The thin spherical shell is an interesting calculation. Before we go straight to it though, it will be helpful to find the moment of inertia of a thick spherical shell and consider the thin shell as the limiting case. The thin spherical shell is like the colored coating on an M&M, while the thick shell is like the chocalate coating on a malt-ball. Mmmm, chocalate...aaaghghghghghhg.

$$I_{thick\ shell} = \int\int\int_{thick\ shell} dm\ s^2$$

This works just like the solid sphere except the limits on the integrals will be different. Which limit is different? It's the radial one, since we no longer start our integration at $r = 0$. Instead we will start integrating at the inner shell R_i and proceed to the outer shell R.

$$\begin{aligned} I_{thick\ shell} &= \int_0^{2\pi}\int_0^{\pi}\int_{R_i}^{R} r^2 \sin^2\theta\rho(r^2 \sin\theta dr d\theta d\phi) \\ &= \frac{2\pi\rho}{5}\frac{4}{3}(R^5 - R_i^5) \end{aligned}$$

where the details of the calculation are very similar to those of the solid sphere. Now subbing in the density $\rho = \frac{M}{\frac{4}{3}\pi(R^3 - R_i^3)}$.

$$I_{thick\ shell} = \frac{2}{5}M\frac{R^5 - R_i^5}{R^3 - R_i^3}$$

which doesn't simplify as nicely as the solid sphere. But now we can find the thin shell using this by taking the limit as $R_i \to R$.

$$\begin{aligned} I_{thin\ shell} &= \lim_{R_i \to R} I_{thick\ shell} \\ &= \frac{2}{5}M \lim_{R_i \to R}\frac{R^5 - R_i^5}{R^3 - R_i^3} \end{aligned}$$

Uh oh. This limit is indeterminate since making the $R_i = R$ substitution yields $\frac{0}{0}$. What do we do in cases like this one? Well, there was L'Hopital's rule (pronounced "Lope-ee-tahls"). Holy Shit!! You mean that stuff we learned in Cacl II is actually useful for something?!?! Yep, just take the derivative of top and bottom with respect to R_i.

$$\begin{aligned} I_{thin\ shell} &= \frac{2}{5}M \lim_{R_i \to R}\frac{-5R_i^4}{-3R_i^2} \\ &= \frac{2}{5}\frac{5}{3}M \lim_{R_i \to R} R_i^2 \\ &= \frac{2}{3}MR_i^2 \end{aligned}$$