Worldsheets, Riemann Surfaces, and Moduli

Jeffrey D. Olson

Department of Physics
University of Texas at Austin
Austin, Texas 78712

jdolson@physics.utexas.edu

Abstract

In this lecture we study the intrinsic geometry of the worldsheet in bosonic string theory and show that this geometry is that of a Riemann surface. We review the basic theory of Riemann surfaces including the uniformization theorem. We also look at the difficulties that the worldsheet geometry imposes on the task of calculating string scattering amplitudes, specifically moduli and conformal Killing vectors.

Contents

1 Introduction

Recall that the worldsheet of the bosonic string is a 2-dimensional manifold M equipped with a (Euclidean signature) metric g_{ab} and embedded in d-dimensional Minkowski spacetime via the maps $X^\mu(\sigma^1, \sigma^2)$. The Polyakov action for the string is given by

$$S = \frac{T}{2}\int_M d^2\sigma\sqrt{g}g^{ab}\partial_a X^\mu \partial_b X_\mu$$

Being a geometric object, the action is invariant under diffeomorphisms of the worldsheet. The group of all such diffeomorphisms is denoted $\mathrm{Diff}(M)$. This action actually possesses another symmetry which is special to the case of 2-dimensions: it is invariant under local rescalings of the metric

$$g_{ab} \mapsto e^{2\omega(\sigma^1,\sigma^2)} g_{ab}$$

where ω is any smooth function on M. These transformations are called **Weyl transformations** and the group of all such is denoted $\mathrm{Weyl}(M)$.

In quantizing the worldsheet these symmetries act like gauge symmetries of our theory. So when we calculate string scattering amplitudes we will need to fix the gauge so as to avoid overcounting surfaces that are gauge equivalent. This naturally leads to a study of the intrinsic geometry of the worldsheet, that is, properties of the worldsheet that are invariant under gauge transformations. Our goal in this lecture is to gain a broad overview this geometry in preparation for calculating string scattering amplitudes. We will also highlight the problems that one encounters in trying to fix the gauge symmetry.

I will assume in this lecture that the reader has a passing familiarity with the topology of surfaces (2-dimensional manifolds). For the uninitiated [1] is an excellent reference.

2 Conformal Geometry

2.1 Conformal Structure

Let M be a smooth manifold and let g_1, g_2 be two Riemannian metrics on M. We say that g_1 and g_2 are **Weyl equivalent** if

$$g_2 = e^{2\omega} g_1$$

for some $\omega \in C^\infty(M)$. This is clearly an equivalence relation on the set of Riemannian metrics on M. An equivalence class of metrics under this relation is known as a **conformal structure** on M. Diffeomorphisms of M which preserve this conformal structure are known as **conformal transformations**. The study of properties invariant under conformal transformations is called **conformal geometry**.

The geometry of the bosonic worldsheet is then that of a conformal surface, i.e. a 2-dimensional smooth manifold equipped with conformal metric. Two surfaces which are conformally equivalent represent the same worldsheet.

2.2 Conformal Flatness

A Riemannian manifold (M, g) is said to be **(locally) conformally flat** if every $x \in M$ has a coordinate neighborhood U in which g in Weyl equivalent to the flat metric:

$$g = e^{2\omega}(dx_1^2 + \cdots + dx_n^2)$$

Such coordinates are called **isothermal coordinates**. The 2-dimensional case is rather special in that one can *always* find isothermal coordinates:

Theorem 1. *Every 2-dimensional Riemannian manifold is locally conformally flat.*

Note that if two metrics are in the same conformal class (i.e. they are Weyl equivalent) then the same isothermal coordinates will work for both metrics. So the definition of conformal flatness depends only on the conformal class of the metric and makes sense in conformal geometry. The restatement of Theorem 1 in conformal geometry is that every conformal surface is conformally flat. It immediately follows that

Corollary 2. *And two conformal surfaces are locally conformally equivalent.*

There is an analogous statement in Riemannian geometry that says any two one-dimensional Riemannian curves are locally isometric. Of course, neither statement holds globally since the manifolds in question may not even be diffeomorphic (e.g. $\mathbb{R}$ and S^1). More fundamentally though, even if they are diffeomorphic they may not be globally equivalent (take two circles of different circumference). The question of when two surfaces are globally conformally equivalent will concern us more later.

3 Riemann Surfaces

Worldsheet geometry has many different formulations, only one of which is the description given in the previous section in terms conformal metrics on a surface. Another (at least in the orientable case) is in terms of complex curves, i.e. one-dimensional complex manifolds, also known as Riemann surfaces.[1] We will establish the equivalence of these two views in §3.2, but first we detour slightly to recall the notion of a complex manifold.

[1]There is a third powerful method which we will not have time to address here: that of algebraic curves. See [2] for more information.

3.1 Complex Structure

Let M be a $2m$-dimensional topological manifold, and let $\{U_i, \phi_i\}$ be a complex-valued atlas on M. We say that the atlas is **complex analytic** if the transition functions $\phi_i \circ \phi_j^{-1}$ are all holomorphic on their domain of definition. A maximal such atlas is called a **complex structure** on M. A manifold together with a complex structure is called a **complex manifold**. The complex dimension of M is defined as $\dim_{\mathbb{C}} M = m$. It is half the dimension of M considered as a real manifold. Clearly, complex manifolds must have even real dimension.

The natural morphisms between complex manifolds are maps whose coordinate expressions

$$\phi_i f \phi_j^{-1} \colon \mathbb{C}^m \to \mathbb{C}^n$$

are holomorphic. Such maps are naturally called **holomorphic maps**. The definition is independent of the special coordinates chosen.

An alternative way of defining a complex structure is by way of a tensor field called the **almost complex structure**, which is a tensor of type $(1, 1)$ with the property that

$$J^i{}_k J^k{}_j = -\delta^i{}_j$$

at each point $p \in M$. A manifold with an almost complex structure may not quite be a complex manifold, hence the qualifier *almost*. However, a necessary and sufficient condition is that the **Nijenhuis tensor** defined by J,

$$N(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY]$$

vanish identically on M. For a proof of this theorem and a proof that this definition of a complex manifold is identical to the one above see any text on complex manifolds.

Before moving on to Riemann surfaces we mention one important fact about complex manifolds.

Theorem 3. *Every complex manifold is orientable.*

This is essentially because as Lie groups, $\mathrm{GL}(n, \mathbb{C})$ is connected, whereas $\mathrm{GL}(n, \mathbb{R})$ is not. Accordingly, a complex structure on a manifold M induces a natural orientation on M.

3.2 Riemann Surfaces as Conformal Surfaces

Now we restrict to the case of one complex dimension (two real dimensions). A one-dimensional complex manifold is called a **Riemann surface**. Like all complex manifolds, Riemann surfaces are orientable.

In complex analysis, bijective holomorphic maps between Riemann surfaces are called conformal maps. The name is not accidental, but rather completely consistent with our earlier definition. On every Riemann surface we can introduce a canonical conformal metric, so that bijective holomorphisms become conformal transformations. In the same way every oriented conformal surface admits a canonical complex structure making it into a Riemann surface. In effect,

Theorem 4. *The category of Riemann surfaces is equivalent to the category of oriented conformal surfaces.*

We prove this in two steps:

Lemma 5. *Every metric g on a oriented surface M determines a unique complex structure on M. Furthermore, the complex structure depends only on the conformal class of the metric.*

Proof. The easiest way to see this is by using the almost complex structure. If g is the metric, then define

$$J^i{}_j = \sqrt{g} g^{ik} \epsilon_{kj}$$

Such a tensor is always global well-defined on an orientable manifold. Using the identity $\epsilon^{ik}\epsilon_{kj} = -\delta^i{}_j$ it is easy to see that $J^2 = -1$ and so defines an almost complex structure on M. One can show that the Nijenhuis tensor vanishes identically in two real dimensions, so that any almost complex structure defines a complex structure. Furthermore, J is clearly invariant under the Weyl transformation $g \mapsto e^{2\omega} g$, and so depends only on the conformal class of the metric.

Alternatively, we can use isothermal coordinates to define the complex structure more directly. Let $\{U_i\}$ be a cover of M by isothermal coordinates. In each coordinate patch g takes the form

$$g = e^{2\omega_i}(dx_i^2 + dy_i^2)$$

where dx_i, dy_i are the coordinates in U_i. On the overlap between two patches $U_i \cap U_j$ we must have

$$e^{2\omega_i}(dx_i^2 + dy_i^2) = e^{2\omega_j}(dx_j^2 + dy_j^2)$$

This leads to two possibilities: either

$$\frac{\partial x_j}{\partial x_i} = +\frac{\partial y_j}{\partial y_i} \qquad \frac{\partial x_j}{\partial y_i} = -\frac{\partial y_j}{\partial x_i}$$

or

$$\frac{\partial x_j}{\partial x_i} = -\frac{\partial y_j}{\partial y_i} \qquad \frac{\partial x_j}{\partial y_i} = +\frac{\partial y_j}{\partial x_i}$$

One notes that the first set of equations are the holomorphic Cauchy-Riemann equations while the second set are the corresponding anti-holomorphic equations. Since our manifold is assumed to be oriented we can use this orientation to ensure that the second possibility never occurs (by orienting x_i and y_i). Now letting $z_i = x_i + iy_i$ in each patch we see that we have a complex-valued atlas with holomorphic transition functions, i.e. a complex structure. As two metrics in the same conformal class define the same isothermal coordinates this structure depends only on the conformal class of the metric. □

Lemma 6. *Every Riemann surface M admits a Riemannian metric compatible with the complex structure of M. The metric is unique up to conformal equivalence.*

Proof. Let f be any non-constant meromorphic function on M (It is a basic theorem of Riemann surfaces that these always exist). Then except for at the poles and critical points (points at which $df = 0$) of f we can always choose the metric to be

$$ds^2 = |df|^2$$

Since the poles and critical points of f form a discrete set $\{p_i\}$ we can cover each with a small coordinate patch z_i and an associated smooth cutoff function ω_i vanishing outside some small neighborhood of p_i. Together with patches the metric takes the form

$$ds^2 = |df|^2 \left(1 - \sum_i \omega_i\right) + \sum_i \omega_i |dz_i|^2$$

Clearly, a different choice of meromorphic function or coordinate patch will only change the metric by an overall conformal factor. It is also not hard to see that every metric compatible with the complex structure must be of this form. □

Thus, for an oriented surface M, one can think of a conformal structure on M as either a conformal metric or as a complex structure. One determines the other. Likewise, conformal maps can be thought of as diffeomorphisms preserving the conformal metric or as bijective holomorphisms. We will use the two notions interchangeably from here on.

3.3 Möbius Transformations

Riemann surfaces have been long studied and hence there is a fairly elaborate theory underlying them including a classification theorem known as uniformization. Before we get to the uniformization theorem itself we begin by classifying the simple connected Riemann surfaces. The result, both beautiful and simple, is that

Theorem 7. *Up to conformal equivalence there are precisely three simply connected Riemann surfaces:*

1. *the Riemann sphere,* $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,
2. *the complex plane,* $\mathbb{C}$, *and*
3. *the unit disk,* $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

For a proof consult [2]. Note the first is not even topologically equivalent to the other two (as it is compact). The second are inequivalent as a result of Liouville's theorem: any bounded, entire function on $\mathbb{C}$ must be constant. Thus, there can be no conformal map from $\mathbb{C}$ onto $\mathcal{D}$. The Riemann Mapping Theorem tells us that any simply connected, open, proper subset of $\mathbb{C}$ is conformally equivalent to $\mathcal{D}$. For many purposes it is convenient to use the upper half plane

$$\mathcal{U} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$$

rather than $\mathcal{D}$ as the model for the third case. We will use the two interchangeably.

Another essential ingredient in our classification of Riemann surfaces will be the automorphism groups of the three simply connected surfaces. Recall that the automorphism group $\text{Aut}(M)$ of a Riemann surface M is the set of conformal diffeomorphisms of that surface, that is, diffeomorphisms preserving the complex structure. Note that every bijective holomorphic map from M to itself is a automorphism since the inverse of a holomorphic map must also be holomorphic.

3.3.1 The Riemann Sphere

Any holomorphic map from a surface M into $\widehat{\mathbb{C}}$ that is not constant ∞ is called a meromorphic function. Thus an automorphism of $\widehat{\mathbb{C}}$ is a bijective meromorphic map. Meromorphisms on $\widehat{\mathbb{C}}$ can also be regarded as $\mathbb{C}$-valued functions that have only poles for singularities. Such a function can always be written as a ratio of two polynomials (with no common zeros)

$$f(z) = \frac{P(z)}{Q(z)}$$

If f is to be bijective then it is necessary that it have only simple zeros and poles. Around any higher order zero or pole f will be multi-valued. Thus both P and Q must be linear polynomials. In other words, f must be of the form

$$f(z) = \frac{az+b}{cz+d}$$

for $a, b, c, d \in \mathbb{C}$. It is easy to see that such a map is bijective iff $ad - bc \neq 0$. Transformations of this type are called **Möbius transformations**. The set of all such transformations forms a group under composition. This group is then the automorphism group of the Riemann sphere, $\mathrm{Aut}(\widehat{\mathbb{C}})$.

There is another useful way to look at the group of Möbius transformations using projective geometry. Recall that complex projective space, $\mathbb{CP}^{n-1}$, is defined as a quotient of $\mathbb{C}^n - \{0\}$ under the equivalence relation $(z_1, \ldots, z_n) \sim (\lambda z_1, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. Likewise, one can define the projective general linear group $\mathrm{PGL}(n, \mathbb{C})$ as the group $\mathrm{GL}(n, \mathbb{C})$ modulo scalar multiples of the identity matrix λI. The natural action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n - \{0\}$ induces a natural action of $\mathrm{PGL}(n, \mathbb{C})$ on $\mathbb{CP}^{n-1}$ in the obvious manner. The reader can check that this operation is well-defined. In the special case $n = 2$, we have an action of $\mathrm{PGL}(2, \mathbb{C})$ on $\mathbb{CP}^1$. But note that the space $\mathbb{CP}^1$ is simply the Riemann sphere. Indeed, according to Theorem 7 every compact, simply-connected, one-dimensional complex manifold is conformally equivalent to the Riemann sphere. Explicitly, the isomorphism $\mathbb{CP}^1 \cong \widehat{\mathbb{C}}$ is given by

$$[z_1, z_2] \leftrightarrow z_1/z_2$$

where $z/0 = \infty$ as usual. Via this identification, the group $\mathrm{PGL}(2, \mathbb{C})$ acts on $\widehat{\mathbb{C}}$ in a natural fashion:

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad \mapsto \quad z' = \frac{az+b}{cz+d}$$

One immediately notes that this is just a Möbius transformation. The group of all Möbius transformations is then isomorphic to $\mathrm{PGL}(2, \mathbb{C})$, and composition in $\mathrm{Aut}(\widehat{\mathbb{C}})$ corresponds to matrix multiplication in $\mathrm{GL}(2, \mathbb{C})$. Note that given any matrix in $\mathrm{GL}(2, \mathbb{C})$ we can always multiply by a constant to get a matrix with positive determinant, so that the group $\mathrm{PGL}(2, \mathbb{C})$ is isomorphic to $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$. In summary, we have

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \mathrm{PGL}(2, \mathbb{C}) = \mathrm{PSL}(2, \mathbb{C})$$

From the last identification it is clear that $\mathrm{Aut}(\widehat{\mathbb{C}})$ is a 3-dimensional complex Lie group.

3.3.2 The Complex Plane

The automorphism or the complex plane are simply going to be those Möbius transformations that fix the point at infinity. Since a general Möbius transformation maps $\infty \to a/c$ we see that ∞ is fixed precisely when $c = 0$. Multiplying by an overall constant so that $d = 1$ gives us the transformation

$$f(z) = az + b$$

where $a \in \mathbb{C}^\times = \mathbb{C} - \{0\}$. This group is simply the group of affine transformations of the plane, $\mathrm{Aff}(1, \mathbb{C})$. It is a 2-dimensional complex Lie group. We have then

$$\mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}(1, \mathbb{C})$$

3.3.3 The Unit Disk

First consider the upper half plane, $\mathcal{U}$. We want to consider the set of all Möbius transformations that map $\mathcal{U}$ onto itself. By continuity, such a transformation must also map the real circle, $\mathbb{R} \cup \{\infty\}$, onto itself. For the same reasons given in the complex case the set of all such transformations will form the group $\mathrm{PGL}(2, \mathbb{R})$, i.e. Möbius transformations for which $a, b, c, d \in \mathbb{R}$. Note, however, that unlike the complex case $\mathrm{PGL}(2, \mathbb{R}) \ncong \mathrm{PSL}(2, \mathbb{R})$ as the former has two disconnected components, according to whether the determinant is positive or negative. The component of the identity is isomorphic to the $\mathrm{PSL}(2, \mathbb{R})$ and maps the upper half plane onto itself, while the other component swaps the upper half plane with the lower half plane. The automorphism group of $\mathcal{U}$ is then $\mathrm{PSL}(2, \mathbb{R})$. Mapping $\mathcal{U}$ to the unit disk $\mathcal{D}$ via

$$z \mapsto \frac{i - z}{i + z}$$

we see that the group $\mathrm{PSL}(2, \mathbb{R})$ maps onto the group $\mathrm{PSU}(1, 1)$ consisting of transformations of the form

$$z \mapsto \frac{az + \bar{b}}{bz + \bar{a}} \qquad |a|^2 - |b|^2 = 1$$

So the automorphism group of $\mathcal{D}$ is the group $\mathrm{PSU}(1, 1)$, a 3-dimensional *real* Lie group:

$$\mathrm{Aut}(\mathcal{D}) = \mathrm{PSU}(1, 1) \cong \mathrm{PSL}(2, \mathbb{R})$$

3.4 Uniformization

Having classified the simply connected Riemann surfaces we move on to the rest. Here the story is very similar to the classification of topological surfaces. Given a 2-dimensional manifold M, one finds a simply-connected surface Σ and a surjective, local homeomorphism

$$p \colon \Sigma \to M$$

such that every point in M has a neighborhood covered by a countable number of "sheets" in Σ. The surface Σ is called the **universal covering space** of M and p is called a **covering map**.

Furthermore, one can always find a group of homeomorphisms G of Σ such that the quotient space Σ/G is homeomorphic to M. The group G is called the **covering group** of M and turns out to be isomorphic to the fundamental group of M, $G \cong \pi_1(M)$. In turn, given any group of homeomorphisms G that acts in a "sufficiently nice" manner on Σ will give rise to a surface Σ/G with the natural projection map as the covering map. One can show that all surfaces arise in this manner. Before discussing the complex case we qualify what we mean by "sufficiently nice".

Let a group G act continuously on a space M. G is said to act **freely discontinuously** at a point $x \in M$ if the exists a neighborhood U of x such that

$$gU \cap U = \varnothing \text{ for all } g \in G - \{1\}$$

Clearly, if G acts freely discontinuously on all of M then it acts freely on M. One can show that if G acts freely discontinuously on a manifold M then the quotient space M/G is also a manifold.

From here the transition to Riemann surfaces is relatively straightforward. Let M be a Riemann surface and let Σ be its universal cover. Since the covering map $p\colon \Sigma \to M$ is a local homeomorphism, it pulls back the complex structure on M to a unique complex structure on Σ so that p is a holomorphic map. The covering group G then becomes a group of conformal automorphisms of Σ, that is, a subgroup of $\mathrm{Aut}(\Sigma)$. Since Theorem 7 tells us all the possibilities for Σ we have established the following general classification theorem:

Theorem 8 (Uniformization Theorem). *Every Riemann surface M is conformally equivalent to Σ/G where*

$$\Sigma = \begin{cases} \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \\ \mathbb{C} \\ \mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\} \end{cases}$$

and G is a subgroup of $\mathrm{Aut}(\Sigma)$ *that acts freely discontinuously on Σ. Furthermore, $G \cong \pi_1(M)$.*

In the topological case it is straightforward to show that two different covering groups, G_1 and G_2, determine homeomorphic surfaces if and only if the are conjugate, that is, $G_2 = gG_1g^{-1}$ for some homeomorphism g. The equivalent statement holds for Riemann surfaces:

Theorem 9. *Two Riemann surfaces are conformally equivalent iff they have same universal cover Σ and their respective covering groups are conjugate in* $\mathrm{Aut}(\Sigma)$.

4 Gauge Fixing

According to Corollary 2 every worldsheet is locally equivalent under the action the gauge group. This means that locally we are free to choose any metric we like, in particular we can always choose a metric proportional to the flat coordinate metric. This is known as **conformal gauge fixing**. The hope being, that in doing so we can eliminate the gauge symmetry in the path integral. There are two obvious problems with this: globally it doesn't work since not every conformal surface is globally equivalent, and locally there may be residual gauge symmetries, that is, transformations that preserve the conformal gauge choice. We will need to investigate both of these problems and their consequences for the path integral more carefully.

4.1 The Gauge Group

Recall that the gauge group of the Polyakov action is generated by the group of Weyl transformations of the metric, Weyl(M), together with diffeomorphisms of the worldsheet, Diff(M).[2] One can show that this group is actually a semidirect product of Weyl by Diff, usually denoted Diff $\ltimes$ Weyl. The multiplication law is given by

$$(f, e^{2\omega})(f', e^{2\omega'}) = (f \circ f', e^{2(\omega \circ f')} e^{2\omega'})$$

The presence of the f' on the RHS is what makes this a semidirect product instead of a direct product.[3]

For a given worldsheet M, the gauge group Diff $\ltimes$ Weyl acts both on the space $\mathcal{E}$ of all embeddings $X\colon M \to \mathbb{R}^d$ as well as the space $\mathcal{G}$ of all Riemannian metrics on M. The action of Diff $\ltimes$ Weyl on $\mathcal{E}$ is given by

$$X \cdot (f, e^{2\omega}) = f^* X = X \circ f$$

while the action on $\mathcal{G}$ is given by

$$g \cdot (f, e^{2\omega}) = e^{2\omega} f^* g$$

Note that in both cases the action is naturally from the right because the pullback reverses direction.

Naïvely the Polyakov path integral is defined over the entire $\mathcal{E} \times \mathcal{G}$ space, but because of the gauge symmetry this is an enormous overcounting. To cancel this overcounting we should divide by the (infinite) volume of the gauge group in the path integral

$$\int_{\mathcal{E} \times \mathcal{G}} \frac{[dX dg]}{V_{\text{Diff} \ltimes \text{Weyl}}} \exp\left(-S[X, g]\right)$$

More rigorously, we would like to integrate over the quotient space $(\mathcal{E} \times \mathcal{G}) / \text{Diff} \ltimes \text{Weyl}$ and avoid the overcounting in the first place, however defining a good measure on the quotient can be difficult.

Classically, at least, we can eliminate the Weyl symmetry in a fairly straightforward manner. To proceed, we quote a theorem on group actions:

Theorem 10. *Let a group G act on a space Y and let K be a normal subgroup of G, then the quotient group G/K acts on the orbit space Y/K in the natural fashion:*

$$(Kg)(Ky) = (Kgy)$$

[2] In the orientated case we should really only consider the group $\text{Diff}^+(M)$ of orientation-preserving diffeomorphisms.

[3] In general, a group G is said to be a semidirect product of K by Q if $K \lhd G$, $Q \leq G$, $KQ = G$, and $K \cap Q = 1$. This is usually denoted $G = K \rtimes Q$. If Q is also a normal subgroup then the semidirect product is just the direct product.

Furthermore, the orbit spaces of these actions are naturally isomorphic:

$$\frac{Y}{G} \cong \frac{Y/K}{G/K}$$

We saw an example of this earlier with the action of $\mathrm{PGL}(n,\mathbb{C})$ on $\mathbb{CP}^{n-1}$. In the present situation $G = \mathrm{Diff} \ltimes \mathrm{Weyl}$, $Y = \mathcal{E} \times \mathcal{G}$ and $K = \mathrm{Weyl}$. Note that by definition of the semidirect product, the Weyl group is a normal subgroup of Diff ⋉ Weyl and the quotient group is isomorphic to Diff:

$$\mathrm{Diff} \cong \frac{\mathrm{Diff} \ltimes \mathrm{Weyl}}{\mathrm{Weyl}}$$

An element of $\mathcal{G}/\,\mathrm{Weyl}$ is just a conformal metric on M, so the quotient

$$\mathcal{J} \cong \mathcal{G}/\,\mathrm{Weyl}$$

is the space of all conformal structures on M. Also note that the Weyl group acts trivially on the space of embeddings $\mathcal{E}$. We then have a natural action of Diff on $\mathcal{E} \times \mathcal{J}$ such that

$$\frac{\mathcal{E} \times \mathcal{G}}{\mathrm{Diff} \ltimes \mathrm{Weyl}} \cong \frac{\mathcal{E} \times \mathcal{J}}{\mathrm{Diff}}$$

4.2 Global Problems – Moduli

The global problem with gauge fixing is that the action of Diff ⋉ Weyl on $\mathcal{G}$ is not *transitive*, i.e. there are metrics in $\mathcal{G}$ which are inequivalent under the action of Diff ⋉ Weyl. If the action were transitive we could simply choose any fixed metric on the worldsheet and eliminate the integration over $\mathcal{G}$ in the path integral.

We define the **moduli space** of a surface M to be the quotient of the space of metrics $\mathcal{G}$ by the action of the Diff ⋉ Weyl group:

$$\mathcal{M} = \frac{\mathcal{G}}{\mathrm{Diff} \ltimes \mathrm{Weyl}}$$

One can think of $\mathcal{M}$ as the space of inequivalent metrics on M. If the action were transitive then $\mathcal{M}$ would be a 1-point set. We shall see later that the moduli space of a compact surface is actually a quotient of a complex manifold. As such it will be labeled by a set of complex parameters called **moduli**.

Working modulo Weyl we can define the moduli space of M to be the quotient of the space of conformal structures on M by the action of the diffeomorphism group.

$$\mathcal{M} = \frac{\mathcal{J}}{\mathrm{Diff}}$$

In this setting $\mathcal{M}$ becomes the space of inequivalent complex structures on M. This definition is, of course, equivalent to the preceding one. In light of Theorem 9 we can also think the moduli space as the set of conjugacy classes of covering groups of M. We shall use this point of view to calculate the moduli space of a genus 1 surface.

4.3 Local Problems – Conformal Killing Vectors

The local problem with gauge fixing is that the action of Diff $\ltimes$ Weyl on $\mathcal{G}$ is not *free*, i.e. there are, in general, some Diff $\ltimes$ Weyl transformations that leave a given metric invariant. This means that even if we fix the metric to a given form (by projecting onto the gauge slice) there will be a residual Diff $\ltimes$ Weyl symmetry that needs to be accounted for. For a given surface (M, g) the subgroup of Diff $\ltimes$ Weyl that preserves (M, g) is called the **conformal Killing group** (CKG) of that surface. The infinitesimal generators of the CKG are naturally called **conformal Killing vectors** (CKV's).

The above is still true modulo Weyl, i.e. the action

$$\mathcal{J} \times \text{Diff} \to \mathcal{J}$$

is not free in general; certain diffeomorphism will preserve the complex structure. In this setting the CKG simply becomes the group of all conformal transformations of M, often denoted $\text{Conf}(M)$. Note that this is a actually an isomorphism as there are no pure Weyl transformations that leave (M, g) invariant. In the language of Riemann surfaces the CKG is the group of transformations preserving the complex structure, i.e. the automorphism group, $\text{Aut}(M)$.

In the path integral, we will typically eliminate the residual symmetry associated with the CKG by using the symmetry to fix the positions of certain vertex operators (or punctures) on the Riemann surface.

5 Moduli Spaces

Having discussed the origin and meaning of moduli spaces, we should look at some actual examples. In this section we will calculate the simplest of moduli spaces: those for the genus 0 and genus 1 surfaces. We shall do so by making use of the uniformization theorem together with Theorem 9 to determine when two Riemann surfaces are conformally equivalent.

5.1 Genus 0

Recall that every element of $\text{PSL}(2, \mathbb{C})$ fixes at least one point of $\widehat{\mathbb{C}}$ (most elements fix two). Hence no subgroup of $\text{PSL}(2, \mathbb{C})$ can act freely discontinuously on $\widehat{\mathbb{C}}$. Therefore,

Theorem 11. *The only Riemann surface with $\widehat{\mathbb{C}}$ as the universal cover is $\widehat{\mathbb{C}}$ itself.*

It follows that any two surfaces of genus 0 are conformally equivalent. In other words the moduli space $\mathcal{M}_0$ of a genus 0 surface is a one-point set. At tree level in string theory we can always use the action of the Diff $\ltimes$ Weyl group to choose a metric of constant curvature $+1$. We will still have the problem of the residual gauge symmetry associated with the CKG. We come back to this in the next section.

5.2 Genus 1

To find the spaces whose universal cover is $\mathbb{C}$ we must find all the subgroups of $\text{Aut}(\mathbb{C})$ that act freely discontinuously on $\mathbb{C}$. We shall see that topologically there are exactly three distinct possibilities:

Theorem 12. *Let M be a Riemann surface whose universal cover is $\mathbb{C}$. Then M is conformally equivalent to $\mathbb{C}$, $\mathbb{C}^\times = \mathbb{C} - \{0\}$, or a torus.*

These correspond to covering groups isomorphic to $1, \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$. In the last case we shall find our first example of a nontrivial moduli space.

Note that an element of $\mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}(1, \mathbb{C})$,

$$z \mapsto az + b$$

acts freely iff $a = 1$. Otherwise there is a fixed point at $z = \frac{b}{1-a}$. Thus if G is to act freely discontinuously on $\mathbb{C}$ it must contain only elements of this form (called parabolic elements). With a little work one can show that G will be discrete iff it is trivial or it is generated by one or two such elements.

If G is generated by a single parabolic element $z \mapsto z + b$ we can always conjugate by $z \mapsto z/b$ to transform G to a group generated by $z \mapsto z + 1$. The Riemann surface $\mathbb{C}/G$ is just the punctured plane $\mathbb{C}^\times$ with the covering map $p \colon \mathbb{C} \to \mathbb{C}^\times$ given by

$$p(z) = \exp(2\pi i z)$$

Now consider the case where G is generated by two parabolic elements

$$z \mapsto z + \omega_1$$
$$z \mapsto z + \omega_2$$

Note that in order for G to be discrete we must have $\mathrm{Im}(\omega_2/\omega_1) \neq 0$. Such a group is isomorphic to a lattice in $\mathbb{C}$ which we denote

$$\Lambda(\omega_1, \omega_2) = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$$

The quotient $\mathbb{C}/\Lambda(\omega_1, \omega_2)$ is clearly isomorphic to a torus, $\mathcal{T}^2$.

In order to determine when two tori are equivalent we need determine when two lattice groups are conjugate in $\mathrm{Aut}(\mathbb{C})$. Before we answer that question, however, we should first determine when two lattices are actually *identical* (This is not obvious as the generators are not unique). For convenience we will always orient our lattices so that $\mathrm{Im}(\omega_2/\omega_1) > 0$.

Lemma 13. *Two oriented lattice bases (ω_1, ω_2) and (ω_1', ω_2') define the same lattice if and only if they are related by an $\mathrm{SL}(2, \mathbb{Z})$ transformation:*

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = A \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \qquad A \in \mathrm{SL}(2, \mathbb{Z})$$

Proof. Both ω_1', and ω_2' must be elements of $\Lambda(\omega_1, \omega_2)$ which means they can be written as linear combinations of ω_1 and ω_2:

$$\begin{pmatrix} \omega_2' \\ \omega_1' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_2 \\ \omega_1 \end{pmatrix} \qquad a, b, c, d \in \mathbb{Z}$$

Likewise, ω_1, and ω_2 must be linear combinations of ω_1', and ω_2'. So the matrix relating them must be invertible. In particular, this implies that $ad - bc = \pm 1$ (The determinant must be a nonzero integer whose inverse is also an integer). Since the bases are both oriented by assumption we can conclude that $ad - bc = 1$. Clearly, any such matrix will send one basis onto another. $\square$

Lemma 14. *The conjugacy class of* $\Lambda(\omega_1, \omega_2)$ *in* $\mathrm{Aut}(\mathbb{C})$ *is the set of lattices of the form* $\Lambda(\alpha\omega_1, \alpha\omega_2)$ *for* $\alpha \in \mathbb{C}^\times$.

Proof. Let $g(z) = \alpha z + \beta$ be an element of $\mathrm{Aut}(\mathbb{C})$ and let $h(z) = z + \omega_i$ be a generator of the covering group. Then

$$ghg^{-1}(z) = z + \alpha\omega_i$$

Thus all lattices conjugate to $\Lambda(\omega_1, \omega_2)$ are generated by $\alpha\omega_1$ and $\alpha\omega_2$ for $\alpha \in \mathbb{C}^\times$. □

In particular we can always choose $\alpha = 1/\omega_1$, so that every lattice is conjugate to one of the form $\Lambda(\tau) \equiv \Lambda(1, \tau)$ where $\tau = \omega_2/\omega_1$. Since we have oriented ω_1 and ω_2, we see that $\mathrm{Im}(\tau) > 0$. By Lemma 13, however, not every such lattice is different. Combining both lemmas we see that two lattices $\Lambda(\tau)$ and $\Lambda(\tau')$ are equivalent if and only if they are related by the Möbius transformation

$$\tau' = \frac{a\tau + b}{c\tau + d}$$

where $a, b, c, d \in \mathbb{Z}$. The group of all such transformations is just $\mathrm{PSL}(2, \mathbb{Z})$, which is often called the **modular group**. Elements of $\mathrm{PSL}(2, \mathbb{Z})$ are called **modular transformations**.

In summary, every compact genus 1 Riemann surface has a covering group $\Lambda(\tau)$ generated by $z \mapsto z + 1$ and $z \mapsto z + \tau$ where $\tau \in \mathcal{U}$. Two such surfaces are conformally equivalent iff τ and τ' are related by a $\mathrm{PSL}(2, \mathbb{Z})$ transformation. The complex parameter τ is known as a **modulus**. The space of possible τ's forms the moduli space for the torus. We have then established that:

Theorem 15. *The moduli space of the torus is* $\mathcal{U}/\mathrm{PSL}(2, \mathbb{Z})$.

The modular group $\mathrm{PSL}(2, \mathbb{Z})$ is known to be generated by two elements S, T such that

$$S^2 = (ST)^3 = 1$$

Specifically we can take $S(z) = -1/z$ and $T(z) = z + 1$. As a subgroup of $\mathrm{PSL}(2, \mathbb{R})$ the modular group clearly acts as a group of automorphisms of the upper half plane.

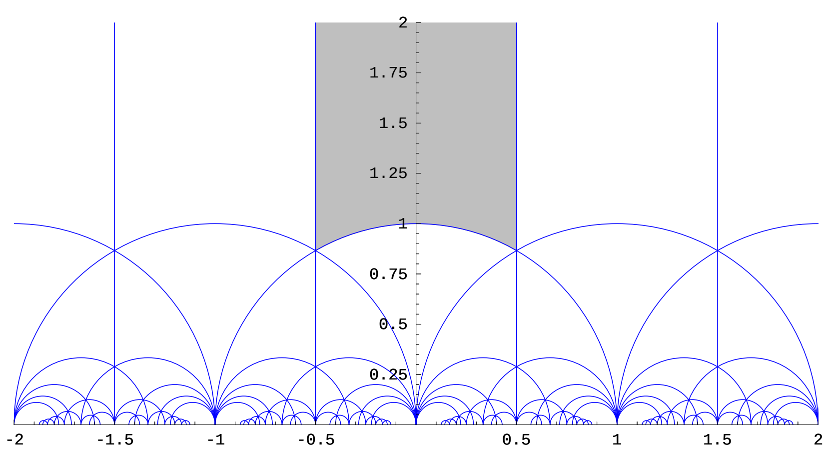

Figure 1: Fundamental region for the action of $\mathrm{PSL}(2, \mathbb{Z})$ on $\mathcal{U}$

To find a fundamental region for the action note that we can apply T repeated until z lies in the range $-\frac{1}{2} \leq z \leq \frac{1}{2}$. We can also apply S if necessary to insure that $|z| > 1$. It turns out this exhausts our freedom so that this region is a fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathcal{U}$. Every element of $\mathrm{PSL}(2,\mathbb{Z})$ will map this fundamental domain onto other (see Figure 1). The moduli space for the torus is then obtained by gluing the edges of this region together as dictated by the actions of S and T.

5.3 Higher Genus

Here life gets really complicated. The reason being that the automorphism group of the upper half plane $\mathrm{Aut}(\mathcal{U}) = \mathrm{PSL}(2,\mathbb{R})$ contains an abundance of subgroups that act freely discontinuously on $\mathcal{U}$. These are the so called fixed point free **Fuchsian groups**. The abelian Fuchsian groups are, at least, relatively simple to describe. This leads to the following (see [2] for a proof):

Theorem 16. *Let M be a Riemann surface whose universal cover is $\mathcal{U}$, and whose fundamental group is abelian. Then M is conformally equivalent to $\mathcal{D}$, $\mathcal{D}^\times = \mathcal{D} - \{0\}$, or an annulus.*

The covering group of the punctured disk $\mathcal{D}^\times$ is generated by a single parabolic transformation (without loss of generality, $z \mapsto z+1$). One can show that every annulus is conformally equivalent to the surface $\mathcal{D}_r = \{z \in \mathbb{C} \mid r < |z| < 1\}$ for some $r \in (0,1)$. Furthermore, annuli with different values of r are conformally inequivalent. The covering group of $\mathcal{D}_r$ is generated by a single transformation $z \mapsto \lambda z$ where $0 < \lambda < 1$. The parameter r is given by $r = \exp(-2\pi^2/\log\lambda)$. In summary,

Theorem 17. *The moduli space of the annulus is given by the open real interval* $(0,1)$.

Other than the 7 exceptional surfaces (which include 2 infinite families) we have mentioned, all Riemann surfaces are covered by the upper half plane $\mathcal{U}$ and have nonabelian fundamental groups.

5.4 Teichmüller Spaces and the Modular Group

We saw that the moduli space $\mathcal{M}_1$ of a genus 1 surface was formed by taking the upper half plane $\mathcal{U}$ and identifying points under the action of the discrete group $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$. Note, however, that the $\mathcal{U}$ is not a covering space of $\mathcal{M}_1$ in the usual sense as Γ_1 does not act freely discontinuously on $\mathcal{U}$. In particular, the points $\tau = i$ and $\tau = e^{2\pi i/3}$ are fixed by certain elements Γ_1. Their stabilizer groups are given by

$$\mathrm{Stab}(i) = \langle S\rangle \cong \mathbb{Z}_2$$
$$\mathrm{Stab}(e^{2\pi i/3}) = \langle ST\rangle \cong \mathbb{Z}_3$$

Of course, points equivalent to i or $e^{2\pi i/3}$ will also have nontrivial stabilizers. But this exhausts the list of "bad points" in $\mathcal{U}$, so the action of Γ_1 on $\mathcal{U}$ is not too terrible: the set of fixed points is discrete and the stabilizer group of each fixed point is finite.

In general, we say that a group G acts **properly discontinuously** on a Hausdorff space M if each point $x \in M$ has a neighborhood U such that

$$gU \cap U = U \qquad \forall g \in \mathrm{Stab}(x)$$
$$gU \cap U = \varnothing \qquad \forall g \in G - \mathrm{Stab}(x)$$

Furthermore, the group $\mathrm{Stab}(x)$ must be finite for each $x \in M$. Note that if G acts freely ($\mathrm{Stab}(x) = 1$ for each x) and properly discontinuously then it acts freely discontinuously. The points for which $\mathrm{Stab}(x)$ is nontrivial are called fixed points or **orbifold singularities**. The quotient space M/G is called an **orbifold space**. It is not necessarily a manifold, although one can show that set of singularities forms a discrete set so that M/G looks like a manifold almost everywhere.

One can generalize this whole scenario to the case of $g > 1$. We define the component of the identity in the group of diffeomorphisms to be those diffeomorphisms homotopic to the identity map

$$\mathrm{Diff}_0(M) = \{f \in \mathrm{Diff}(M) \mid f \sim \mathrm{id}_M\}$$

It is, as always, a closed, normal subgroup of $\mathrm{Diff}(M)$ so we can form the quotient space $\mathrm{Diff}(M)/\mathrm{Diff}_0(M)$ whose elements are homotopy classes of diffeomorphisms. Since we are dealing with oriented surfaces we really only want to consider orientation preserving diffeomorphisms $\mathrm{Diff}^+(M)$. We can then form the quotient space $\mathrm{Diff}^+/\mathrm{Diff}_0$ (as all diffeomorphism homotopic to the identity are orientation preserving). For a compact surface of genus g ($M = \Sigma_g$) this group is called the **mapping class group** or the **modular group** of genus g,

$$\Gamma_g = \mathrm{Diff}^+/\mathrm{Diff}_0$$

One can think of the mapping class group as the group of global diffeomorphisms of Σ_g, that is, diffeomorphisms *not* continuously connected to the identity.

The utility of these groups are that they allow us to break up the construction of the moduli space $\mathcal{M}_g$ into two stages. We first identify the space of complex structures $\mathcal{J}_g$ under the action of Diff_0. This operation is smooth and produces a finite dimensional manifold. Secondly we identify under action of the discrete group Γ_g to produce the actual moduli space $\mathcal{M}_g$.

The space of complex structures on Σ_g modulo non-global diffeomorphisms is called the **Teichmüller space** of Σ_g and is denoted

$$\mathcal{T}_g = \mathcal{J}_g/\mathrm{Diff}_0$$

One can show that $\mathcal{T}_g$ is actually a simply connected, complex manifold of finite dimension

$$\dim_{\mathbb{C}} \mathcal{T}_g = \begin{cases} 0 & g = 0 \\ 1 & g = 1 \\ 3g - 3 & g \geq 2 \end{cases}$$

The moduli space is then obtained from the Teichmüller space by identifying under the action of the mapping class group Γ_g

$$\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$$

As mentioned above this action is not necessarily free, rather Γ_g acts properly discontinuously on $\mathcal{M}_g$ so that the moduli space is an orbifold. The points in the moduli space corresponding to orbifold singularities represent Riemann surfaces with enhanced symmetry groups. Since the Teichmüller space is simply connected we can regard it as the orbifold universal covering of $\mathcal{M}_g$ and identify Γ_g with the orbifold fundamental group of $\mathcal{M}_g$. Because of the singularities involved these notions are slightly different then the usual definitions of universal cover and fundamental group. (For a rigorous treatment of these matters and other topics in this section see [3]).

It is easy to see how this scheme generalizes the genus 1 situation. For the torus one can show that $\mathcal{T}_1 = \mathcal{U}$ and $\Gamma_1 = \mathrm{PSL}(2,\mathbb{Z})$ so the moduli space is the orbifold $\mathcal{U}/\mathrm{PSL}(2,\mathbb{Z})$ as we deduced earlier. When we compute the automorphism group of the torus in the next section we shall see how the orbifold singularities at $\tau = i$ and $\tau = e^{2\pi i/3}$ represent surfaces with enhanced symmetry.

6 Automorphism Groups

In our discussion of gauge fixing we determined that it was important to understand which gauge transformations would leave the conformal structure on the worldsheet invariant. In the language of Riemann surfaces, these transformations form the automorphism group $\mathrm{Aut}(M)$ of the surface M. It turns out that for most Riemann surfaces the automorphism group is a discrete group. Such a surface has no CKV's. In fact, only the seven exceptional Riemann surfaces (those with abelian fundamental groups) mentioned in the previous section have continuous automorphism groups and hence CKV's.

The automorphism group of a surface M can be calculated from knowledge of the universal covering map $p\colon \Sigma \to M$ as follows:

Theorem 18. *Let M be a Riemann surface and let Σ be its universal cover and G be its covering group. Then*

$$\mathrm{Aut}(M) \cong N(G)/G$$

where $N(G)$ is the normalizer of G in $\mathrm{Aut}(\Sigma)$.

Recall that the normalizer of a group G in a group H is the set of all elements in H which fix H under conjugation:

$$N(G) = \{h \in H \mid hGh^{-1} = G\}$$

It is the largest subgroup of H in which G is normal. The proof of Theorem 18 is not hard and is left to the reader.[4]

Let $\mathrm{Aut}_0(M)$ be the connected component of the identity in $\mathrm{Aut}(M)$. Our claim (though we will not prove it) is that only for the exceptional surfaces is $\mathrm{Aut}_0(M)$ nontrivial. We have already calculated $\mathrm{Aut}(M)$ for the three simply connected surfaces in §3.3. We handle the remaining four cases below.

The punctured plane – $\mathbb{C}^\times$. The subgroup of $\mathrm{Aut}(\mathbb{C}) = \mathrm{Aff}(1,\mathbb{C})$ that fixes 0 is given by $z \mapsto az$ for $a \in \mathbb{C}^\times$, so that $\mathrm{Aut}_0(\mathbb{C}^\times) = \mathbb{C}^\times$. In addition, we have the discrete transformation that interchanges 0 and ∞, $z \mapsto 1/z$. $\mathrm{Aut}(\mathbb{C}^\times)$ is then a $\mathbb{Z}_2$-extension of $\mathbb{C}^\times$.

The punctured disk – $\mathcal{D}^\times$. From the definition of $\mathrm{PSU}(1,1)$ it is clear that the subgroup fixing 0 is given by $z \mapsto e^{i\theta}z$ for $\theta \in \mathbb{R}$. There are no discrete transformations

[4] Meaning, as always, that the details are tedious and overtax the patience of the author.

to speak of so that $\mathrm{Aut}(\mathcal{D}^\times) = \mathrm{Aut}_0(\mathcal{D}^\times) = \mathrm{U}(1)$.

The annulus – $\mathcal{D}_r$. We saw the covering group G of the annulus was generated by $g(z) = \lambda z$ for some $\lambda \in \mathbb{R}^+$. Any element h in the normalizer $N(G)$ must satisfy

$$hgh^{-1} = g^{\pm 1}$$

The component of the identity in $N(G)$ will consist of those elements satisfying $hgh^{-1} = g$. It is not hard to see that these elements are all of the form $z \mapsto kz$ for some $k \in \mathbb{R}^+$. Under the covering map $p\colon \mathcal{U} \to \mathcal{D}_r$

$$z \mapsto \exp\left(2\pi i \frac{\log z}{\log \lambda}\right)$$

and h will map to an element of the form $z \mapsto e^{i\theta} z$ for $\theta \in \mathbb{R}$, so that $\mathrm{Aut}_0(\mathcal{D}_r) = \mathrm{U}(1)$. However, if h satisfies $hgh^{-1} = g^{-1}$ then h will lead to a discrete automorphism of $\mathcal{D}_r$. In particular, choosing h to be $h(z) = -1/z$ gives under projection the discrete map $z \mapsto r/z$ which inverts the annulus, swapping the inner and outer boundaries. This exhausts the possibilities so that $\mathrm{Aut}(\mathcal{D}_r)$ is a $\mathbb{Z}_2$-extension of $\mathrm{U}(1)$.

The torus – $T^2(\tau)$. Recall that the covering group of the torus in $\mathbb{C}$ is a lattice group $\Lambda(\tau)$ generated by $z \mapsto z + 1$ and $z \mapsto z + \tau$. The normalizer of $\Lambda(\tau)$ will consist of those elements in $\mathrm{Aut}(\mathbb{C})$ that map (by conjugation) the generators of $\Lambda(\tau)$ onto another set of generators. Recall that conjugation by $\alpha z + \beta$ sends $z + \omega$ onto $z + \alpha\omega$. So $\alpha z + \beta$ will be an element of $N(G)$ if and only if

$$\begin{pmatrix} \alpha\tau \\ \alpha \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

for some $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. Clearly, all elements with $\alpha = 1$ will satisfy this equation (with $A = 1$). So $N(G)$ contains, at the very least, all translations $z \mapsto z + \beta$, and $\mathrm{Aut}(T^2)$ contains T^2 as a normal subgroup. If fact, this is the entire identity component.

$$\mathrm{Aut}_0(T^2) = T^2$$

The other permissible values of α will lead to discrete transformations of the torus. One such value is clearly $\alpha = -1$ with $A = -1$. So $\mathrm{Aut}(T^2)$ is at least a $\mathbb{Z}_2$-extension. For most values of the modulus τ these are the only possibilities. To see this note that τ must be fixed by the element in $\mathrm{PSL}(2,\mathbb{Z})$ corresponding to A. The general theory of Möbius transformations tells us that A must be elliptic; that is, it must satisfy $0 \le \mathrm{Tr}^2 A < 4$. As the elements of A are in $\mathbb{Z}$ the only possibilities are $\mathrm{Tr}^2 A = 0, 1$. The first case, $\mathrm{Tr}^2 A = 0$ can occur only when $\tau = i \pmod{\mathrm{PSL}(2,\mathbb{Z})}$ in which case $\alpha = \pm i$. The case $\mathrm{Tr}^2 A = 1$ can occur only when $\tau = e^{2\pi i/3} \pmod{\mathrm{PSL}(2,\mathbb{Z})}$. We find that $\alpha = e^{\pm 2\pi i/3}$ corresponds to $\mathrm{Tr}\, A = -1$ and $\alpha = e^{\pm \pi i/3}$ corresponds to $\mathrm{Tr}\, A = 1$.

In general we see that α must be an n^{th} root of unity, $\alpha^n = 1$, where

$$n = \begin{cases} 6 & \tau = e^{2\pi i/3} \quad \mod \mathrm{PSL}(2,\mathbb{Z}) \\ 4 & \tau = i \qquad\quad\;\; \mod \mathrm{PSL}(2,\mathbb{Z}) \\ 2 & \text{otherwise} \end{cases}$$

The group $\mathrm{Aut}(T^2)$ will correspondingly be a $\mathbb{Z}_n$-extension of T^2.

We summarize the information on the exceptional Riemann surfaces in Table 1. There μ denotes the number of real moduli and κ denotes the number of real CKV's, which is equal to $\dim_{\mathbb{R}} \mathrm{Aut}(M)$.

	π_1	μ	κ	Aut_0	$\mathrm{Aut}\,/\,\mathrm{Aut}_0$
$\widehat{\mathbb{C}}$	1	0	6	$\mathrm{PSL}(2,\mathbb{C})$	1
$\mathbb{C}$	1	0	4	$\mathrm{Aff}(1,\mathbb{C})$	1
$\mathbb{C}^\times$	$\mathbb{Z}$	0	2	$\mathbb{C}^\times$	$\mathbb{Z}_2$
$T^2(\tau)$	$\mathbb{Z}\oplus\mathbb{Z}$	2	2	$T^2(\tau)$	$\mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_6$
$\mathcal{D}$	1	0	3	$\mathrm{PSU}(1,1)$	1
$\mathcal{D}^\times$	$\mathbb{Z}$	0	1	$\mathrm{U}(1)$	1
$\mathcal{D}_r$	$\mathbb{Z}$	1	1	$\mathrm{U}(1)$	$\mathbb{Z}_2$

Table 1: The Exceptional Riemann Surfaces

As far as the other Riemann surfaces are concerned, one can show that $N(G)$ is itself a discrete group, so that $\mathrm{Aut}(M)$ will be discrete as well. For compact Riemann surfaces one can do much better. There is a theorem due to Schwarz and Hurwitz that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is actually *finite* with order n bounded above by

$$n \leq 84(g-1)$$

7 Klein Surfaces

Throughout most of this lecture we have only considered worldsheets that appear in closed, oriented string theory, i.e. oriented surfaces without boundary. We briefly highlight in this section how one handles the general surfaces that can appear in open, unoriented string theory.

How one adds boundaries to the theory is not too hard to imagine. One can simply introduce a complex structure onto a surface with boundary using the closed upper half plane $\overline{\mathcal{U}}$ rather than $\mathbb{C}$ as the model space. This defines the notion of a **Riemann surface with boundary**.

Handling the unoriented case is slightly more problematic. Since complex manifolds are always orientable there is no way to introduce a complex structure onto a unorientable surface. Nor can we introduce an almost complex structure as such a tensor cannot be globally defined. But note that we are not too far off from a complex structure: given a conformal metric we can still introduce isothermal coordinates around each point. The transition functions between isothermal coordinate patches are either holomorphic or anti-holomorphic on each component of their domain. It is the anti-holomorphic possibility that is the obstruction to the introduction of a complex structure. We are naturally led, then, to introduce a structure that allows both holomorphic and anti-holomorphic transition functions.

7.1 Dianalytic Structure

Let M be a $2m$-dimensional manifold, and let $\{U_i, \phi_i\}$ be a complex-valued atlas on M. We say that the atlas is **dianalytic** if the transition functions $\phi_i \circ \phi_j^{-1}$ are either holomorphic or anti-holomorphic on each component of their domain of definition. A maximal such atlas is called a **dianalytic structure** on M.

A 2-dimensional surface M together with a dianalytic structure on M is called a **Klein surface**. Note that Riemann surfaces are special cases of Klein surfaces. We can also define the notion of a **Klein surface with boundary** in complete analogy with the orientable case.

As every conformal metric defines isothermal coordinates it also defines a dianalytic structure as well. Conversely, every Klein surface admits a unique conformal metric compatible with its dianalytic structure. We have then the following generalization of Theorem 4:

Theorem 19. *The category of Klein surfaces (with boundary) is equivalent to the category of conformal surfaces (with boundary).*

It is a classical result, that every compact unorientable surface without boundary can be written as the connected sum of p (real) projective planes, $\mathbb{RP}^2$. We will define the genus of such a surface to be $g = p - 1$. Every compact unorientable surface with boundary can be obtained from one without by removing b non-intersecting open disks.

7.2 Uniformization

To be completed.

7.3 Moduli Spaces and Automorphisms Groups

To be completed.

References

[1] J. M. Lee, *Introduction to Topological Manifolds*. No. 202 in Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[2] H. M. Farkas and I. Kra, *Riemann Surfaces*. No. 71 in Graduate Texts in Mathematics. Springer-Verlag, New York, 2nd ed., 1992.

[3] R. Hain, "Moduli of Riemann Surfaces, Transcendental Aspects," `math.AG/0003144`.

[4] N. L. Alling and N. Greenleaf, *Foundations of the Theory of Klein Surfaces*. No. 219 in Lecture Notes in Mathematics. Springer-Verlag, New York, 1971.

[5] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory*. Cambridge University Press, Cambridge, 1987. 2 volumes.

[6] B. Hatfield, *Quantum Field Theory of Point Particles and Strings*. Westview Press, Boulder, Colorado, 1992.

[7] J. Polchinski, *String Theory*. Cambridge University Press, Cambridge, 1998. 2 volumes.