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# **On Hamiltonian and Action Principle Formulations of Plasma Dynamics**

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**Abstract.** A general discussion of Hamiltonian and action principle formulations for fluid and plasma models is given. A procedure, based on Hamilton's principle of mechanics but adapted for continua, for the construction of action principles for fluid and kinetic models is given. The transformation from action principles in terms of the Lagrangian variable description to the Eulerian variable description in terms of noncanonical Poisson brackets is described. Two examples are developed: ideal MHD and Braginskii's fluid model with gyroviscosity.

**Keywords:** action principle, variational principle, Hamiltonian, plasma, noncanonical, Poisson bracket, magnetohydrodynamics, gyroviscosity, Braginskii **PACS:** 47.10.Df,47.65.-d,52.30.-q,95.30.Qd,52.25.Dg

# **1. INTRODUCTION**

The Hamiltonian and action principle (HAP) formulations of various plasma models derive from the fact that the basic electromagnetic interaction possesses HAP formulations. It is well-known that the ultimate plasma description, the relativistic 2N-body problem of N electrons and N ions interacting via the Maxwell field has HAP formulations, which is the case for any fundamental interaction. The HAP formulation of this *parent model* gives rise to that of the most commonly used ideal plasma models such as the magnetohydrodynamics (MHD), the Vlasov description, BBGKY hierarchy, etc. (for my own reviews see [1, 2, 3, 4]).

The HAP theory of continua, by which we mean general fluid or kinetic models that describe media, has a rich pedigree that includes, for example, Lagrange (1788), Clebsch (1857, 1859), Kirchoff (1876) in olden times, as well as the more modern works of Eckart, Newcomb, Lin, and Arnold in the 1960s, and Zakharov and others in the 1970s. In the 1980s there was a burst of activity, motivated in large part by [5].

Researchers who care about HAP formulations do so for various reasons; for example, some believe it is a basic form of physical law and are uncomfortable if it has not been identified, some use it as a means for consistent approximation, while others may use it to obtain an unambiguous definition of energy. Whatever the reason, it is clear that the HAP formulation imposes a structure that singles out a class of models, and it is at least reasonable to expect physical models to lie in this class.

Dissipation is often considered to be important in realistic physical systems, so it is natural to ask what role it plays in our discussion of HAP formulations. To begin with it should be noted that the parent model is nondissipative, as evidenced by its HAP formulations. Dissipation comes about because we choose, usually out of necessity, to

consider reduced or simplified descriptions. These simplified descriptions can be traced to approximations based on time scale separation, and the dissipative terms that arise in this way are accompanied by phenomenological constants such as viscosity and resistivity. We will not be concerned with this reduction process, but concentrate on the structure possessed by equations when such phenomenological constants are set to zero. We argue that the resulting equations should possess HAP formulations. Indeed, this is the case for MHD, the Vlasov description, and other important equations of fluid mechanics and plasma physics. This assures that fake or unphysical dissipation has not crept somehow into our model.

An important and active area of plasma physics, where HAP formulations have been considered, is reduced fluid modeling. These kinds of models are two or nearly twodimensional magnetofluid models that have been developed to incorporate important physics into computable closed dynamical systems. One of the early examples is reduced MHD [6], but many models for many purposes have been obtained over the years, e.g. [7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Sometimes models like these arise from systematic asymptotic expansion, while in other cases they have been proposed in a more ad hoc manner. Many reduced fluid models have been shown, after the fact, to possess Hamiltonian formulations, which can serve as a means of selecting good models from bad. (See [17] for a general theory of models of this type.) One idea emphasized here is a way to derive them directly by using a more basic HAP formulation at the outset.

So, in these lectures we will describe some general properties of HAP formulations, with an emphasis on magnetofluid models. In Sec. 2, we review Hamilton's principle of mechanics, which motivates the action principles for continua. In Sec. 3, the Lagrangian and Eulerian descriptions of a fluid are reviewed, the former being more amenable to an action principle formulation. In Sec. 4, we describe a general procedure for building action principles. We propose this approach as an alternative to the model building that is often used. In Sec. 5, two examples are considered: ideal MHD and a two-dimensional gyrofluid model. Finally, in Sec. 6, we describe the transformation from the action principle description of Sec. 4 to a Lagrangian variable Hamiltonian description, and then the transformation to an Eulerian variable description in terms of noncanonical Poisson brackets, the final goal of the theory.

# 2. HAMILTON'S PRINCIPLE AND THE FUNCTIONAL DERIVATIVE

The *procedure* for constructing Hamilton's principle for a mechanical system is described in mechanics textbooks as follows: the first step is to determine the configuration space, i.e. assign the coordinates, angles or displacements, necessary to describe the system's configuration. These coordinates are called the generalized coordinates usually denoted by  $q^{\ell}(t)$ , where  $\ell = 1, 2, ..., N$  and N is the number of degrees of freedom of the system. The second step is to construct the kinetic potential, the Lagrangian, which is the difference between kinetic and potential energies, L := T - V. Given the Lagrangian,

the action functional is given by

$$S[q] = \int_{t_0}^{t_1} dt \, L(q, \dot{q}, t) \,. \tag{1}$$

The action is an example of a functional, i.e. a quantity with a domain composed of functions and a range of real numbers. In this case, given a path q(t), S[q] returns a real number upon substitution into L and integration over time.

Different paths usually give different numbers. In Hamilton's principle one fixes the beginning and end points of the path,  $q(t_0)$  and  $q(t_1)$ , and searches over such a space of paths for extremal values. Extremal means that the functional derivative of the action vanishes,  $\delta S[q]/\delta q^{\ell} = 0$ , where the functional derivative is defined by

$$\begin{split} \delta S[q; \delta q] &= \left. \frac{dS[q + \varepsilon \delta q]}{d\varepsilon} \right|_{\varepsilon = 0} = \int_{t_0}^{t_1} dt \left( \frac{\partial L}{\partial q^{\ell}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{\ell}} \right) \delta q^{\ell} \\ &=: \left. \left\langle \frac{\delta S[q]}{\delta q^{\ell}}, \delta q^{\ell} \right\rangle. \end{split}$$
(2)

Here  $\delta q(t)$  is an arbitrary perturbation of a path q(t) and, because of this arbitrariness, the only way for  $\delta S$  to vanish for all  $\delta q(t)$  is to have the quantity within the parentheses vanish, i.e.

$$\frac{\delta S[q]}{\delta q^{\ell}} = 0 \quad \Leftrightarrow \quad \frac{\partial L}{\partial q^{\ell}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^{\ell}} = 0.$$
(3)

Thus, the extremal path satisfies Lagrange's equations of motion.

In the second line of (2),  $\langle , \rangle$  denotes the integration over time. We employ two operations in this second line that defines the functional derivative: first we *vary*, which amounts to linearization in  $\delta q$  and then we *isolate*, which means all derivatives (in general any operators) are removed from  $\delta q$ . The notation is reminiscent of the variation of an ordinary function of space,  $\delta f = \nabla f \cdot \delta \mathbf{x}$ . Thus we see that  $\langle , \rangle$  is analogous to  $\delta \mathbf{x}$ , and  $\delta S[q]/\delta q$  is analogous to  $\nabla f$ , i.e. the functional derivative is a gradient in function space.

#### **3. LAGRANGIAN AND EULERIAN DESCRIPTIONS**

The two principal ways of describing the state of a fluid are the particle-like description, usually called the Lagrangian or material variable description, and the more common Eulerian or spatial description (see e.g. [18, 19, 20, 2]). Because the Lagrangian description treats a fluid as a continuum of particles, it most naturally is amenable to the HAP formulations of particle mechanics. For this reason we describe it first here, before proceeding to a brief review of the Eulerian description. Finally in this section we consider the Euler-Lagrange map that relates the two descriptions.

The Lagrangian variable is a coordinate that gives the position of a fluid element or parcel, as it is sometimes called, at time t. We denote this variable by  $q = q(a,t) = (q^1, q^2, q^3)$ , which is measured relative to some origin O and cartesian coordinates are assumed, as depicted in Fig. 1. Here  $a = (a^1, a^2, a^3)$  denotes the *fluid element* 

*label*, which is often defined to be the position of the fluid element at the initial time, a = q(a, 0), but this need not always be the case. (Note, the label *a* is a continuum analog of the discrete index that labels a generalized coordinate in a finite degree-of-freedom system.) If *D* is a domain that is fully occupied by the fluid, then at each fixed time  $t, q: D \rightarrow D$  is assumed to be 1-1 and onto. Not much is really known about the mathematical properties of this function, but we will assume that it is as smooth as it needs to be for the operations performed.



**FIGURE 1.** In the Lagrangian variable description, a fluid element labeled by a is located at q(a,t) relative to some origin O. In the Eulerian variable description fluid quantities are defined at an observation point r in the domain D that the fluid occupies.

Important quantities are the deformation matrix,  $\partial q^i / \partial a^j =: q^i_{,j}$  and its Jacobian determinant  $\mathscr{J} := \det(q^i_{,j})$ , which in three and two dimensions, respectively, is given by

$$\mathscr{J} = \frac{1}{6} \varepsilon_{kjl} \varepsilon^{imn} q^k_{,i} q^j_{,m} q^l_{,n}, \qquad (4)$$

$$= \frac{1}{2} \varepsilon_{kj} \varepsilon^{il} q^{k}_{,i} q^{j}_{,l}, \qquad (5)$$

where  $\varepsilon_{ijk} = \varepsilon^{ijk}$  and  $\varepsilon_{ij} = \varepsilon^{ij}$  are the three and two-dimensional unit, purely antisymmetric, (Levi-Civita) tensor densities. When we restrict to two dimensions i, j = 1, 2. We assume a fluid element is uniquely determined by its label for all time. Thus,  $\mathscr{J} \neq 0$  and we can invert q = q(a,t) to obtain a = a(q,t), the label of an element that reaches q at

time t. In general for such coordinate transformations we have

$$q_{,k}^i a_{,j}^k = a_{,k}^i q_{,j}^k = \delta_{,j}^i,$$

i.e. the inverse of the deformation matrix is  $a_{,j}^k = \partial a^k / \partial q^j$  and repeated indices are summed. It should be clear from the context when the quantities q and a denote dependent or independent variables. Using q(a,t) or its inverse, quantities such as  $a_{,j}^k$  can be written either as a function of q or a.

The locus of points of material surfaces that move with the fluid are clearly determined by q. An initial volume element  $d^3a$  maps into a volume element at time t according to

$$d^3q = \mathcal{J}d^3a, \tag{6}$$

and components of an area form map according to

$$(d^2q)_i = \mathscr{J}a^j_{,i}(d^2a)_j, \qquad (7)$$

where  $\mathcal{J}a_{,i}^{j}$  is the transpose of the cofactor matrix of  $q_{,i}^{j}$ , which in three and two dimensions, respectively, is given by

$$\mathscr{J}a^{i}_{,k} = \frac{1}{2} \varepsilon_{kjl} \varepsilon^{imn} q^{j}_{,m} q^{l}_{,n}$$
(8)

$$\mathscr{J}a^{i}_{,k} = \varepsilon_{kj}\varepsilon^{il}q^{j}_{,l}.$$
<sup>(9)</sup>

Other useful identities are

$$\frac{1}{\mathscr{J}}\frac{\partial\mathscr{J}}{\partial q_{,j}^{i}} = a_{,i}^{j}, \qquad (10)$$

$$\frac{\partial(\mathscr{J}a^{i}_{,k})}{\partial a^{i}} = 0, \qquad (11)$$

where (10), the standard rule for differentiation of determinants, follows from (4) or (5), and (11) follows from (8) or (9) by the antisymmetry of  $\varepsilon_{ijk}$  or  $\varepsilon_{ij}$ .

Typical *Eulerian variables* are the velocity field v(r,t) and the mass density  $\rho(r,t)$ . It is most important to distinguish the Lagrangian position and label variables from the Eulerian observation point,  $r = (x, y, z) = (x^1, x^2, x^3)$ , an independent variable that does not move with the fluid, but like *a* and *q* is an element of *D*. Confusion exists in the literature because some authors insist on using the same symbol for the Lagrangian coordinate and the Eulerian observation point.

The Lagrangian and Eulerian descriptions must clearly be related and, indeed, knowing q(a,t) we can obtain v(r,t). If one were to insert a velocity probe into a fluid at (r,t) then one would measure the velocity of the fluid element that happend to be at that postition at that time. Thus it is clear that  $\dot{q}(a,t) = v(r,t)$ , where the overdot means the time derivative is taken at constant a. But, which fluid element will be at r at time t? Evidently r = q(a,t), which upon inversion yields the label of that element that will be measured,  $a = q^{-1}(r,t) =: a(r,t)$ . Thus, the Eulerian velocity field is given by

$$v(r,t) = \dot{q}(a,t)|_{a=a(r,t)}$$
 (12)

Properties can be attached to fluid elements, just as a given mass is identified with a given particle in mechanics. For a continuum system it is natural to attach a mass density,  $\rho_0(a)$ , to the element labeled by a. Whence the element of mass in a given volume is given by  $\rho_0 d^3 a$  and this amount of mass is preserved by the flow, i.e.  $\rho(r,t)d^3r = \rho_0 d^3 a$ . Thus, from (6) we obtain  $\rho_0 = \rho \mathscr{J}$ . Other quantities could be attached to a fluid element; for the ideal fluid, entropy per unit mass, s(r,t), is such a quantity. The assumption that each fluid element is isentropic then amounts to  $s = s_0$ . Similarly, for MHD a magentic field,  $B_0(a)$ , can be attached, and then the frozen flux condition yields  $B \cdot d^2r = B_0 \cdot d^2a$ , and from (7) we obtain  $\mathscr{J}B^i = q_{,j}^i B_0^j$ . Thus, one set of Eulerian magnetofluid variables is  $\{\rho, s, v, B\}$ .

Sometimes it is convenient to use another set of Eulerian density variables:  $\{\rho, \sigma, M, B\}$ , where  $\sigma = \rho s$  is the entropy per unit volume, and *M* is the momentum density. These Eulerian variables can be represented by using the Dirac delta function to 'pluck out' the fluid element that happens to be at the Eulerian observation point *r* at time *t*. For example, the mass density  $\rho(r,t)$  is obtained by

$$\rho(r,t) = \int_{D} d^{3} a \rho_{0}(a) \,\delta(r - q(a,t)) = \frac{\rho_{0}}{\mathscr{J}} \bigg|_{a=a(r,t)} \,. \tag{13}$$

The density one observes at *r* at time *t* will be the one attached to the fluid element that happens to be there then, and this fluid element has a label given by solving r = q(a,t). The second equality of (13) is obtained by using the three-dimensional version of the delta function identity  $\delta(f(x)) = \sum_i \delta(x - x_i)/|f'(x_i)|$ , where  $f(x_i) = 0$ . Similarly, the entropy per unit volume is given by

$$\boldsymbol{\sigma}(\boldsymbol{r},t) = \int_{D} d^{3} a \, \boldsymbol{\sigma}_{0}(a) \, \boldsymbol{\delta}\left(\boldsymbol{r} - \boldsymbol{q}\left(\boldsymbol{a},t\right)\right) = \left. \frac{\boldsymbol{\sigma}_{0}}{\mathscr{I}} \right|_{\boldsymbol{a}=\boldsymbol{a}(\boldsymbol{r},t)},\tag{14}$$

which is consistent with  $\sigma_0(a) = \rho_0(a)s_0(a)$  and  $s(r,t) = s_0(a)|_{a=a(r,t)}$ , where the latter means *s* is constant along a Lagrangian orbit. Proceeding, the momentum density,  $M = (M_1, M_2, M_3)$ , is related to the Lagrangian canonical momentum (defined in Sec. 6) by

$$M(r,t) = \int_{D} d^{3}a \Pi(a,t) \,\delta\left(r - q(a,t)\right) = \frac{\Pi(a,t)}{\mathscr{I}} \bigg|_{a=a(r,t)} \,. \tag{15}$$

For MHD,  $\Pi(a,t) = (\Pi_1, \Pi_2, \Pi_3) = \rho_0 \dot{q}$ . Lastly,

$$B^{i}(r,t) = \int_{D} d^{3}a q^{i}_{,j}(a,t) B^{j}_{0}(a) \,\delta\left(r - q(a,t)\right) = q^{i}_{,j}(a,t) \frac{B^{j}_{0}(a)}{\mathscr{I}} \bigg|_{a=a(r,t)}, \quad (16)$$

for the components of the magnetic field.

In the remainder of this subsection we record some other useful identities. The chain rule reveals the components of the Eulerian gradient are given by

$$\frac{\partial}{\partial x^k} = a^i_{,k} \frac{\partial}{\partial a^i} \bigg|_{a=a(r,t)}.$$
(17)

Thus, with r = q(a,t) the time derivative of any function  $f(a,t) = \tilde{f}(r,t) = \tilde{f}(q(a,t),t)$  maps into Eulerian variables according to the usual expression

$$\dot{f}\Big|_{a=a(r,t)} = \frac{\partial \tilde{f}}{\partial t} + \dot{q}^{i}(a,t) \frac{\partial \tilde{f}}{\partial x^{i}}\Big|_{a=a(r,t)} = \frac{\partial \tilde{f}}{\partial t} + v \cdot \nabla \tilde{f}(r,t).$$
(18)

Recall, an overdot denotes the time derivative at constant a,  $\partial/\partial t$  denotes the time derivative at constant r, and  $\nabla$  is the Eulerian derivative, i.e.  $\partial/\partial r$  with components  $\partial/\partial x^i$ . Evidently,  $s_0(a)$  being constant on a fluid element means s(r,t) is advected.

Similarly, using Eq. (10) we obtain

$$\mathcal{J} = \frac{\partial \mathcal{J}}{\partial q_{,j}^{i}} \dot{q}_{,j}^{i} = \mathcal{J} a_{,i}^{j} \dot{q}_{,j}^{i}, \qquad (19)$$

which upon evaluation at a = a(r,t) gives a formula due to Euler [18],

$$\frac{\partial \tilde{\mathcal{J}}}{\partial t} + v \cdot \nabla \tilde{\mathcal{J}} = \tilde{\mathcal{J}} \nabla \cdot v.$$
(20)

Using  $\rho \mathscr{J} = \rho_0$  and (20), it follows that the density satisfies the continuity equation. We will leave it as an exercise to show that the frozen-in flux condition,  $\mathscr{J}B^i = q^i_{,j}B^j_0$ , assures that *B* satisfies

$$\frac{\partial B}{\partial t} = -v \cdot \nabla B + v \cdot \nabla B - B \nabla \cdot v.$$
(21)

### 4. BUILDING ACTION PRINCIPLES FOR CONTINUA

Now we propose a systematic method for building action functionals for continua, models of continuous media that describe fluids, plasmas, gasses, etc. In plasma physics conventional construction of such models begins with some basic equations, usually less complete than the parent model, and then simpler models are obtained either by systematic asymptotic expansion or by model building where terms are chosen based on the physics such terms are believed to describe. Here we propose a general alternative approach, where one builds action functionals from scratch. The idea is to develop intuition about which kinds of terms in a kinetic potential give rise to various physical phenomena. Usually in modern physics action functionals are built with certain symmetry properties firmly in mind; however, continuous media models possess their own challenges and, in particular, an additional closure principle is required. Although in the present paper we are interested in action principles for fluid theories, the procedure we describe is very general. We have broken down the construction of such actions into four steps that we describe below.

#### Step 1: Domain and Coordinates

In this first step one must decide on the domain D. For a fluid it would be either one, two, or three-dimensional,  $D \subset \mathbb{R}^{1,2,3}$ , while for a complete kinetic theory  $D \subset \mathbb{R}^6$ 

or a guiding center theory might have  $D \subset \mathbb{R}^5$ . Here we concentrate on the case of two and three-dimensional fluids and suppose there is a Lagrangian (trajectory) variable  $q: D \to D$ . To be complete one needs to specify more about the nature of the functions q(a,t), where recall that the label  $a \in D$ . The word diffeomorphism is often used in this connection, even when serious mathematics is not being undertaken. Indeed, the kinds of theories we wish to construct are difficult partial or partial integrodifferential equations and uniqueness and existence theories are likely to be lacking. As physicists we slog on and assume that q is invertible for each t and possesses a sufficient number of derivatives for our purposes, at least in some sense that will not be further considered. This first step is the continuum analog of the first step of the procedure mentioned above that is described in ordinary mechanics textbooks.

# Step 2: Observables

In the second step one defines a set of *observables*. These are the Eulerian variables of our theory, and for the fluids considered here they will be defined to be functions of the Eulerian observation point  $r \in D$  and the time t. For example, for MHD one could choose either the set  $\{\rho, s, v, B\}$  or the set  $\{\rho, \sigma, M, B\}$  described in Sec. 3, or, e.g. the entropy variable could be replaced by a pressure. At this point we see a departure from conventional quantum field theories, in that at a most basic level we introduce the two kinds of variables of Steps 1 and 2. A basic assumption in our construction is that the observables be completely determined by the functions q(a,t), but the converse is not required.

#### Step 3: Eulerian Closure Principle

When constructing an action, various candidate terms for the kinetic potential may be considered, and because our action principles will be reminiscent of Hamilton's principle of mechanics, these terms will involve the variable q(a,t) and derivatives of this quantity with respect to both its arguments. However, not all such terms will be acceptable – we require a *closure principle* which ultimately means that our theory must be 'Eulerianizable.' More specifically, we require that each term of our action must be expressible entirely in terms of our set of observables. For example, the kinetic energy functional for a fluid must satisfy

$$T[q] := \frac{1}{2} \int_{D} d^{3}a \,\rho_{0}(a) |\dot{q}|^{2} = \frac{1}{2} \int_{D} d^{3}r \,\rho |v|^{2} \,.$$
(22)

This closure principle has important consequences: it leads us to equations of motion that can be written entirely in terms of our set of observables, i.e. an Eulerian variable description, and it leads to noncanonical Poisson brackets for our Eulerian theory, quantities that we discuss in Sec. 6.

#### Step 4: Conventional Symmetries

The final step is to build in conventional symmetries, a step that in practice would be done in concert with Step 3. For example, when building a nonrelativistic system one might want to assure symmetry under the Galilean group or for a relativistic system the Poincaré group. One would need to decide the tensorial nature of the dependent and independent variables and how they are related, allowing for both internal and space-time symmetries. However, in some situations one may want to break Galilean symmetry; for example, this would be the case for an anisotropic models that are common when modeling plasmas with strong applied magnetic fields.

### 5. TWO MAGNETOFLUID EXAMPLES

Now we apply the general procedure of Sec. 4 to two magnetofluid examples: ideal MHD and a compressible gyrofluid model that is a two-dimensional version of that given in [21] (see also [22]). For both of these examples, Steps 1 and 2 were already accomplished in Sec. 3. We will chose the observables to be the set { $\rho$ , *s*, *v*, *B*}. So, it only remains to construct the appropriate kinetic potential; this is where the particular physics of interest is built into the model.

### 5.1. MHD Action

The action for ideal MHD appears to have been first given by Newcomb in [19]. Other relevant references are [20, 23, 24]. We will pretend we don't know the answer and proceed as if we were building a theory from the ground up.

Since our fluid will have motion, it will have kinetic energy, a quantity that should be invariant under Galilean frame shifts and rotations, we choose the function T[q] of (22) to be part of our action. That Galilean invariance gives rise to a term like (22) is well-known (see e.g. page 73 of [25]).

Proceeding, if we restrict our theory to be one that involves only first derivatives of q, then it is natural to consider a generalized internal energy functional of the following form:

$$V[q] := \int_{D} d^{3}a \rho_{0} W\left(\rho_{0}/\mathscr{J}, s_{0}, |q_{,j}^{i}B_{0}^{j}|/\mathscr{J}\right) = \int_{D} d^{3}r \rho W(\rho, s, |B|), \qquad (23)$$

where the function W is at present arbitrary. Form the second equality of (23) it is clear that V[q] satisfies the closure principle.

On physical grounds we expect our fluid to possess internal and magnetic energies. For the fluid internal energy we make the usual assumption of local thermal equilibrium, which is described by a function  $U(\rho, s)$ , an internal energy per unit mass. This quantity is written in Eulerian form, since we expect the fluid at each Eulerian observation point to be in thermal equilibrium. From U we compute the temperature and pressure according to the usual differentiations,  $T = U_s$  and  $p = \rho^2 U_\rho$ , where the subscript means partial differentiation with the other variable held fixed. In electricity and magnetism the

magnetic energy density is  $\int d^3r |B|^2/2$ , so this is a natural choice. Thus we obtain the following generalized internal energy functional:

$$V[q] := \int_{D} d^{3}a \left( \rho_{0} U(\rho_{0}/\mathscr{J}, s_{0}) + q^{i}_{,j} q^{i}_{,k} B^{j}_{0} B^{k}_{0} / (2\mathscr{J}) \right)$$
  
$$= \int_{D} d^{3}r \left( \rho U(\rho, s) + |B|^{2} / 2 \right).$$
(24)

Given (24) the action follows immediately,

$$S_{MHD}[q] = \int_{t_0}^{t_1} dt \int_D d^3 a \left[ \frac{\rho_0 |\dot{q}|^2}{2} - \rho_0 U\left(\frac{\rho_0}{\mathscr{I}}, s_0\right) - \frac{q_{,j}^i q_{,k}^i B_0^j B_0^k}{2\mathscr{I}} \right].$$
(25)

Upon calculating  $\delta S_{MHD}/\delta q = 0$ , one obtains an equation of the form of Newton's second law,  $\rho_0 \ddot{q}^i = \mathscr{F}^i$ , where we will not write out the force density  $\mathscr{F}$ . Upon making use of the identities of Sec. 3, Newton's second law becomes

$$\rho\left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right) - B \cdot \nabla B + \nabla \left(p + \frac{|B|^2}{2}\right) = 0.$$
(26)

We refer the reader to the literature cited above for the details of these calculations.

There are many possibilities here for alternative or more general formulations. For example, one could make a special choice for U that gives rise to  $p = \kappa(s)\rho^{\gamma}$ , and then use this relation to eliminate the entropy per unit mass s in terms of the pressure p. Another possibility would be to allow |B| dependence in U and in this way obtain the anisotropic pressure tensor of the CGL equations [19, 1].

#### 5.2. Gyrofluid Action

If one imagines that the fluid is composed of charged particle executing their Lamor gyrations in a magnetic field, then a model with some kind of internal angular momentum seems appropriate. We will construct such an action principle with the same observables as used above for MHD,  $\{\rho, s, v, B\}$ , but we will restrict our theory to two spatial dimensions.

The action of (25) contains a term quadratic in  $\dot{q}$ , but no linear term. Gyroscopic Lagrangians for finite degree-of-freedom systems possess such linear terms, which give rise to systems with Coriolis or magnetic forces. Thus it is natural to consider for gyrofluids a term of the from

$$G[q] := \int_D d^2 a \, \Pi^* \cdot \dot{q} = \int_D d^2 r \, M^* \cdot v \,, \tag{27}$$

where  $\Pi^*$  could be a function of q and its *a*-derivatives and  $\mathcal{J}M^* = \Pi^*$ . Again we require the closure principle, which means that upon transforming to our set of observables, G must have the form shown in the second equality of (27), where  $M^*$  is

expressible in terms of the observables and their Eulerian derivatives. This transpires if we assume the components of  $\Pi^*$  have the form

$$\Pi^{*i} = \mathscr{J}\varepsilon^{ij}a^m_{,j}\frac{\partial L^*}{\partial a^m},\tag{28}$$

where  $L^*(\rho, s, B, \nabla \rho, \nabla \rho, \nabla s, \nabla B, ...)$ , i.e.  $L^*$  is a function of the observables and their Eulerian derivatives. From (28) and (17) we obtain  $\Pi^* / \mathscr{J} = \nabla \times L^* = M^*$ , and thus we see that the quantity  $M^*$  is divergence-free, like the magnetization current of electricity and magnetism.

Momentum transport by gyroviscosity arises from microscopic charged particle gyration [26, 27], and so it is natural to think that the mass and charge of the important species (ions for a single fluid model) would enter. With dimensional analysis one is led to the following form:

$$\Pi^{*i}/\mathscr{J} = -\frac{m}{2e}\nabla\left(\frac{p}{|B|}\right) \times \hat{b}, \qquad (29)$$

where  $\hat{b}$  is the direction of the magnetic field and it determines the plane of variation of the two-dimensional theory. The factor of 1/2 in (29) was included for reasons that will soon become clear.

Given the above, we propose the following gyrofluid action:

$$S_{GF} := S_{MHD} + \int_{t_0}^{t_1} dt \, G[q] \,. \tag{30}$$

Again,  $\delta S_{GF}/\delta q = 0$  produces a form of Newton's second law of the form  $\rho_0 \ddot{q} = \mathscr{F}$ , but now the force density includes a new term due to *G*. We will not give the details of this (tedious but straightforward) calculation, but remark that the new term turns out to be exactly the gyroviscosity tensor of Braginskii [21].

It was first shown by the author about twenty years ago (with the help of his then student Raul Acevedo ) that the above action gives the Braginskii equations [28, 29, 30]. In a series of papers, Bill Newcomb (now deceased) [31, 32, 33] developed a theory of incompressible gyrofluids. In correspondence with him we learned that the incompressible limit of the theory given here reduces to his theory [34].

# 6. HAMILTONIAN DESCRIPTION

There is an intimate relationship between action principles and Hamilton's equations; it is for this reason we put them under the same heading, HAP. For convex Lagrangians, this relationship is particularly straightforward and determined by a transformation referred to as the Legendre transform. Our two magnetofluid actions have convex Lagrangians. Thus, we briefly review the Legendre transform first for finite systems and then use this to motivate the transform for our infinite systems in terms of the Lagrangian variable description. Lastly, we show how to transform the Lagrangian variable Hamiltonian description to an Eulerian variable Hamiltonian description in terms of noncanonical Poisson brackets.

# 6.1. Legendre Transform and Hamiltonian Form

The Hamiltonian description incorporates phase space coordinates composed of q and a canonical momenta defined by

$$p_{\ell} = \frac{\partial L}{\partial \dot{q}^{\ell}}, \qquad \ell = 1, 2, \dots, N,$$
(31)

where the latter can be viewed as a coordinate replacing the velocity coordinate  $\dot{q}$ . To effect the Legendre transform one must be able to solve (31) for  $\dot{q}$  as a function of p, and by the inverse function theorem this requires the convexity condition

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}^\ell \partial \dot{q}^m}\right) \neq 0, \tag{32}$$

a condition that is satisfied for usual kinetic energy functions. Assuming one has solved for  $\dot{q}(p)$  the Hamiltonian is given by

$$H = p_\ell \dot{q}^\ell - L, \tag{33}$$

where  $\dot{q}$  is eliminated in terms of p and Newton's second order in time equations are replaced by Hamilton's equations

$$\dot{z}^{\alpha} = J_{c}^{\alpha\beta} \frac{\partial H}{\partial z^{\beta}} = \{ z^{\alpha}, H \}, \qquad (34)$$

where z = (q, p), the indices  $\alpha, \beta = 1, 2, \dots, 2N$ , and

$$J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}.$$
 (35)

In the second equality of (34) we have written Hamilton's equations in terms of the Poisson bracket defined by

$$\{f,g\} := \frac{\partial f}{\partial z^{\alpha}} J_c^{\alpha\beta} \frac{\partial g}{\partial z^{\beta}} = \frac{\partial f}{\partial q^{\ell}} \frac{\partial g}{\partial p_{\ell}} - \frac{\partial g}{\partial p_{\ell}} \frac{\partial f}{\partial q^{\ell}}, \tag{36}$$

which is a convenient way to define Hamiltonian systems. In fact, the essence of being Hamiltonian lies in certain properties of the Poisson bracket.

The more common way of writing (34) in terms of the Poisson bracket is

$$\dot{q}^{\ell} = \{q^{\ell}, H\} = \frac{\partial H}{\partial p_{\ell}} \quad \text{and} \quad \dot{p}_{\ell} = \{p_{\ell}, H\} = -\frac{\partial H}{\partial q^{\ell}},$$
(37)

and we will do this for our infinite-dimensional systems. For our infinite systems, the Lagrangian has the form

$$L = T[q] - V[q], \qquad (38)$$

the canonical momentum (density) is defined by

$$\Pi = \frac{\delta L}{\delta \dot{q}},\tag{39}$$

and the Hamiltonian functional is constructed as follows:

$$H[q,\Pi] = \int_D d^3 a \, \dot{q} \cdot \Pi - L, \qquad (40)$$

where, assuming convexity,  $\dot{q}$  is eliminated in terms of  $\Pi$ . The Poisson bracket in infinite dimensions, defined on functions *F*, *G*, say, is given by

$$\{F,G\} = \int_D d^3 a \left(\frac{\delta F}{\delta q} \cdot \frac{\delta G}{\delta \Pi} - \frac{\delta G}{\delta q} \cdot \frac{\delta F}{\delta \Pi}\right). \tag{41}$$

For MHD the canonical momentum is given by  $\Pi = \rho_0 \dot{q}$ , while for the gyrofluid model the canonical momentum is given by  $\Pi = \rho_0 \dot{q} + \Pi^*$ . Hamilton's equations for MHD are

$$\dot{q} = \{q, H\} = \frac{\delta H}{\delta \Pi} = \Pi / \rho_0 \quad \text{and} \quad \dot{\Pi} = \{\Pi, H\} = -\frac{\delta H}{\delta q} = \mathscr{F}, \quad (42)$$

where the Hamiltonian is  $H = T[\Pi] + V[q]$ . For the gyrofluid model the situation is more complicated and we will make a few comments about his below. In going from the first to the second equality in each of the above equations we have used

$$\frac{\delta q^{i}(a)}{\delta q^{j}(a')} = \delta^{i}_{j}\delta(a-a') \tag{43}$$

and similarly for  $\Pi$ .

#### **6.2.** Noncanonical Poisson Brackets

Now we transform the above Lagrangian variable Hamiltonian description to one in terms of Eulerian variables. We will see that Eulerian variables are not canonical variables and this affects the form of the Hamiltonian structure – in particular the Poisson bracket. The idea that hydrodynamics and MHD are Hamiltonian theories described in terms of noncanonical variables with a corresponding noncanonical Poisson bracket was introduced by Morrison and Greene in [5]. In this description, the dynamics is described by means of a noncanonical Poisson bracket that has Lie algebraic properties (see below), but does not have the usual canonical form and possesses degeneracy that gives rise to invariants known as Casimir invariants.

It is straightforward, but a bit tedious, to derive the noncanonical Poisson bracket; so, we only sketch the derivation. The basic tool used is the chain rule for functional derivatives, which we describe.

Suppose the functionals F and G that enter the Poisson bracket of (41) obtain their dependence on  $(q,\Pi)$  through the Eulerian variables, i.e.

$$F[q,\Pi] = \bar{F}[\rho, s, \nu, B].$$
(44)

Then, variation of F gives

$$\delta F = \int_{D} d^{3}a \left( \frac{\delta F}{\delta q} \cdot \delta q + \frac{\delta F}{\delta \Pi} \cdot \delta \Pi \right) = \delta \bar{F}$$
  
= 
$$\int_{D} d^{3}r \left( \frac{\delta \bar{F}}{\delta \rho} \delta \rho + \frac{\delta \bar{F}}{\delta s} \delta s + \frac{\delta \bar{F}}{\delta v} \cdot \delta v + \frac{\delta \bar{F}}{\delta B} \cdot \delta B \right).$$
(45)

Using (13) and similar expressions for the other Eulerian variables we obtain expressions like

$$\delta \rho = -\int_D d^3 a \,\rho_0 \nabla \delta(r-q) \cdot \delta q \,, \tag{46}$$

which we insert in (45) to obtain

$$\int_{D} d^{3}a \left( \frac{\delta F}{\delta q} \cdot \delta q + \frac{\delta F}{\delta \Pi} \cdot \delta \Pi \right) = -\int_{D} d^{3}r \frac{\delta \bar{F}}{\delta \rho} \int_{D} d^{3}a \rho_{0} \nabla \delta(r-q) \cdot \delta q + \dots$$
(47)

Interchanging the order of integration and removing the  $\int_D d^3 a$  by equating coefficients of  $\delta q$  gives an expression of the following form:

$$\frac{\delta F}{\delta q} = \mathscr{O}_{\rho} \frac{\delta \bar{F}}{\delta \rho} + \mathscr{O}_{s} \frac{\delta \bar{F}}{\delta s} + \mathscr{O}_{v} \frac{\delta \bar{F}}{\delta v} + \mathscr{O}_{B} \frac{\delta \bar{F}}{\delta B}, \tag{48}$$

where the  $\mathcal{O}$ 's are operators involving integration over  $d^3r$  and Dirac delta functions. Upon inserting (48) and a similar expression for  $\delta F/\delta\Pi$  into (41), doing some rearrangement, and dropping the overbars, yields

$$\{F,G\} = -\int_{D} d^{3}r \left[ \left( \frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta \nu} - \frac{\delta G}{\delta \rho} \nabla \cdot \frac{\delta F}{\delta \nu} \right) + \left( \frac{\nabla \times \nu}{\rho} \cdot \frac{\delta G}{\delta \nu} \times \frac{\delta F}{\delta \nu} \right) + \frac{\nabla s}{\rho} \cdot \left( \frac{\delta F}{\delta s} \frac{\delta G}{\delta \nu} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta \nu} \right) + B \cdot \left[ \frac{1}{\rho} \frac{\delta F}{\delta \nu} \cdot \nabla \frac{\delta G}{\delta B} - \frac{1}{\rho} \frac{\delta G}{\delta \nu} \cdot \nabla \frac{\delta F}{\delta B} \right] + B \cdot \left[ \nabla \left( \frac{1}{\rho} \frac{\delta F}{\delta \nu} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left( \frac{1}{\rho} \frac{\delta G}{\delta \nu} \right) \cdot \frac{\delta F}{\delta B} \right] \right].$$
(49)

This bracket together with the Hamiltonian

$$H = \int_{D} d^{3}r \left( \frac{\rho |\nu|^{2}}{2} + \rho U(\rho, s) + \frac{|B|^{2}}{2} \right),$$
 (50)

which we know by the closure principle can be written in terms of Eulerian variables alone, casts MHD into the Eulerian Hamiltonian form

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial s}{\partial t} = \{s, H\}, \quad \frac{\partial v}{\partial t} = \{v, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.$$
(51)

Using the density Eulerian variables,  $\{\rho, \sigma, M, B\}$ , the transformation gives instead

$$\{F,G\} = -\int_{D} d^{3}r \left[ M_{i} \left( \frac{\delta F}{\delta M_{j}} \frac{\partial}{\partial x^{j}} \frac{\delta G}{\delta M_{i}} - \frac{\delta G}{\delta M_{j}} \frac{\partial}{\partial x^{j}} \frac{\delta F}{\delta M_{i}} \right) + \rho \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left( \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right)$$

$$+ B \cdot \left[ \frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] + B \cdot \left[ \nabla \left( \frac{\delta F}{\delta M} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left( \frac{\delta G}{\delta M} \right) \cdot \frac{\delta F}{\delta B} \right] \right],$$
(52)

where dyadic notation is used; for example,  $B \cdot [\nabla(M) \cdot C] = \sum_{i,j} B_i C_j \partial_i M_j$ , for vectors B, M, and C. The equations of motion for ideal MHD in terms of these variables follow similarly.

The noncanonical Poisson brackets of both (49) and (52) no longer have the canonical form, yet certain algebraic properties are preserved by the transformation to Eulerian variables, viz. the Lie algebraic properties of bilinearity, antisymmetry, and the Jacobi identity. Thus, what it really means to be Hamiltonian is to have the equations of motion expressible in terms of a Poisson bracket with these properties. This is a coordinate independent definition and part of a bigger story – one beyond the scope of the present lectures.

The gyrofluid model in terms of the variable M, obtained from the canonical momentum of (15), possesses a Poisson bracket identical to that of (52). However, M is no longer related simply to the usual Eulerian velocity that arises from  $\dot{q}$ . If one transforms to this Eulerian variable one obtains a different and complicated noncanonical Poisson bracket (see [22]), one that is simplified by a change of coordinates that was termed the gyromap in [10]. Further details of the gyrofluid Hamiltonian structure and its action principle derivation will be published elsewhere.

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