

Note that X is not a good conformal field. It goes as $\log(z)$ as $z \to \infty$. However

$$j = i\partial X$$

is a perfectly good one:

$$\begin{aligned} T(z)j(w) &= -\frac{1}{2}\,\partial X \partial X(z) \quad i\partial X(w) \\ &= \frac{i\partial X(z)}{(z-w)^2} = \frac{j(w)}{(z-w)^2} + \frac{\partial_w j(w)}{z-w} \end{aligned}$$

This implies that j is a primary field with a conformal weight of $h = 1$. Another primary field is

$$:e^{ipX}:$$

whose OPE with $T(z)$ is given by

$$\begin{aligned} T(z)e^{ipX(w,\bar{w})} &= -\frac{1}{2}\partial X \partial X(z) \quad e^{ipX(w,\bar{w})} \\ &= :\partial X(z)\frac{ip}{z-w}e^{ipX(w,\bar{w})}: + \frac{1}{2}\frac{p^2}{(z-w)^2}e^{ipX(w,\bar{w})} \\ &= \frac{\frac{p^2}{2}e^{ipX(w,\bar{w})}}{(z-w)^2} + \frac{\partial_w e^{ipX(w,\bar{w})}}{z-w} + \cdots \end{aligned}$$

And so $:e^{ipX}:$ is a primary field with conformal weight $h = p^2/2$. The OPE of two of these fields is

$$\begin{aligned} e^{ip_1 X(z,\bar{z})}e^{ip_2 X(w,\bar{w})} &= e^{-(ip_1)(ip_2)\log|z-w|^2}:e^{i(p_1 X(z,\bar{z}) + p_2 X(w,\bar{w}))}: \\ &= |z-w|^{2p_1 p_2}\left[e^{i(p_1+p_2)X(w,\bar{w})} + (z-w)ip_1\partial X e^{i(p_1+p_2)X(w,\bar{w})} + \cdots\right] \end{aligned}$$

5.2.2 Free Massless Majorana Fermions

We now study the Euclidean (Wick-rotated) analogs of Majorana fermions in Minkowski space. The Euclidean ρ matrices are given by

$$\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \rho^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Note that ρ_2 is related to our earlier definition of ρ_0 by $\rho_2 = -i\rho_0$.

$$\Psi = \begin{pmatrix} \psi \\ \tilde{\psi} \end{pmatrix} \qquad -i\not{\partial} = \begin{pmatrix} 0 & \partial \\ \bar{\partial} & 0 \end{pmatrix}$$

The action for this theory is

$$S = \frac{1}{4\pi}\int d^2z \left(\psi\bar{\partial}\psi + \tilde{\psi}\partial\tilde{\psi}\right)$$

with

$$\contraction{}{\psi}{(z)\,}{\psi}\psi(z)\,\psi(w) = -\frac{1}{z-w}$$

Exercise 5.6. *Use Wick's Theorem to compute the central charge c and the conformal weight h of ψ. Answer: $c = 1/2, h = 1/2$.*

5.3 Representations of the Virasoro Algebra

A primary, or *highest weight*, state $|\mathcal{O}\rangle$ satisfies

$$\begin{aligned} L_0|\mathcal{O}\rangle &= h|\mathcal{O}\rangle \\ L_n|\mathcal{O}\rangle &= 0 \qquad\qquad n \geq 1 \end{aligned}$$

The descendant, or secondary, states are

$$\begin{aligned} &L_{-1}|\mathcal{O}\rangle \\ &L_{-2}|\mathcal{O}\rangle, \quad L_{-1}^2|\mathcal{O}\rangle \\ &L_{-3}|\mathcal{O}\rangle, \quad L_{-2}L_{-1}|\mathcal{O}\rangle, \quad L_{-1}^3|\mathcal{O}\rangle \\ &\vdots \end{aligned}$$

The number of states at level N is equal to $P(N)$, the number of partitions of the number N, whose generating function is given by

$$\sum_{N=0}^{\infty} P(N)q^N = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)}$$

with $P(0) \equiv 1$. A primary state together with all of its descendants is known as a *conformal family* and furnishes a representation of the Virasoro algebra. In general, this representation is not unitary. We would like a unitary representation (in which all states have a positive norm). Consider

$$\begin{aligned} \|L_{-n}|\mathcal{O}\rangle\|^2 &= \langle\mathcal{O}|L_{-n}^\dagger L_{-n}|\mathcal{O}\rangle \\ &= \langle\mathcal{O}|\,[L_n, L_{-n}]\,|\mathcal{O}\rangle \\ &= 2n\langle\mathcal{O}|L_0|\mathcal{O}\rangle + \frac{c}{12}(n^3-n)\langle\mathcal{O}|\mathcal{O}\rangle \\ &= \left(2nh + \frac{c}{12}(n^3-n)\right)\langle\mathcal{O}|\mathcal{O}\rangle \end{aligned}$$

We can ask the question, for which values of c and h is this positive for all n. When $n = 1$ the central term vanishes and we see that we must have

$$h \geq 0$$

When $n \gg 1$ the n^3 term dominates and we must have

$$c \geq 0$$

for positivity. When $h = 0$ we can see that $L_{-1}|\mathcal{O}\rangle = 0$ and so the primary state $\mathcal{O}$ may be identified with the vacuum (which corresponds to inserting the identity operator at the origin):

$$h = 0 \implies |\mathcal{O}\rangle = |0\rangle$$

Lemma 5.2. *For $c = 0$ the representation is trivial.*

Proof. Look at

$$L_{-2n}|h\rangle \qquad L_{-n}^2|h\rangle$$

The 2×2 matrix of inner products has

$$\det = 4n^3h^2(4h - 5n)$$

which goes negative for $h \neq 0$ and $n \gg 1$. □

If $\mathcal{O}$ has a conformal weight of $(h, 0)$ then $\mathcal{O}$ is holomorphic:

$$\left[\bar{L}_{-1}^{(\bar{z})}, \mathcal{O}(z, \bar{z})\right] = \bar{\partial}\mathcal{O} = 0$$

And so we can expand

$$\mathcal{O}(z) = \sum_{r \in \mathbb{Z} - h} \mathcal{O}_r z^{-r-h} \qquad \mathcal{O}_r = \oint \frac{dz}{2\pi i} z^{h+r-1}\mathcal{O}(z)$$

$$\mathcal{O}_r|0\rangle = 0 \qquad \text{for } r \geq -h + 1$$

Now consider the matrix of inner products at level $N = 2$

$$A = \begin{pmatrix} \langle\mathcal{O}|L_2L_{-2}|\mathcal{O}\rangle & \langle\mathcal{O}|L_2L_{-1}^2|\mathcal{O}\rangle \\ \langle\mathcal{O}|L_1^2L_{-2}|\mathcal{O}\rangle & \langle\mathcal{O}|L_1^2L_{-1}^2|\mathcal{O}\rangle \end{pmatrix}$$

Using the Virasoro algebra (7) we can calculate the matrix elements

$$\begin{aligned} \langle\mathcal{O}|\,[L_2, L_{-2}]\,|\mathcal{O}\rangle &= \langle\mathcal{O}|4L_0 + \frac{c}{12}(2^3 - 2)|\mathcal{O}\rangle \\ &= 4h + \frac{c}{2} \end{aligned}$$

$$\begin{aligned} \langle\mathcal{O}|L_1^2L_{-1}^2|\mathcal{O}\rangle &= \langle\mathcal{O}|\,(L_1 2L_0L_{-1} + L_1L_{-1}L_1L_{-1})\,|\mathcal{O}\rangle \\ &= \langle\mathcal{O}|\,(L_1L_{-1}2(h+1) + L_1L_{-1}2h)\,|\mathcal{O}\rangle \\ &= 4h(2h+1) \end{aligned}$$

$$\begin{aligned} \langle\mathcal{O}|L_2L_{-1}^2|\mathcal{O}\rangle &= \langle\mathcal{O}|3L_1L_{-1}|\mathcal{O}\rangle \\ &= 6h = \langle\mathcal{O}|L_1^2L_{-2}|\mathcal{O}\rangle \end{aligned}$$

Thus

$$A = \begin{pmatrix} 4h + \frac{c}{2} & 6h \\ 6h & 4h(2h+1) \end{pmatrix}$$

In a unitary theory the matrix of inner products, and thus its determinant, must be non-negative. The determinant of A is given by

$$\begin{aligned}\det A &= \left(4h + \frac{c}{2}\right)(4h(2h+1)) - 36h^2 \\ &= 4h\left(8h^2 + (c-5)h + \frac{c}{2}\right) \\ &= \frac{h}{8}\left(16h - 5 + c - \sqrt{(1-c)(25-c)}\right)\left(16h - 5 + c + \sqrt{(1-c)(25-c)}\right)\end{aligned}$$

When

$$h = \frac{1}{16}\left(5 - c \pm \sqrt{(1-c)(25-c)}\right)$$

$\det A = 0$ and we have a null vector

$$|\chi\rangle = \left(L_{-2} - \frac{3}{2(2h+1)}L_{-1}^2\right)|\mathcal{O}\rangle$$

Note that the roots are complex for $1 < c < 25$ and for $c \geq 25$ the roots only occur for $h \leq 0$, so the interesting case is $0 < c \leq 1$. For the special case $c = 1/2$

$$\det A = \frac{h}{8}(16h - 8)(16h - 1)$$

with roots $h = 0, \frac{1}{2},$ and $\frac{1}{16}$.

5.3.1 The Kac Formula

The determinant of the matrix of inner products at level N is known. Its roots are given by the Kac formula:

$$\begin{aligned}h_{n,m} &= \frac{c-1}{24} + \frac{1}{4}\left(n\alpha_+ + m\alpha_-\right)^2 \\ \alpha_\pm &= \frac{1}{\sqrt{24}}\left(\sqrt{1-c} \pm \sqrt{25-c}\right)\end{aligned}$$

where n and m are positive integers such that $nm \leq N$. The associated null vector has conformal weight

$$h = h_{n,m} + nm$$

and so occurs at level nm. For $0 < c \leq 1$ the values of h are real. It helps to parameterize

$$c = 1 - \frac{6}{s(s+1)}$$

under which

$$h_{n,m} = \frac{((s+1)n - sm)^2 - 1}{4s(s+1)}$$

Note that this is invariant under the symmetry

$$n \to s - n$$
$$m \to s + 1 - m$$

Since we are interested in (nontrivial) unitary representations of the Virasoro algebra we consider only

$$c > 0, \quad h \geq 0$$

For $1 < c < 25$, the roots $h_{n,m}$ are complex and for $c \geq 25$ they are negative. So $c > 1$ agrees with Kac *a priori* (the norms are strictly positive). For $c = 1$ we get vanishing norms for

$$h = \frac{k^2}{4}$$

but for $k \in \mathbb{Z}$ the norm never goes negative. It is only for $0 < c < 1$ that we have a potential problem with negative norms. The solution due to Friedan, Qiu, and Shenker is that only for discrete values of s is the representation unitary:

$$c = 1 - \frac{6}{s(s+1)} \qquad s = 3, 4, 5, \ldots$$

with $s(s-1)/2$ distinct roots

$$h_{n,m} = \frac{((s+1)n - sm)^2 - 1}{4s(s+1)}$$

where $1 \leq n \leq s - 1$ and $1 \leq m \leq s - n$.

5.4 The Ising Model

The smallest nonzero central charge for which the representation is unitary is $s = 3$ or $c = 1/2$. The roots are

$$h_{1,1} = 0, \quad h_{2,1} = \frac{1}{2}, \quad h_{1,2} = \frac{1}{16}$$

Recall that the CFT of the free massless Majorana fermion had a central charge of $c = 1/2$. As discussed the root $h_{1,1}$ is related to the vacuum state $|0\rangle$, and hence the identity operator. And we expect that $h_{2,1}$ will be related to ψ. What, then, is $h_{1,2}$ related to?

Exercise 5.7. *Consider the free massless Majorana fermion. Using*

$$T(z) = \frac{1}{2}\psi\partial\psi$$

compute

$$\left[L_{-2}^{(w)}, \psi(w)\right]$$

Show that this is a multiplier of $\partial^2\psi$.

It turns out that this CFT describes the critical theory of the Ising model in statistical mechanics. Hence it will be fruitful to study that theory.

The classic Ising model in statistical mechanics consists of a two dimensional square lattice of spins each taking on values

$$\sigma_i = \pm 1$$

with nearest neighbor interactions. There is a second order phase transition as the temperature increases. On coarse scales we can imagine the spin lattice as a continuous magnetization field $\sigma(x)$. In the disordered phase

$$\langle \sigma(x)\sigma(0)\rangle \underset{|x|\to\infty}{\sim} e^{-|x|/\xi}$$

where ξ is the correlation length. At the critical temperature $\xi \to \infty$ and instead the correlation functions decay as a power law

$$\langle \sigma(x)\sigma(0)\rangle \sim \frac{1}{|x|^{d-2+\eta}}$$

where η is the critical exponent[2].

Besides the magnetization another important observable is the energy operator:

$$\epsilon_i = \frac{1}{\#\text{n.n.}} \sum_{\substack{\text{nearest}\\ \text{neighbors}}} \sigma_i \sigma_j$$

where #n.n is the number of nearest neighbors (e.g. four on a 2D square lattice). At criticality

$$\langle \epsilon(x)\epsilon(0)\rangle \sim \frac{1}{|x|^{2(d-1/\nu)}}$$

Onsager's exact solution to the Ising model gives the critical exponents

$$\eta = \frac{1}{4}, \quad \nu = 1$$

σ is an operator with scaling dimension

$$\Delta_\sigma = h_\sigma + \bar{h}_\sigma = \frac{1}{8}$$

while ϵ has scaling dimension

$$\Delta_\epsilon = h_\epsilon + \bar{h}_\epsilon = 1$$

The $c = 1/2$ CFT (the free massless Majorana fermion) has just this primary field content

$$\begin{aligned} \sigma &= \mathcal{O}_{1,2} & h_\sigma &= \bar{h}_\sigma = \frac{1}{16} \\ \epsilon &= \mathcal{O}_{2,1} = \psi\bar{\psi} & h_\epsilon &= \bar{h}_\epsilon = \frac{1}{2} \end{aligned}$$

[2]Something was mentioned here about an anomalous dimension

5.4.1 The Four Point Function

Conformal invariance allows us to fully determine the four point function of fields in degenerate representations. Take, for example, the spin field operator, $\sigma(z,\bar{z})$, in the Ising model which has conformal weight

$$h = \bar{h} = \frac{1}{16}$$

Recall that PSL$(2,\mathbb{C})$ invariance allowed us to determine

$$\langle \sigma_1(z_1,\bar{z}_1)\sigma_2(z_2,\bar{z}_2)\sigma_3(z_3,\bar{z}_3)\sigma_4(z_4,\bar{z}_4)\rangle = |z_{13}|^{-1/4}|z_{24}|^{-1/4}\langle\sigma|\sigma(1)\sigma(x,\bar{x})|\sigma\rangle$$

where

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

In general, the x dependance in completely undetermined. Here, however, we can use the fact that σ has a null descendent at level two. Inserting this into the correlation function gives

$$0 = \langle\sigma|\sigma(1)\left(L_{-2}^{(x)} - \frac{4}{3}L_{-1}^{(x)2}\right)\sigma(x,\bar{x})|\sigma\rangle$$

or

$$\langle\sigma|\sigma(1)L_{-2}^{(x)}\sigma(x,\bar{x})|\sigma\rangle = \frac{4}{3}\frac{\partial^2}{\partial x^2}\langle\sigma|\sigma(1)\sigma(x,\bar{x})|\sigma\rangle$$

Now insert a stress tensor and taking the contour integral around x.

$$\begin{aligned}
&\oint_x \frac{dz}{2\pi i}\frac{z(1-z)}{z-x}T(z)\sigma(x,\bar{x}) \\
&\qquad = \oint_x \frac{dz}{2\pi i}\left[\frac{x(1-x)}{z-x} - (2x-1) - (z-x)\right]T(z)\sigma(x,\bar{x}) \\
&\qquad = x(1-x)L_{-2}^{(x)}\sigma(x,\bar{x}) - (2x-1)\partial\sigma(x,\bar{x}) - \frac{1}{16}\sigma(x,\bar{x})
\end{aligned}$$

From the above differential equation this is just

$$= \left[\frac{4}{3}x(1-x)\frac{\partial^2}{\partial x^2} - (2x-1)\frac{\partial}{\partial x} - \frac{1}{16}\right]\sigma(x,\bar{x})$$

Now take the contour integral around 0:

$$\oint_0 \frac{dz}{2\pi i}\frac{z(1-z)}{z-x}T(z)|\sigma\rangle = -\frac{1}{16x}|\sigma\rangle$$

and then around 1:

$$\oint_1 \frac{dz}{2\pi i}\frac{z(1-z)}{z-x}T(z)\sigma(1) = -\frac{1}{16(1-x)}\sigma(1)$$

and then around ∞:

$$\begin{aligned}
\langle\sigma|\oint_\infty \frac{dz}{2\pi i}\frac{z(1-z)}{z-x}T(z) &= \langle\sigma|\oint_0 \frac{dy}{2\pi i}\frac{y(1-y)}{1-xy}T(y) \\
&= \frac{1}{16}\langle\sigma|
\end{aligned}$$

where we used the transformation $y = 1/z$ in the first equality. Combining these by contour deformation we have

$$0 = \left[\frac{4}{3}x(1-x)\frac{\partial^2}{\partial x^2} - (2x-1)\frac{\partial}{\partial x} - \frac{1}{16x(1-x)}\right] \langle\sigma|\sigma(1)\sigma(x,\bar{x})|\sigma\rangle$$

and similarly for $\bar{x}$. Simplifying a bit we can write

$$\langle\sigma|\sigma(1)\sigma(x,\bar{x})|\sigma\rangle = |x(1-x)|^{-1/4} f(x,\bar{x})$$

where $f(x)$ satisfies the differential equation

$$x(1-x)f''(x) - \frac{1}{2}(2x-1)f'(x) + \frac{1}{16}f(x) = 0$$

and likewise for $f(\bar{x})$. This is a hypergeometric equation with regular singular points at 0, 1, and ∞. The solutions then are hypergeometric functions, but they happen to simplify in this case to algebraic ones:

$$f_{1,2}(x) = \left(1 \pm \sqrt{1-x}\right)^{1/2}$$

So we expect the general solution to $f(x,\bar{x})$ to be of the form

$$f(x,\bar{x}) = \sum_{i,j=1}^{2} a_{ij} f_i(x) f_j(\bar{x})$$

We would like to choose the a_{ij} so that $f(x,\bar{x})$ is single-valued. The solution is

$$a_{ij} = a\delta_{ij}$$

for some constant a. So finally we have

$$\langle\sigma|\sigma(1)\sigma(x,\bar{x})|\sigma\rangle = a|x(1-x)|^{-1/4}\left(\left|1+\sqrt{1-x}\right| + \left|1-\sqrt{1-x}\right|\right)$$

Letting $x \to 0$ and expanding

$$\begin{aligned} a|x(1-x)|^{-1/4}\left|1+\sqrt{1-x}\right| &= 2a|x|^{-1/4}|1+\cdots|^2 \\ a|x(1-x)|^{-1/4}\left|1-\sqrt{1-x}\right| &= \frac{a}{2}|x|^{3/4}|1+\cdots|^2 \end{aligned}$$

Now comparing this to the OPE of two σ's

$$\sigma(z,\bar{z})\sigma(w,\bar{w}) = \frac{1}{|z-w|^{1/4}}[1+\cdots] + c_{\sigma\sigma\epsilon}|z-w|^{3/4}[\epsilon(w,\bar{w})+\cdots]$$

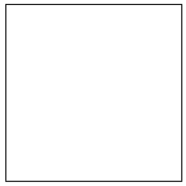
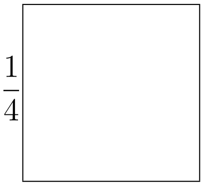
we can read off the value of a

$$a = \frac{1}{2}$$

as well as the coefficient of $\epsilon(w,\bar{w})$ that appears in the OPE

$$c_{\sigma\sigma\epsilon}^2 = \frac{1}{4} \implies c_{\sigma\sigma\epsilon} = \frac{1}{2}$$

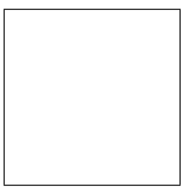
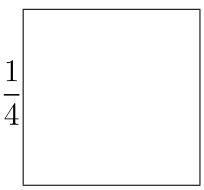
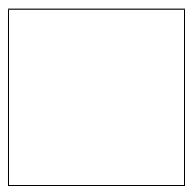
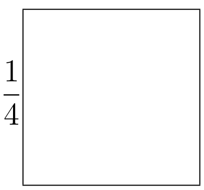
We indicate this result schematically by

 $+\frac{1}{4}$ 

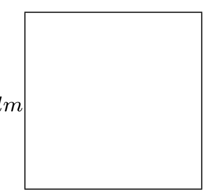
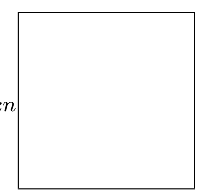
Now let $x \to 1$. For convenience define $y = 1 - x$.

$$\begin{aligned}
\lim_{x\to 1}\langle\sigma|\sigma(1)\sigma(x,\bar{x})|\sigma\rangle &= \lim_{y\to 0}\frac{1}{2}|y(1-y)|^{-1/4}\left(|1+\sqrt{y}|+|1-\sqrt{y}|\right) \\
&= \frac{1}{2}|y|^{-1/4}\left|1+\frac{y}{8}+\cdots\right|^2\left(\left|1+\frac{\sqrt{y}}{2}-\frac{y}{8}\right|^2+\left|1-\frac{\sqrt{y}}{2}-\frac{y}{8}\right|^2\right) \\
&= \frac{1}{2}|y|^{-1/4}\left[2+\frac{|y|}{2}+\cdots\right] \\
&= |y|^{-1/4}(1+\cdots)+\frac{1}{4}|y|^{3/4}(1+\cdots)
\end{aligned}$$

Note that cancellation between terms gives us the exact same result we obtained by taking $x \to 0$ limit. Schematically,

 $+\frac{1}{4}$  $=$  $+\frac{1}{4}$ 

This equality is, in fact, demanded by the associativity of the operator algebra, and is known as the *crossing symmetry* of the four point function. In general,

$\sum_m c_{ijm}c_{klm}$  $= \sum_n c_{jln}c_{ikn}$ 

This process generalizes to the n-point function:

$$\langle\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle$$

Namely, the n-point function will be

- a single-valued, real, analytic function of the z_i's,
- independent of the order of the operators $\mathcal{O}_i$,
- independent of the decomposition of the punctured sphere into 3-punctured spheres.

In this example (and for all $c < 1$ minimal models) we have a closed operator algebra with a finite number of conformal blocks. Specifically, we have the identity block, the block corresponding to the spin field operator, and the block corresponding to the energy operator

$$[1], \quad [\sigma], \quad [\epsilon]$$

with

$$\begin{aligned} [\sigma][\sigma] &= [1] + [\epsilon] \\ [\sigma][\epsilon] &= [\sigma] \\ [\epsilon][\epsilon] &= [1] \end{aligned}$$

The last two lines are determined by the $\sigma \to -\sigma, \epsilon \to \epsilon$ symmetry and the low/high temperature duality, $\sigma \leftrightarrow \mu$ under which $\epsilon \to -\epsilon$. Of course,

$$[*][1] = [*]$$

Life is not so easy for $c \geq 1$ where, in general, there are an infinite number of conformal blocks. In these situations we can

- look for extended chiral algebras (with the operators organized into a finite number of irreducible representations of this larger algebra),
- study free field theories,
- or muddle through.

Examples of extended chiral algebra's are the superconformal algebra and current algebras, which we now take up.

5.5 Current Algebras

Consider an OPE of the form

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{if^{ab}{}_c J^c(w)}{z-w} + \cdots$$

where the indices take on values in a finite dimensional representation of a Lie algebra $\mathfrak{g}$ (associated to a compact Lie group). The $f^{ab}{}_c$ are the structure constants of $\mathfrak{g}$. We normalize the generators of the Lie algebra such that the highest root has length$^2 = 2$.

The mode expansions are

$$J^a(z) = \sum_{n=-\infty}^{\infty} J^a_n z^{-n-1}$$

where

$$J^a_n = \oint \frac{dz}{2\pi i} z^n J^a(z)$$

Primary fields associated to some finite dimensional representation R of $\mathfrak{g}$ obey

$$J^a(z)\mathcal{O}(w,\bar{w}) = \frac{t^a_{(R)}\mathcal{O}(w,\bar{w})}{z-w} + \cdots$$

where $t^a_{(R)}$ is the matrix of the representation and $\mathcal{O}(w,\bar{w})$ is a vector of operators (fields).

$$\begin{aligned} J^a_n|\mathcal{O}\rangle &= 0 \qquad\qquad n \geq 1 \\ J^a_0|\mathcal{O}\rangle &= t^a_{(R)}|\mathcal{O}\rangle \end{aligned}$$

Inserting J^a into a correlation function gives

$$\langle J^a(z)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle = \sum_{i=1}^{n}\frac{t^a_{(R_i)}}{z-z_i}\langle\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle$$

This is obtained by taking the contour integral

$$\oint_C \frac{dz}{2\pi i}\frac{1}{w-z}\langle J^a(z)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_n(z_n,\bar{z}_n)\rangle$$

where C encloses all points.

The energy-momentum tensor is given by

$$T(z) = \frac{1/2}{k+c_A/2}{:}J^aJ^a{:}(z)$$

with

$${:}J^aJ^a{:}(w) = \lim_{z\to w}\left(J^a(z)J^a(w) - \frac{k|\mathfrak{g}|}{(z-w)^2}\right)$$

where $|\mathfrak{g}|$ is the dimension of the Lie algebra $\mathfrak{g}$, and c_A is the *quadratic Casimir in the adjoint representation*:[3]

$$f^{acd}f^{bcd} = c_A\delta^{ab}$$

Exercise 5.8. *Show that*

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + \cdots$$

and

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \cdots$$

where the central charge c is given by

$$c = \frac{k|\mathfrak{g}|}{k+c_A/2}$$

$\mathfrak{g}$	$\lvert\mathfrak{g}\rvert$	κ	index
$\mathfrak{so}(n)$	$\frac{1}{2}n(n-1)$	$n-2$	$\ell_n = 2$
$\mathfrak{su}(n)$	n^2-1	n	$\ell_n = 1$
$\mathfrak{sp}(n)$	$2n^2+n$	$n+1$	$\ell_{2n} = 1$
$\mathfrak{g}_2$	14	4	$\ell_7 = 2$
$\mathfrak{f}_4$	52	9	$\ell_{26} = 6$
$\mathfrak{e}_6$	78	12	$\ell_{27} = 6$
$\mathfrak{e}_7$	133	18	$\ell_{56} = 12$
$\mathfrak{e}_8$	248	30	$\ell_{248} = 30$

Table 1: Common Lie Algebras

For convenience we define the *dual Coxeter number* as

$$\kappa = c_A/\text{length}^2(\text{highest root})$$

which with our normalization is $c_A/2$. Table 1 lists some common Lie algebras, their dimensions, dual Coxeter numbers, and the index of their fundamental representation.

For a representation R we define

$$t^a_{(R)} t^a_{(R)} = c_R \mathbb{1}$$
$$\mathrm{tr}_R\, t^a t^a = d_R c_R$$
$$\mathrm{tr}_R\, t^a t^b = \ell_R \delta^{ab}$$

where c_R is the quadratic Casimir in the representation R, d_R is the dimension of R, and ℓ_R is the index of R. These are related by

$$c_R = \frac{\ell_R \lvert\mathfrak{g}\rvert}{d_R}$$

If R_1 and R_2 are two representations then

$$\ell_{R_1 \oplus R_2} = \ell_{R_1} + \ell_{R_2}$$
$$\ell_{R_1 \otimes R_2} = d_{R_1}\ell_{R_2} + d_{R_2}\ell_{R_1}$$

and

$$c_{R_1 \oplus R_2} = \frac{(\ell_{R_1} + \ell_{R_2})\lvert\mathfrak{g}\rvert}{d_{R_1} + d_{R_2}}$$
$$c_{R_1 \otimes R_2} = c_{R_1} + c_{R_2}$$

[3]Polchinski seems to define this as

$$-f^{ac}{}_d f^{bd}{}_c = c_A \delta^{ab}$$

The lengths of the roots are related to the dual Coxeter number by

$$\kappa = \frac{1}{r_{\mathfrak{g}}}\left(n_L + \left(\frac{S}{L}\right)^2 n_S\right)$$

where $r_{\mathfrak{g}}$ is the rank of $\mathfrak{g}$, and n_L (n_S) is the length of the long (short) root.

$$\left(\frac{S}{L}\right)^2 = \begin{cases} \frac{1}{2} & \text{for SO}(2n+1),\ \text{Sp}(n),\ \text{and F}_4 \\ \frac{1}{3} & \text{for } G_2 \end{cases}$$

The algebras $\text{SU}(n), \text{E}_6, \text{E}_7$, and E_8 are said to be *simply laced* as the roots all have the same length.

Returning to the energy-momentum tensor, the Virasoro generator $L_0^{(w)}$ obeys

$$\begin{aligned} L_0^{(w)}\mathcal{O}(w,\bar{w}) &= \oint_w \frac{dz}{2\pi i}(z-w)T(z)\mathcal{O}(w,\bar{w}) \\ &= \frac{1/2}{k+c_A/2}\oint_w \frac{dz}{2\pi i}(z-w){:}J^a(z)J^a(z){:}\mathcal{O}(w,\bar{w}) \\ &= \frac{1/2}{k+c_A/2}\oint_w \frac{dz}{2\pi i}(z-w)\frac{t_R^a t_R^a}{(z-w)^2}\mathcal{O}(w,\bar{w}) \\ &= \frac{c_R/2}{k+c_A/2}\mathcal{O}(w,\bar{w}) \end{aligned}$$

giving a conformal weight of $\frac{c_R/2}{k+c_A/2}$ for $\mathcal{O}$.

$$\begin{aligned} L_{-1}^{(w)}\mathcal{O}(w,\bar{w}) &= \frac{1/2}{k+c_A/2}\oint_w \frac{dz}{2\pi i}{:}J^a(z)J^a(z){:}\mathcal{O}(w,\bar{w}) \\ &= \frac{1}{k+c_A/2}\oint_w \frac{dz}{2\pi i}\frac{1}{z-w}J^a(z)t_R^a\mathcal{O}(w,\bar{w}) \\ &= \frac{1}{k+c_A/2}J^a_{-1}{}^{(w)}t_R^a\mathcal{O}(w,\bar{w}) \\ &= \partial_w\mathcal{O}(w,\bar{w}) \end{aligned}$$

Now take the contour integral

$$0 = \oint_C \frac{dz}{2\pi i}\frac{1}{(z-w_i)}\left\langle J^a(z)\frac{1}{k+c_A/2}t^a_{R_i}\mathcal{O}_1(w_1,\bar{w}_1)\cdots\mathcal{O}_n(w_n,\bar{w}_n)\right\rangle$$

where C encloses all points. This gives the *Knizhnik-Zamolodchikov equation*:

$$0 = \left[\frac{\partial}{\partial w_i} + \frac{1}{k+c_A/2}\sum_{j\neq i}\frac{t^a_{R_j}t^a_{R_i}}{w_j-w_i}\right]\langle\mathcal{O}_1(w_1,\bar{w}_1)\cdots\mathcal{O}_n(w_n,\bar{w}_n)\rangle$$

5.5.1 Example: $\mathrm{SU}_k(2)$

An $\mathrm{SU}(2)_k$ current algebra has a central charge of

$$c = \frac{3k}{k+2}$$

The representations are labeled by

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

The conformal weight of a primary state in the representation j is

$$h_j = \frac{j(j+1)}{k+2}$$

With our normalization convention

$$f^{ijk} = \sqrt{2}\epsilon^{ijk}$$

The quadratic Casimir in the fundamental representation is

$$c_{\mathrm{fund}} = \frac{3}{2} = (J_0^1)^2 + (J_0^2)^2 + (J_0^0)^2$$

The usual SU(2) generators are given by

$$J^{\pm} = \frac{1}{\sqrt{2}}(J^1 \pm J^2)$$
$$J^3 = \frac{1}{\sqrt{2}} J^0$$

and so

$$c_{\mathrm{fund}} = J_0^+ J_0^- + J_0^- J_0^+ + 2(J_0^3)^2$$

The OPE's of these generators are

$$J^3(z)J^3(w) = \frac{k/2}{(z-w)^2}$$
$$J^+(z)J^-(w) = \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}$$

The mode expansions satisfy

$$\left(J_n^{\pm}\right)^{\dagger} = J_{-n}^{\mp} \qquad \left(J_n^3\right)^{\dagger} = J_{-n}^3$$

and

$$\left[J_n^+, J_m^-\right] = 2J_{n+m}^3 + kn\delta_{n,-m}$$

We will label the primary states by $|j,m\rangle$. In order for the representation to be unitary we must have

$$\begin{aligned} J_0^-|j,-j\rangle &= 0 \\ J_0^+|j,\quad j\rangle &= 0 \end{aligned}$$

Consider

$$\begin{aligned} \|J_{-1}^-|j,-j\rangle\|^2 &= \langle j,-j|J_1^+J_{-1}^-|j,-j\rangle \\ &= \langle j,-j|(k+2J_0^3)|j,-j\rangle \\ &= (k-2j)\langle j,-j|j,-j\rangle \end{aligned}$$

So we must have

$$j \leq \frac{k}{2}$$

in order for the representation to be unitary.

One can show that the generators

$$J_1^+, \quad J_{-1}^-, \quad \tilde{J}_0^3 = J_0^3 + \frac{k}{2}$$

satisfy a SU(2) algebra. Considerations of unitarity show that we must have

$$k \in \mathbb{Z}^+$$

otherwise $(J_{-1}^-)^p$ will create a state of negative norm. The following conditions are then necessary and sufficient for unitarity

$$\begin{aligned} &k \in \mathbb{Z}^+ \\ &k \in \mathbb{Z}^+/2 \\ &j \leq \frac{k}{2} \end{aligned}$$

From our experience with SU(2) we expect the *fusion rule*

$$[\,j_1\,] \times [\,j_2\,] = \sum_{j_3=|j_1-j_2|}^{j_1+j_2} [\,j_3\,]$$

but this is not quite right because of the unitarity restriction. The correct rule is

$$[\,j_1\,] \times [\,j_2\,] = \sum_{j_3=|j_1-j_2|}^{\substack{\min(k/2-(j_1+j_2),\\ j_1+j_2)}} [\,j_3\,]$$

from the restriction $j \leq k/2$. This can be derived by inserting a null vector into the four point function, or by a more powerful method to be discussed later.

5.5.2 The Free Boson on a Circle

Consider the level one current algebra $\mathrm{SU}(2)_1$. The only allowed representations are $j = 0, 1/2$. The central charge is $c = 1$. It turns out that this theory can be related to our friend the free boson

$$S = \frac{1}{2\pi} \int d^2z \partial X \bar{\partial} X$$

Only now we let X be periodic (the free boson on a circle)

$$X(z, \bar{z}) \sim X(z, \bar{z}) + 2\pi R$$

The Virasoro primary field

$$\mathcal{O}_p(z, \bar{z}) = :e^{ipX}:$$

is only well defined for

$$p = \frac{n}{R}, \qquad n \in \mathbb{Z}$$

Recall that

$$J(z) = i\partial X \qquad T(z) = \frac{1}{2}:JJ(z):$$

$$J(z)J(w) = \frac{1}{(z-w)^2}$$

$$J(z)\mathcal{O}_p(w, \bar{w}) = \frac{p}{z-w}\mathcal{O}_p(w, \bar{w}) + \cdots$$

and similarly for

$$\bar{J}(\bar{z}) = i\bar{\partial} X$$

For the boson on circle we have a new primary field

$$\mathcal{O}_w(w, \bar{w})$$

which “winds” m times around the circle (with $m \in \mathbb{Z}$):

$$X(e^{2\pi i}z, e^{-2\pi i}\bar{z})\mathcal{O}_w(w, \bar{w}) = (X(z, \bar{z}) + 2\pi m R)\mathcal{O}_w(w, \bar{w})$$

The winding number m is a topological invariant. The contour integral

$$\begin{aligned} \oint_w dX \mathcal{O}_w(w, \bar{w}) &\equiv \oint_w (dz\partial X + d\bar{z}\bar{\partial} X)\mathcal{O}_w(w, \bar{w}) \\ &= 2\pi m R \end{aligned}$$

gives

$$\begin{aligned} \oint \frac{dz}{2\pi i} J(z)\mathcal{O}_w(w, \bar{w}) &= \frac{1}{2} m R \, \mathcal{O}_w(w, \bar{w}) \\ -\oint \frac{d\bar{z}}{2\pi i} \bar{J}(\bar{z})\mathcal{O}_w(w, \bar{w}) &= -\frac{1}{2} m R \, \mathcal{O}_w(w, \bar{w}) \end{aligned}$$

More generally we have primary fields with both momentum and winding:

$$\begin{aligned}
\oint \frac{dz}{2\pi i} J(z)\mathcal{O}_{n,m}(w,\bar{w}) &= \left(\frac{n}{R} + \frac{1}{2}mR\right)\mathcal{O}_{n,m}(w,\bar{w}) \\
&\equiv p_L\,\mathcal{O}_{n,m}(w,\bar{w}) \\
-\oint \frac{d\bar{z}}{2\pi i} \bar{J}(\bar{z})\mathcal{O}_{n,m}(w,\bar{w}) &= \left(\frac{n}{R} - \frac{1}{2}mR\right)\mathcal{O}_{n,m}(w,\bar{w}) \\
&\equiv p_R\,\mathcal{O}_{n,m}(w,\bar{w})
\end{aligned}$$

where L and R stand for left and right respectively. Note

$$\begin{aligned}
p &= \frac{1}{2}(p_L + p_R) = \frac{n}{R} \\
w &= \frac{1}{2}(p_L - p_R) = \frac{1}{2}mR
\end{aligned}$$

The left and right moving conformal weights of these primaries will, in general, be different:

$$\begin{aligned}
L_0^{(z)}\mathcal{O}_{n,m}(z,\bar{z}) &= \frac{1}{2}{p_L}^2\mathcal{O}_{n,m}(z,\bar{z}) \\
\bar{L}_0^{(z)}\mathcal{O}_{n,m}(z,\bar{z}) &= \frac{1}{2}{p_R}^2\mathcal{O}_{n,m}(z,\bar{z})
\end{aligned}$$

Now take

$$R = \sqrt{2}$$

The conformal weights are

$$\begin{aligned}
h_{n,m} &= \frac{1}{2}\left(\frac{n}{R} + \frac{1}{2}mR\right)^2 = \frac{1}{4}(n+m)^2 \\
\bar{h}_{n,m} &= \frac{1}{2}\left(\frac{n}{R} - \frac{1}{2}mR\right)^2 = \frac{1}{4}(n-m)^2
\end{aligned}$$

so the fields $\mathcal{O}_{\pm1,\pm1}$ have weights

$$\begin{aligned}
h &= \frac{1}{2}{p_L}^2 = 1 \qquad p_L = \pm\sqrt{2} \\
\bar{h} &= \frac{1}{2}{p_R}^2 = 0 \qquad p_R = 0
\end{aligned}$$

and so are holomorphic with conformal dimension 1. Call these fields $J^\pm(z)$.

$$J(z)J^\pm(w) = \frac{p_L}{z-w}J^\pm(w)$$

If we set

$$J^3(z) = \frac{1}{\sqrt{2}} J(z)$$

then we have

$$J^3(z) J^3(w) = \frac{1}{2} J(z) J(w) = \frac{1/2}{(z-w)^2}$$

Note

$$\mathcal{O}_{n,m}(z,\bar{z}) \mathcal{O}_{n',m'}(w,\bar{w}) = (z-w)^{p_L p'_L} (\bar{z}-\bar{w})^{p_R p'_R} :\mathcal{O}_{n,m}(z,\bar{z}) \mathcal{O}_{n',m'}(w,\bar{w}):$$

with

$$\begin{aligned} :\mathcal{O}_{n,m}(z,\bar{z}) \mathcal{O}_{n',m'}(w,\bar{w}): = \mathcal{O}_{n+n',m+m'}(w,\bar{w}) & \\ + (z-w) p_L :J(w) \mathcal{O}_{n+n',m+m'}(w,\bar{w}): & \\ + (\bar{z}-\bar{w}) p_R :\bar{J}(\bar{w}) \mathcal{O}_{n+n',m+m'}(w,\bar{w}): + \cdots & \end{aligned}$$

This is the same formula we had before, but now we must distinguish between p_L and p_R. Let us compute

$$\begin{aligned} J^+(z) J^-(w) &= \mathcal{O}_{1,1}(z,\bar{z}) \mathcal{O}_{-1,-1}(w,\bar{w}) \\ &= \frac{1}{(z-w)^2} \left[1 + (z-w)\sqrt{2} J(w) + \cdots \right] \end{aligned}$$

Recalling that $\mathcal{O}_{0,0}$ is the identity operator. Thus we have

$$J^+(z) J^-(w) = \frac{1}{(z-w)^2} + \frac{2J^3}{z-w} + \cdots$$

and so $J^{\pm}$, J^3 form an SU(2) current algebra with $k = 1$. That is, $\mathrm{SU}(2)_1$ can be realized as a free boson on a circle of radius $R = \sqrt{2}$, as advertised.

5.6 Ghost Systems

5.6.1 Weyl Fermion

Recall that the action for the free Weyl (complex) fermion[4]

$$S = \frac{1}{2\pi} \int d^2 z \psi^\dagger \partial_{\bar{z}} \psi$$

with

$$\overbrace{\psi^\dagger(z) \psi}(w) = -\frac{1}{z-w}$$

The stress tensor is given by

$$T = \frac{1}{2} (\psi^\dagger \partial \psi - \partial \psi^\dagger \psi)$$

[4]I've used $\psi^\dagger$ here for the complex conjugate of ψ instead of $\bar{\psi}$ to avoid confusion with right movers.

and the central charge is 1.

$$J = \psi^\dagger \psi$$

under which

$$q(\psi) = -q(\psi^\dagger) = 1$$

and

$$h(\psi) = h(\psi^\dagger) = \frac{1}{2}$$

We could just as well contemplate as system for which

$$h(\psi) \neq h(\psi^\dagger)$$

as long as their sum was unity

$$h(\psi) + h(\psi^\dagger) = 1$$

5.6.2 bc Ghosts

Consider the action

$$S = -\frac{1}{2\pi}\int d^2 z b \partial_{\bar{z}} c$$

with

$$\langle b(z)\, c(w) \rangle = \frac{1}{z-w}$$

and the stress tensor

$$T(z) = (1-\lambda)(\partial b)c - \lambda b(\partial c)$$

Computing the OPE's of T with b and c one obtains

$$\begin{aligned} T(z)b(w) &= \frac{\lambda b}{(z-w)^2} + \frac{\partial b}{z-w} + \cdots \\ T(z)c(w) &= \frac{(1-\lambda)c}{(z-w)^2} + \frac{\partial c}{z-w} + \cdots \end{aligned}$$

showing that the conformal weights of b and c are λ and $1-\lambda$ respectively.

Exercise 5.9. *Show that the OPE of T with itself is given by*

$$T(z)T(w) = \frac{-(6\lambda^2 - 6\lambda + 1)}{(z-w)^4} + \frac{2T}{(z-w)^2} + \frac{\partial T}{z-w}$$

The central charge of the system is then given by

$$\begin{aligned} c &= -2(6\lambda^2 - 6\lambda + 1) \\ &= 1 - 12(\lambda - \frac{1}{2})^2 \end{aligned}$$

Note that for

$$\lambda = \frac{1}{2} \quad c = 1$$
$$\lambda = 2 \quad c = -26$$

Without loss of generality we can take $\lambda \geq \frac{1}{2}$ (so that the conformal weight of b is greater than or equal to that of c). Note that for $\lambda \geq 1$

$$h(c) \leq 0 \quad \text{and} \quad c < 1$$

and so the theory in non-unitary. Recall that

$$\|L_{-n}|\mathcal{O}\rangle\|^2 = \Big(2nh + \frac{c}{12}(n^3 - n)\Big)\langle\mathcal{O}|\mathcal{O}\rangle$$

and positivity for $n \gg 1$ implied that $c \geq 0$, while positivity for $n = 1$ implied that $h \geq 0$. The case $h = 0$ implied that $\mathcal{O}$ could be identified with the identity operator. Actually for bc systems, things are a bit more subtle. To see this expand

$$b(z) = \sum_n b_n z^{-n-\lambda}$$
$$c(z) = \sum_n c_n z^{-n-1+\lambda}$$

with

$$\{c_n, b_m\} = \delta_{n+m,0}$$

Regularity as $z \to 0$ implies

$$b_n|0\rangle = 0 \qquad n \geq 1 - \lambda$$
$$c_n|0\rangle = 0 \qquad n \geq \lambda$$

Now look at $z \to \infty$. Under $y = 1/z$

$$b(z) \to \left(\frac{\partial y}{\partial z}\right)^{\lambda} b(y) = (-1)^{\lambda} \sum_n b_n y^{-n+\lambda}$$
$$c(z) \to \left(\frac{\partial y}{\partial z}\right)^{1-\lambda} c(y) = (-1)^{1-\lambda} \sum_n c_n y^{-n+1-\lambda}$$

So

$$\langle 0|b_n = 0 \qquad n \leq \lambda - 1$$
$$\langle 0|c_n = 0 \qquad n \leq -\lambda$$

We can then construct

$$\langle 0|\overbrace{c_{\lambda-1}c_{\lambda-2}\ldots c_{1-\lambda}}^{2\lambda-1}|0\rangle \neq 0$$

If we take

$$\langle 0|0\rangle = 0$$

and

$$\langle\tilde{0}| \equiv \langle 0|c_{\lambda-1}\dots c_{1-\lambda}$$

then we can define

$$\langle\tilde{0}|0\rangle = 1$$

So we have $2\lambda - 1$ oscillator modes which do not annihilate either vacuum. More generally, on a Riemann surface of genus g

$$(\#\text{ of } b \text{ zero modes}) - (\#\text{ of } c \text{ zero modes}) = (2\lambda - 1)(g - 1)$$

where a zero mode is a normalizable solution to

$$\bar{\partial} f = 0$$

and f transforms as a tensor of weight λ (or $1-\lambda$), i.e. $f(z)(dz)^\lambda$ is single-valued under $z \to z'(z)$. For a bosonic string, $\lambda = 2$ is of interest

$$1 = \langle 0|c_1 c_0 c_{-1}|0\rangle$$

The ghost number current for the bc system is given by

$$J(z) = -{:}bc{:}(z)$$

under which

$$q(c) = -q(b) = 1$$

The OPE of two current operators is given by

$$\begin{aligned} J(z)J(w) &= bc\ \ bc + bc\ \ bc + bc\ \ bc \\ &= \frac{1}{(z-w)^2} + \frac{b(z)c(w)}{z-w} + \frac{c(z)b(w)}{z-w} + \cdots \\ &= \frac{1}{(z-w)^2} + \partial bc + \partial cb + \cdots \end{aligned}$$

the last line holding because of the cancellation between the leading terms in single pole terms. We also have

$$\partial J = -\partial bc - b\partial c$$

so

$$\begin{aligned} T(z) &= \frac{1}{2}{:}JJ{:} + (\lambda - \frac{1}{2})\partial J \\ &= -\lambda b\partial c + (1-\lambda)\partial bc \end{aligned}$$

We now consider the bosonization of the bc ghost system.

$$J = i\partial\sigma$$

$$S_\sigma = \frac{1}{2\pi}\int d^2z \partial\sigma\bar{\partial}\sigma$$

The stress tensor is a little funky:

$$\begin{aligned} T &= \frac{1}{2}{:}JJ{:} + (\lambda - \frac{1}{2})\partial J \\ &= -\frac{1}{2}\partial\sigma\partial\sigma + i(\lambda - \frac{1}{2})\partial^2\sigma \end{aligned}$$

In either formulation $J(z)$ is not a conformal primary

$$T(z)J(w) = \frac{1-2\lambda}{(z-w)^3} + \frac{J}{(z-w)^2} + \frac{\partial J}{z-w} + \cdots$$

or

$$\delta_\epsilon J(w) + \oint_w \frac{dz}{2\pi i}\epsilon(z)T(z)J(w) = (\frac{1}{2} - \lambda)\epsilon''(w) + \epsilon' J + \epsilon\partial J$$

Exercise 5.10. *Check that under the finite transformation $z \to y(z)$*

$$J(z) \to \frac{dy}{dz}J(y) + (\frac{1}{2} - \lambda)\frac{d^2y/dz^2}{dy/dz}$$

The conserved charge is given by

$$Q_{bc} = J_0 = \oint_C \frac{dz}{2\pi i}J(z)$$

where C encloses all insertions. Now contract the contour around ∞:

$$y = \frac{1}{z} \qquad dz \to -\frac{1}{y^2}dy \qquad \frac{d^2y/dz^2}{dy/dz} = \frac{2y^3}{-y^2} = -2y$$

$$J_0 = -\oint \frac{dy}{2\pi i}\left(J(y) + \frac{1-2\lambda}{y}\right)$$

so

$$\langle 0|J_0 = -(1-2\lambda)\langle 0|$$

So the sum of the charges of all the insertions is equal to $(2\lambda - 1)$. Note that the charge of the vacuum on the left is not necessarily 0.

σ may look like a scalar field but it transforms inhomogeneously under conformal transformations.

$$i\partial_z\sigma \to \frac{dy}{dz}\, i\partial_y\sigma + \left(\frac{1}{2} - \lambda\right)\frac{d^2y/dz^2}{dy/dz}$$

We assume the same for right-movers. Thus,

$$\sigma \to \sigma - i\left(\frac{1}{2} - \lambda\right)\log\left|\frac{dy}{dz}\right|^2$$

If one replaces σ with $i\sigma$ everywhere this is a real shift and so makes a little more sense. Infinitesimally (and keeping only the holomorphic part)

$$\delta\sigma = \epsilon\partial\sigma + \frac{i}{2}(2\lambda - 1)\partial\epsilon$$

So

$$\begin{aligned}\delta S &= \int d^2z\, \partial\left(\epsilon\partial\sigma + \frac{1}{2}(2\lambda-1)\partial\epsilon\right)\delta\sigma + \partial\sigma\bar{\partial}\left(\epsilon\partial\sigma + \frac{i}{2}(2\lambda-1)\partial\epsilon\right)\\ &= \int d^2z\, \partial\left(\frac{1}{2}\epsilon\partial\sigma\bar{\partial}\sigma + \frac{i}{2}(2\lambda-1)\partial\epsilon\bar{\partial}\sigma + \frac{i}{2}(2\lambda-1)\epsilon\partial\bar{\partial}\sigma\right)\\ &\quad + \bar{\partial}\left(\frac{1}{2}\epsilon\partial\sigma\partial\sigma + \frac{i}{4}(2\lambda-1)\partial\epsilon\partial\sigma\right)\\ &\quad + \epsilon\bar{\partial}\left(-\frac{1}{2}\partial\sigma\partial\sigma + \frac{i}{2}(2\lambda-1)\partial^2\sigma\right)\end{aligned}$$

Hence

$$T(z) = -\frac{1}{2}\partial\sigma\partial\sigma + \frac{i}{2}(2\lambda-1)\partial^2\sigma$$

Calculating the OPE's we have

$$\begin{aligned}T(z)e^{i\sigma}(w) &= \frac{(1-\lambda)e^{i\sigma}}{(z-w)^2} + \frac{\partial(e^{i\sigma})}{z-w} + \cdots\\ T(z)e^{-i\sigma}(w) &= \frac{\lambda e^{-i\sigma}}{(z-w)^2} + \frac{\partial(e^{-i\sigma})}{z-w} + \cdots\end{aligned}$$

Giving the conformal weights

$$\begin{aligned}h(e^{i\sigma}) = \bar{h}(e^{i\sigma}) &= 1-\lambda \quad \Longrightarrow \quad e^{i\sigma} = c\bar{c}\\ h(e^{-i\sigma}) = \bar{h}(e^{-i\sigma}) &= \lambda \quad\quad\;\; \Longrightarrow \quad e^{-i\sigma} = b\bar{b}\end{aligned}$$

In general,

$$T(z)e^{i\rho\sigma}(w) = \frac{\frac{1}{2}\rho(\rho - 2\lambda + 1)e^{i\rho\sigma}}{(z-w)^2} + \frac{\partial(e^{i\rho\sigma})}{z-w} + \cdots$$

If you do not like inhomogeneous transforms of σ then define

$$\tilde{\sigma} = \sigma - \frac{i}{2}(2\lambda - 1)\log\sqrt{g}$$

The log term will transform inhomogeneous to cancel the inhomogeneous transformation of σ. $\tilde{\sigma}$ then transforms like a true scalar.

$$S_\sigma = \frac{1}{2\pi}\int d^2z\partial\sigma\bar{\partial}\sigma$$

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