MHD equilibrium variational principles with symmetry

2010 Plasma Phys. Control. Fusion 52 055001

(http://iopscience.iop.org/0741-3335/52/5/055001)

Plasma Phys. Control. Fusion **52** (2010) 055001 (22pp) doi:10.1088/0741-3335/52/5/055001

MHD equilibrium variational principles with symmetry

T Andreussi[1], P J Morrison[2] and F Pegoraro[3]

[1] Dipartimento di Ingegneria Aerospaziale L Lazzarino, Università di Pisa, Pisa, Italy
[2] Department of Physics and Institute for Fusion Studies, University of Texas, Austin, TX 78712, USA
[3] Dipartimento di Fisica E Fermi and CNISM, Pisa, Italy

E-mail: tommaso.andreussi@ing.unipi.it, morrison@physics.utexas.edu and pegoraro@df.unipi.it

Received 8 September 2009, in final form 6 January 2010
Published 31 March 2010
Online at stacks.iop.org/PPCF/52/055001

Abstract
The chain rule for functionals is used to reduce the noncanonical Poisson bracket for magnetohydrodynamics (MHD) to one for axisymmetric and translationally symmetric MHD and hydrodynamics. The procedure for obtaining Casimir invariants from noncanonical Poisson brackets is reviewed and then used to obtain the Casimir invariants for the considered symmetrical theories. It is shown why extrema of the energy plus Casimir invariants correspond to equilibria, thereby giving an explanation for the *ad hoc* variational principles that have existed in plasma physics. Variational principles for general equilibria are obtained in this way.

1. Introduction

The development of ideal magnetohydrodynamics (MHD) followed early attempts to understand astrophysical plasma phenomena and to model experiments in different scientific fields, ranging from research in magnetic fusion to that associated with innovative plasma based technologies such as electric propulsion and MHD generators. Like ideal hydrodynamics (HD), which is described by Euler's fluid equations, the MHD equations are typically expressed in terms of Eulerian variables, which facilitates the study of stationary (time-independent equilibrium) flows. Some insight into plasma and fluid behaviour using MHD and HD has been achieved, but because of the presence of nonlinear terms it is difficult to obtain solutions, even for the vastly simpler problems of equilibrium configurations and their nearby (linear) dynamics. Thus further simplifications of the MHD model are often introduced, and one class of such simplifications arises by applying symmetry constraints that reflect both technological choices and reasonable approximations for many phenomena.

Assuming axial symmetry in toroidal geometry, Grad and Rubin [1] and Shafranov [2] (see also [3]) obtained an equation for static MHD equilibria, and the identical equation was obtained earlier by Bragg and Hawthorne [4] in the HD context (see also [5–8]). In order to describe stationary equilibria, i.e. equilibria with time-independent flow, the Grad–Shafranov (GS) equation has been successively extended and generalized equilibrium (GE) equations have been obtained [9]. Over the years both the GS and GE equations have been studied. In addition, equations for general MHD equilibria with one ignorable coordinate have also been obtained [10] as well as relativistic GE equations [11] and equations for equilibria with translation symmetry, which permit the description of self-similar flows and the study of transitions between different flow regimes (see [10, 12]).

Contemporaneous with the above developments, variational formulations for various MHD equilibria have been constructed in an *ad hoc* manner. For example, a variational formulation of hydromagnetic equilibria, including the velocity field, was extensively treated for the first time by Woltjer [13–16]. His procedure was to bound the energy of the system with a sufficient number of dynamical constraints—however, a complete set of constraints was found only for the axisymmetric case. A similar approach was adopted by Taylor [17, 18] in order to deduce the final state of plasma relaxation under constraints for both toroidal and spherical topologies and to show that, if the plasma flow and internal energy can be neglected, the invariance of the total magnetic helicity produces specific force-free configurations. Another example is the variational principle for GE that was obtained in [9] and further investigated in [19–21] and references therein. A main goal of this paper is to show how to derive a variational principle for GE from the noncanonical Hamiltonian description of MHD introduced in [22, 23] with the imposition of symmetry, a generalization of the case of reduced MHD where this procedure was first carried out [24, 25]. Thus it is seen that the Lagrangian of previous work is in fact a Hamiltonian and, as is expected from mechanics, equilibria are extrema of this Hamiltonian.

Since both the HD and MHD models are ideal, i.e. dissipation free, a Hamiltonian description is to be expected (see, e.g., [26]). First attempts to define an Eulerian version of Hamilton's equations conflicted with the fact that the Eulerian variables, unlike the Lagrangian displacement variable and its conjugate momentum density, do not constitute a set of canonical variables. One consequence of this is the occurrence of the Casimir dynamical invariants, such as helicity and cross helicity, that have been used in constructing variational principle for equilibria (see, e.g., [27–29]). Thus it is observed that the collection of dynamical invariants and associated variational principles that have been obtained for a myriad of plasma kinetic and fluid models all derive from a basic Hamiltonian theory.

The idea that HD and MHD are Hamiltonian theories described in terms of noncanonical variables with a corresponding noncanonical Poisson bracket was introduced by Morrison and Greene in [22, 23]. In this description, the dynamics is described by means of a noncanonical Poisson bracket that has Lie algebraic properties (see below), but does not have the usual canonical form and possesses degeneracy that gives rise to the Casimir invariants. Over the years this idea has been extended to essentially all kinetic and fluid models in Eulerian variables, including, for example, the BBGKY hierarchy [30] and recent reduced fluid models for reconnection and tokamak dynamics such as [31, 32]. Various kinds of derivations are possible within this context, such as that recently given for the Charney–Hasegawa–Mima equation in [33].

The Hamiltonian structure for systems that describe continuous media in terms of a generic set of Eulerian variables, $\xi(x) = (\xi_1, \xi_2, \ldots, \xi_m)$, has the following form:

$$\frac{\partial \xi}{\partial t} = \{\xi, H\}, \tag{1}$$

where H represents the Hamiltonian of the system and usually corresponds to the energy; the noncanonical Poisson bracket $\{\cdot, \cdot\}$ is given by

$$\{F, G\} = \int \frac{\delta F}{\delta \xi_i} \mathcal{J}_{ij} \frac{\delta G}{\delta \xi_j} \, \mathrm{d}^n r, \tag{2}$$

where i and j are summed $1, 2, \ldots, m$, F and G are functionals, an example being the total energy of the system; $\mathrm{d}^n r$ is the volume element, where for fluids $n = 2$ or 3; $\delta F/\delta \xi_i$ denotes the functional derivative defined by

$$\delta F = \int \delta \xi_i \frac{\delta F}{\delta \xi_i} \, \mathrm{d}^n r;$$

and the operator $\mathcal{J}$ generally depends on ξ and is degenerate, but endows the bracket of (2) with the properties of antisymmetry, $\{F, G\} = -\{G, F\}$, bilinearity, $\{F + \lambda K, G\} = \{F, G\} + \lambda \{K, G\}$, for all real numbers λ, and the Jacobi identity,

$$\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} \equiv 0,$$

where the above are to be satisfied for all functionals F, G and K.

The degeneracy in $\mathcal{J}$ arises from the fact that the transformation from Lagrangian to Eulerian variables is not invertible (see, e.g., [34, 35])), and this in turn gives rise to the Casimir invariants, which satisfy

$$\{C, F\} = 0 \Leftrightarrow \mathcal{J}_{ij} \frac{\delta C}{\delta \xi_j} = 0 \tag{3}$$

for all functionals F. The Casimir functionals C, which are determined by the noncanonical Poisson bracket alone, clearly commute with any Hamiltonian. Thus, the Hamiltonian that generates the dynamics in the form of (1) is not unique and can be replaced by $H + C$, because $\{\xi, H + C\} = \{\xi, H\}$. Thus, from (1) written with Hamiltonian $\mathfrak{F} := H + C$,

$$\frac{\partial \xi_i}{\partial t} = \mathcal{J}_{ij} \frac{\delta \mathfrak{F}}{\delta \xi_j},$$

we see the extrema of $\mathfrak{F}$ correspond to equilibria.

Although the above explicit connection between extrema of $\mathfrak{F}$ and equilibria was not known in the early years of plasma physics, it was known for many cases by direct calculation that extremization of $H + C$ for various known Cs gave rise to equilibria, and that this kind of variational principle could be used for addressing stability by computing the second variation of $\mathfrak{F}$ (see Kruskal and Oberman [36]). Now we understand this to be an infinite-dimensional generalization of Dirichlet's principle of Hamiltonian mechanics where $\mathfrak{F}$ serves as a Lyapunov functional (see, e.g., [28]). The name Casimir invariant in this context was introduced in the early 1980s, in analogy with the invariant associated with the magnitude of angular momentum for the group of rotations. Many examples of energy-Casimir stability for fluid and plasma equilibria have been worked out and reviewed in [28, 37–39], but the references particularly relevant to this paper are [30, 40] where the procedure was first carried out for MHD, that of [41] where a variational principle for translationally symmetric MHD was obtained, but it was not shown that the invariants used were in fact Casimir invariants, and [42] where Casimir invariants and a variational principle with the assumption of the existence of flux surfaces were obtained (see also [43]). A refined approach by considering a type of constrained variations, called dynamically accessible, a terminology introduced in [44, 45], has also been given in [42]. In [46] this same approach is used to study the linear stability of Hall MHD equilibria with flow.

In section 2 we briefly summarize the noncanonical Hamiltonian description of MHD. In the next five sections we show how to reduce this description to one that imposes axial

symmetry on the generated dynamics and the ramifications that follow from this imposition. This will lead us to an energy-Casimir variational principle that produces the GE equations of axisymmetric MHD. In section 3 a representation of the magnetic field in terms of a flux function is introduced to enforce axial symmetry. A similar representation is also introduced for the velocity field in terms of a stream function and momentum potential. This is done in order to facilitate the HD limit ($\boldsymbol{B} \to 0$), but is not necessary for the MHD description. In section 4 it is shown how to transform the Poisson bracket of section 2 into the axisymmetric variables of section 3, and then in section 5 condition (3) is solved for the axisymmetric Poisson bracket to obtain the Casimir invariants. These invariants are then used in section 6 to obtain the variational principle for the GE equations. Next, in section 7, the hydrodynamic limit is presented: the HD Poisson bracket for axisymmetric configurations is deduced from the general MHD form by neglecting the variables associated with the magnetic field and the Casimirs are determined for this model. Section 8 concerns translational symmetry, and here the previous results are adapted to this case for both the HD and MHD models. Finally, we conclude in section 9.

2. Noncanonical Hamiltonian dynamics of MHD

The MHD equations are often written in terms of Eulerian variables as follows:

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}), \tag{4}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} - \frac{1}{\rho} \boldsymbol{\nabla} p + \frac{1}{4\pi\rho} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{5}$$

$$\frac{\partial s}{\partial t} = -\boldsymbol{v} \cdot \boldsymbol{\nabla} s, \tag{6}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{B} \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{B} \cdot \boldsymbol{\nabla} \boldsymbol{v} - \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{B}, \tag{7}$$

where ρ is the plasma density, $\boldsymbol{v}$ is the velocity field, s is the plasma entropy per unit mass and $\boldsymbol{B}$ is the magnetic field. The pressure p in (5) and the plasma temperature, T, can be expressed in terms of the plasma internal energy per unit mass, $U(\rho, s)$, by means of the thermodynamic relationships

$$p = \rho^2 U_\rho, \qquad T = U_s, \tag{8}$$

where U is a function of ρ and s and subscripts indicate partial derivatives. With an appropriate choice for U one can eliminate the entropy equation and replace it by a pressure or temperature equation.

The noncanonical Poisson bracket of Morrison and Greene [22, 23], which is the Eulerian counterpart to the Lagrangian variable canonical Hamiltonian description of MHD given by Newcomb [47], is given by the following:

$$\begin{aligned}\{F, G\} = -\int_V \Bigg\{ & \frac{\delta F}{\delta \rho} \boldsymbol{\nabla} \cdot \frac{\delta G}{\delta \boldsymbol{v}} - \frac{\delta G}{\delta \rho} \boldsymbol{\nabla} \cdot \frac{\delta F}{\delta \boldsymbol{v}} + \frac{\boldsymbol{\nabla} \times \boldsymbol{v}}{\rho} \cdot \left(\frac{\delta G}{\delta \boldsymbol{v}} \times \frac{\delta F}{\delta \boldsymbol{v}} \right) \\ & + \frac{\boldsymbol{\nabla} s}{\rho} \cdot \left(\frac{\delta F}{\delta s} \frac{\delta G}{\delta \boldsymbol{v}} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta \boldsymbol{v}} \right) + \boldsymbol{B} \cdot \left[\left(\frac{1}{\rho} \frac{\delta F}{\delta \boldsymbol{v}} \cdot \boldsymbol{\nabla} \right) \frac{\delta G}{\delta \boldsymbol{B}} - \left(\frac{1}{\rho} \frac{\delta G}{\delta \boldsymbol{v}} \cdot \boldsymbol{\nabla} \right) \frac{\delta F}{\delta \boldsymbol{B}} \right] \\ & + \boldsymbol{B} \cdot \left[\left(\boldsymbol{\nabla} \frac{1}{\rho} \frac{\delta F}{\delta \boldsymbol{v}} \right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} - \left(\boldsymbol{\nabla} \frac{1}{\rho} \frac{\delta G}{\delta \boldsymbol{v}} \right) \cdot \frac{\delta F}{\delta \boldsymbol{B}} \right] \Bigg\} \mathrm{d}^3 r. \end{aligned} \tag{9}$$

The bracket of (9) together with the Hamiltonian

$$H[\xi] = \int_V \left[\frac{1}{2}\rho v^2 + \rho U + \frac{1}{8\pi}B^2\right] \mathrm{d}^3 r \tag{10}$$

casts the MHD equations of (4)–(7) into the form of (1), where the generic Eulerian variable ξ constitutes an element of the set $\{\rho, \boldsymbol{v}, s, \boldsymbol{B}\}$. To show the equivalence of (4)–(7) to (1) the functional derivatives,

$$\begin{aligned} \frac{\delta H}{\delta \rho} &= \frac{1}{2}v^2 + U + \rho U_\rho, \qquad & \frac{\delta H}{\delta \boldsymbol{v}} &= \rho \boldsymbol{v}, \\ \frac{\delta H}{\delta s} &= \rho U_s, \qquad & \frac{\delta H}{\delta \boldsymbol{B}} &= \frac{1}{4\pi}\boldsymbol{B}, \end{aligned} \tag{11}$$

together with

$$\frac{\delta \xi_i(\boldsymbol{x}')}{\delta \xi_i(\boldsymbol{x})} = \delta(\boldsymbol{x}' - \boldsymbol{x}),$$

where $\boldsymbol{x} = (x, y, z)$ denotes the usual Cartesian coordinates, are used. See [27] for further details.

Sometimes it is more convenient to use a different set of Eulerian variables, in particular, the densities $\zeta := \{\rho, \boldsymbol{M}, \sigma, \boldsymbol{B}\}$, where

$$\boldsymbol{M} = \rho \boldsymbol{v}, \qquad \sigma = \rho s.$$

The Hamiltonian in terms of these new variables is

$$H[\zeta] = \int_V \left[\frac{M^2}{2\rho} + \rho U + \frac{B^2}{8\pi}\right] \mathrm{d}^3 r. \tag{12}$$

Using the chain rule for functional derivatives

$$\begin{aligned} &\left.\frac{\delta F}{\delta \rho}\right|_{v,s} = \left.\frac{\delta F}{\delta \rho}\right|_{M,\sigma} + \frac{\boldsymbol{M}}{\rho}\cdot\frac{\delta F}{\delta \boldsymbol{M}} + \frac{\sigma}{\rho}\frac{\delta F}{\delta \sigma}, \\ &\frac{\delta F}{\delta \boldsymbol{v}} = \rho\frac{\delta F}{\delta \boldsymbol{M}} \qquad \text{and} \qquad \frac{\delta F}{\delta s} = \rho\frac{\delta F}{\delta \sigma}, \end{aligned}$$

it was shown in [22] that in terms of the set ζ, the bracket of (9) obtains Lie–Poisson form:

$$\begin{aligned} \{F, G\} = -\int_V \Bigg[&\rho\left(\frac{\delta F}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\frac{\delta F}{\delta \rho}\right) + M_i\left(\frac{\delta F}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j}\frac{\partial}{\partial x^j}\frac{\delta F}{\delta M_i}\right) \\ &+ \sigma\left(\frac{\delta F}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\frac{\delta F}{\delta \sigma}\right) + \boldsymbol{B}\cdot\left(\frac{\delta F}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{M}}\cdot\boldsymbol{\nabla}\frac{\delta F}{\delta \boldsymbol{B}}\right) \\ &+ \boldsymbol{B}\cdot\left(\boldsymbol{\nabla}\frac{\delta F}{\delta \boldsymbol{M}}\cdot\frac{\delta G}{\delta \boldsymbol{B}} - \boldsymbol{\nabla}\frac{\delta G}{\delta \boldsymbol{M}}\cdot\frac{\delta F}{\delta \boldsymbol{B}}\right)\Bigg] \mathrm{d}^3 r, \end{aligned} \tag{13}$$

i.e. it is linear with respect to each dynamical variable. With the bracket of (13), equations (4)–(7) are equivalent to

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \{\rho, H\}, \qquad & \frac{\partial \boldsymbol{M}}{\partial t} &= \{\boldsymbol{M}, H\}, \\ \frac{\partial \sigma}{\partial t} &= \{\sigma, H\}, \qquad & \frac{\partial \boldsymbol{B}}{\partial t} &= \{\boldsymbol{B}, H\} \end{aligned} \tag{14}$$

which is equivalent to MHD in conservation form.

3. Axial symmetry

Introducing the cylindrical coordinate system (r, ϕ, z), $\boldsymbol{B}$ and $\boldsymbol{M}$ can be expressed in terms of their azimuthal components, B_ϕ and M_ϕ, and their poloidal components, $\boldsymbol{B}_p$ and $\boldsymbol{M}_p$, respectively. We assume the azimuthal angle ϕ is an ignorable coordinate and introduce scalar fields ψ, χ and Υ to express $\boldsymbol{B}$ and $\boldsymbol{M}$ as follows:

$$\boldsymbol{B} = B_\phi \hat{\boldsymbol{\phi}} + \boldsymbol{\nabla}\psi \times \boldsymbol{\nabla}\phi, \tag{15}$$

$$\boldsymbol{M} = M_\phi \hat{\boldsymbol{\phi}} + \boldsymbol{\nabla}\chi \times \boldsymbol{\nabla}\phi + \boldsymbol{\nabla}\Upsilon, \tag{16}$$

where $\hat{\boldsymbol{\phi}} = r\boldsymbol{\nabla}\phi$ is the unit vector in the azimuthal direction. As noted above, it is not necessary to introduce the variables χ and Υ for $\boldsymbol{M}_p$, and indeed the Casimirs we eventually obtained are unchanged by this choice, but we do this for convenience in considering the HD limit. With the above assumptions, $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \cdot \boldsymbol{M} = \Delta\Upsilon$, where Δ denotes the Laplacian, and since $\boldsymbol{B} \cdot \boldsymbol{\nabla}\psi = 0$, (15) implies that each magnetic flux surface coincides with a constant ψ surface. For two generic magnetic flux surfaces, the difference between the values of ψ is equal to the flux of the magnetic field enclosed between them and thus ψ is usually called the magnetic flux function.

In order to simplify expressions below, we introduce

$$D := \boldsymbol{\nabla} \cdot \boldsymbol{M} = \Delta\Upsilon \tag{17}$$

and assume Δ has an inverse so that Υ is given by

$$\Upsilon = \Delta^{-1} (\boldsymbol{\nabla} \cdot \boldsymbol{M}) . \tag{18}$$

Similarly, the function χ is related to $\Omega := \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla} \times \boldsymbol{M}$, through the equation

$$\Omega = -\boldsymbol{\nabla} \cdot \left(|\boldsymbol{\nabla}\phi|^2 \boldsymbol{\nabla}\chi \right) := \mathcal{L}\chi , \tag{19}$$

where the elliptic operator $\mathcal{L}$ is formally self-adjoint. We assume it has an inverse so that

$$\chi = \mathcal{L}^{-1} (\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla} \times \boldsymbol{M}) . \tag{20}$$

Thus, by the above, we have defined a noninvertible coordinate change: given $\zeta_{\mathrm{AS}} := \{\rho, \chi, \Upsilon, M_\phi, \sigma, \psi, B_\phi\}$ we can construct $\zeta = \{\rho, \boldsymbol{M}, \sigma, \boldsymbol{B}\}$ but in general not vice versa. In the next section we will see that in spite of this noninvertibility, the Poisson bracket in terms of the variables ζ_{AS} can be constructed.

4. Axisymmetric Poisson bracket

Now, with the assumptions of section 3, we reduce the Poisson bracket of (13) to one that generates axisymmetric dynamics. To this end we must map the functional derivatives with respect to elements of the set ζ to ones with respect to elements of the set ζ_{AS}. This requires making use of the chain rule for functional derivatives, which we describe in detail.

Suppose $F[\zeta]$ is a functional of the MHD density variables. By functional we mean a quantity, usually involving a volume integration, that gives a real number when evaluated on particular choices for the fields ζ (see [27, 28] for more details). The axisymmetric Poisson bracket will naturally depend on functionals of the form $\bar{F}[\zeta_{\mathrm{AS}}]$. To relate functional derivatives we suppose $\bar{F}$ obtains its dependence on ζ_{AS} through functionals of ζ, i.e.

$$\bar{F}[\zeta_{\mathrm{AS}}] = F[\zeta] \tag{21}$$

with (15) and (16) inserted into the right-hand side of (21). Variation of (21) gives

$$\int_V \frac{\delta F}{\delta \boldsymbol{M}} \cdot \delta \boldsymbol{M} \, \mathrm{d}^3 r = \int_V \left[\frac{\delta \bar{F}}{\delta M_\phi} \delta M_\phi + \frac{\delta \bar{F}}{\delta \chi} \delta\chi + \frac{\delta \bar{F}}{\delta \Upsilon} \delta\Upsilon \right] \mathrm{d}^3 r, \tag{22}$$

while variation of (18) and (20) gives

$$\delta\chi = \mathcal{L}^{-1}\left(\nabla\phi\cdot\nabla\times\delta\boldsymbol{M}\right) \qquad \text{and} \qquad \delta\Upsilon = \Delta^{-1}\left(\nabla\cdot\delta\boldsymbol{M}\right). \tag{23}$$

Using (23) and the fact that both $\mathcal{L}$ and Δ are formally self-adjoint, (22) can be rewritten as follows:

$$\int_V \frac{\delta F}{\delta\boldsymbol{M}}\cdot\delta\boldsymbol{M}\,\mathrm{d}^3r = \int_V\left[\frac{\delta F}{\delta M_\phi}\hat{\phi}\cdot\delta\boldsymbol{M} + \nabla\left(\mathcal{L}^{-1}\frac{\delta F}{\delta\chi}\right)\times\nabla\phi\cdot\delta\boldsymbol{M} - \nabla\Delta^{-1}\frac{\delta F}{\delta\Upsilon}\cdot\delta\boldsymbol{M}\right]\mathrm{d}^3r, \tag{24}$$

where the last two terms have been integrated by parts with surface terms dropped. In the following we either consider variations such that $\delta\boldsymbol{M}$ vanishes at the domain boundaries or assume that natural conditions hold, thus neglecting surface terms [21, 48]. On the right-hand side of (24) we have dropped the 'bars' on the functionals; we do this henceforth since the proper arguments are clear from context. Considering the arbitrariness of the variation $\delta\boldsymbol{M}$, expression (24) yields

$$F_M = F_{M_\phi}\hat{\phi} + \nabla\left(\mathcal{L}^{-1}F_\chi\right)\times\nabla\phi - \nabla\left(\Delta^{-1}F_\Upsilon\right) \tag{25}$$

where a compact subscript notation for the functional derivatives,

$$\frac{\delta F}{\delta\zeta_{\mathrm{AS}}} = F_{\zeta_{\mathrm{AS}}},$$

has been introduced.

Moreover, we can use definitions (18) and (19) in order to deduce expressions for the functional derivatives, i.e.

$$\int_V F_\Upsilon\delta\Upsilon\,\mathrm{d}^3r = \int_V F_D\delta D\,\mathrm{d}^3r = \int_V \Delta F_D\delta\Upsilon\,\mathrm{d}^3r \tag{26}$$

and

$$\int_V F_\chi\delta\chi\,\mathrm{d}^3r = \int_V F_\Omega\delta\Omega\,\mathrm{d}^3r = \int_V \mathcal{L}F_\Omega\delta\chi\,\mathrm{d}^3r, \tag{27}$$

where again we have exploited the self-adjointness of the two operators Δ and $\mathcal{L}$. Because $\delta\Upsilon$ and $\delta\chi$ are arbitrary, we conclude

$$F_\Upsilon = \Delta F_D \quad \text{and} \quad F_\chi = \mathcal{L}F_\Omega. \tag{28}$$

Inverting and substituting (28) into (25) yield

$$F_M = F_{M_\phi}\hat{\phi} + \nabla F_\Omega\times\nabla\phi - \nabla F_D, \tag{29}$$

and it is straightforward to prove the relationships

$$\Delta F_D = F_\Upsilon = -\nabla\cdot F_M, \qquad \mathcal{L}F_\Omega = F_\chi = \nabla\phi\cdot\nabla\times F_M, \qquad \text{and} \qquad F_{M_\phi} = F_M\cdot\hat{\phi}. \tag{30}$$

In a similar way, we can prove that the functional derivative with respect to B_ϕ and ψ satisfies

$$F_{B_\phi} = F_B\cdot\hat{\phi} \qquad \text{and} \qquad F_\psi = \nabla\phi\cdot\nabla\times F_B. \tag{31}$$

Now upon substitution of the expressions for the functional derivatives of (29) and (31) into the Poisson bracket (13) and enforcing $\partial/\partial\phi = 0$ produces

$$\begin{aligned}
\{F, G\}_{\mathrm{AS}} = & -\int_V \Big\{ \rho \left(\left[G_\rho, F_\Omega\right] - \left[F_\rho, G_\Omega\right] \right) - \rho \left(\nabla F_D \cdot \nabla G_\rho - \nabla G_D \cdot \nabla F_\rho \right) \\
& + r M_\phi \left(\left[\frac{G_{M_\phi}}{r}, F_\Omega \right] - \left[\frac{F_{M_\phi}}{r}, G_\Omega \right] \right) \\
& - r M_\phi \left(\nabla F_D \cdot \nabla \frac{G_{M_\phi}}{r} - \nabla G_D \cdot \nabla \frac{F_{M_\phi}}{r} \right) \\
& + \Omega \left[G_\Omega, F_\Omega\right] - \Omega \left(\nabla G_\Omega \cdot \nabla F_D - \nabla F_\Omega \cdot \nabla G_D \right) + \frac{\Omega}{|\nabla\phi|^2} \left[G_D, F_D\right] \\
& + \Upsilon \left(\left[G_\Upsilon, F_\Omega\right] - \left[F_\Upsilon, G_\Omega\right] \right) - \Upsilon \left(\nabla F_D \cdot \nabla G_\Upsilon - \nabla G_D \cdot \nabla F_\Upsilon \right) \\
& - |\nabla\phi|^2 \nabla\chi \cdot \left(\nabla F_\Omega G_\Upsilon - \nabla G_\Omega F_\Upsilon \right) - \chi \left(\left[F_D, G_\Upsilon\right] - \left[G_D, F_\Upsilon\right] \right) \\
& + \sigma \left(\left[G_\sigma, F_\Omega\right] - \left[F_\sigma, G_\Omega\right] \right) - \sigma \left(\nabla F_D \cdot \nabla G_\sigma - \nabla G_D \cdot \nabla F_\sigma \right) \\
& + \frac{B_\phi}{r} \left(\left[r G_{B_\phi}, F_\Omega \right] - \left[r F_{B_\phi}, G_\Omega \right] \right) \\
& - \frac{B_\phi}{r} \left(\nabla \left(r G_{B_\phi} \right) \cdot \nabla F_D - \nabla \left(r F_{B_\phi} \right) \cdot \nabla G_D \right) \\
& + \psi \left(\left[G_\psi, F_\Omega\right] - \left[F_\psi, G_\Omega\right] \right) - \psi \left(\nabla F_D \cdot \nabla G_\psi - \nabla G_D \cdot \nabla F_\psi \right) \\
& + \psi \left(\left[r G_{B_\phi}, \frac{F_{M_\phi}}{r} \right] - \left[r F_{B_\phi}, \frac{G_{M_\phi}}{r} \right] \right) + \psi \left(G_\Upsilon F_\psi - F_\Upsilon G_\psi \right) \Big\} \, \mathrm{d}^3 r,
\end{aligned} \tag{32}$$

where

$$(\nabla A \times \nabla B) \cdot \nabla\phi = [A, B] = \nabla \cdot (B \nabla\phi \times \nabla A) \,. \tag{33}$$

is used to simplify the notation. Details of this calculation can be found in appendix A. Expression (32) represents the noncanonical Poisson bracket for axisymmetric MHD.

The Hamiltonian in terms of the variables ζ_{AS} is given by

$$H_{\mathrm{AS}}\left[\zeta_{\mathrm{AS}}\right] = \int_V \left(\frac{M_\phi^2}{2\rho} + \frac{|\nabla\chi|^2}{2\rho r^2} + \frac{|\nabla\Upsilon|^2}{2\rho} + \frac{[\Upsilon, \chi]}{\rho} + \rho U + \frac{|\nabla\psi|^2}{8\pi r^2} + \frac{B_\phi^2}{8\pi} \right) \mathrm{d}^3 r. \tag{34}$$

With the Hamiltonian (34), the equations of motion for axisymmetric MHD can be expressed in terms of the Poisson bracket (32) as follows:

$$\frac{\partial \zeta_{\mathrm{AS}}}{\partial t} = \{\zeta_{\mathrm{AS}}, H_{\mathrm{AS}}\}_{\mathrm{AS}}. \tag{35}$$

The calculations implied by the above are carried out in detail in appendix B, and in this way the equations for axisymmetric MHD in the ζ_{AS} variables are explicitly obtained.

5. Axisymmetric Casimirs

Now we seek the Casimir invariants associated with the axisymmetric MHD bracket (32), i.e. functionals that satisfy

$$\{F, C\}_{\mathrm{AS}} = 0 \tag{36}$$

for all functionals F. Casimir invariants are constants of motion that are built into the phase space, which for MHD is a function space, and the set of Casimirs, $\{C_i | i = 1, 2, \ldots\}$, define families of subspaces to which the solution is confined in the course of the MHD dynamics.

With (32) we see that (36) implies

$$\int_V \left[F_\rho \mathfrak{C}_1 + \frac{F_{M_\phi}}{r}\mathfrak{C}_2 + F_\sigma \mathfrak{C}_3 + rF_{B_\phi}\mathfrak{C}_4 + F_\psi \mathfrak{C}_5 + F_\Omega \mathfrak{C}_6 + F_D \mathfrak{C}_7 \right] \mathrm{d}^3 r = 0,$$

where the functions $\mathfrak{C}_i$ are given by

$$\mathfrak{C}_1 = [\rho, C_\Omega] - \nabla \cdot (\rho \nabla C_D)\,, \tag{37}$$

$$\mathfrak{C}_2 = [\sigma, C_\Omega] - \nabla \cdot (\sigma \nabla C_D)\,, \tag{38}$$

$$\mathfrak{C}_3 = \left[rM_\phi, C_\Omega\right] - \nabla \cdot \left(rM_\phi \nabla C_D\right) + \left[\psi, rC_{B_\phi}\right], \tag{39}$$

$$\mathfrak{C}_4 = \left[\frac{B_\phi}{r}, C_\Omega\right] - \nabla \cdot \left(\frac{B_\phi}{r}\nabla C_D\right) + \left[\psi, \frac{C_{M_\phi}}{r}\right], \tag{40}$$

$$\mathfrak{C}_5 = [\psi, C_\Omega] - \nabla \cdot (\psi \nabla C_D) + \psi C_\Upsilon, \tag{41}$$

$$\begin{aligned}\mathfrak{C}_6 = {} & \left[\rho, C_\rho\right] + \left[rM_\phi, \frac{C_{M_\phi}}{r}\right] + [\sigma, C_\sigma] + \left[\frac{B_\phi}{r}, rC_{B_\phi}\right] + \left[\psi, C_\psi\right] \\ & + [\Omega, C_\Omega] - \nabla \cdot (\Omega \nabla C_D) + [\Upsilon, C_\Upsilon] + \nabla \cdot \left(C_\Upsilon |\nabla\phi|^2 \nabla\chi\right),\end{aligned} \tag{42}$$

$$\begin{aligned}\mathfrak{C}_7 = {} & \nabla \cdot \left(\rho \nabla C_\rho\right) + \nabla \cdot \left(rM_\phi \nabla \frac{C_{M_\phi}}{r}\right) + \nabla \cdot (\sigma \nabla C_\sigma) + \nabla \cdot \left(\frac{B_\phi}{r}\nabla rC_{B_\phi}\right) \\ & + \nabla \cdot \left(\psi \nabla C_\psi\right) + \nabla \cdot (\Omega, \nabla C_\Omega) + \left[r^2\Omega, C_D\right] + \nabla \cdot (\Upsilon \nabla C_\Upsilon) + [\chi, C_\Upsilon] \\ & + \triangle \left([\Upsilon, C_\Omega] - \nabla \cdot (\Upsilon \nabla C_D) + [\chi, C_D] + |\nabla\phi|^2 \nabla\chi \cdot \nabla C_\Omega - \psi C_\psi\right).\end{aligned} \tag{43}$$

Since each term in the bracket must vanish separately, it can be seen that this implies a set of Casimir conditions

$$\mathfrak{C}_i = 0 \qquad \text{for } i = 1\text{–}7.$$

A further simplification of the Casimir conditions is possible. In fact, (37) is equivalent to

$$\nabla \cdot (\rho \nabla C_\Omega \times \nabla\phi - \rho \nabla C_D) = 0, \tag{44}$$

and thus the term inside the divergence satisfies

$$\rho \left(\nabla C_\Omega - \frac{\nabla\phi}{|\nabla\phi|^2} \times \nabla C_D\right) = \nabla A, \tag{45}$$

where A is an arbitrary function. If the plasma density is assumed to be strictly positive everywhere inside the domain, then (45) can be divided by ρ to give

$$\nabla C_\Omega - r^2 \nabla\phi \times \nabla C_D = \frac{1}{\rho}\nabla A. \tag{46}$$

Because the curl of (46) only has an azimuthal component, this equation yields

$$\triangle C_D = C_\Upsilon = \nabla\phi \cdot \nabla A \times \nabla \frac{1}{\rho} = \left[A, \frac{1}{\rho}\right]. \tag{47}$$

Moreover, upon multiplying equation (46) by $|\nabla\phi|^2$ and taking its divergence, we obtain

$$\mathcal{L}C_\Omega = C_\chi = -\nabla \cdot \left(|\nabla\phi|^2 \frac{1}{\rho}\nabla A\right). \tag{48}$$

Thus, the single condition (37) is satisfied by every functional C for which

$$C_\chi = -\nabla \cdot \left(|\nabla\phi|^2 \frac{1}{\rho}\nabla A\right), \qquad C_\Upsilon = \left[A, \frac{1}{\rho}\right], \tag{49}$$

where A is an arbitrary function.

Using (46) in the Casimir conditions (38)–(41) produces the following:

$$0 = \left[\frac{\sigma}{\rho}, A\right], \tag{50}$$

$$0 = \left[\frac{rM_\phi}{\rho}, A\right] + \left[\psi, rC_{B_\phi}\right], \tag{51}$$

$$0 = \left[\frac{B_\phi}{r\rho}, A\right] + \left[\psi, C_{M_\phi}/r\right], \tag{52}$$

$$0 = \frac{1}{\rho}\left[\psi, A\right], \tag{53}$$

while conditions (42) and (43) represent the curl and the divergence of the equation

$$\rho\nabla C_\rho + \sigma\nabla C_\sigma + rM_\phi\nabla\frac{C_\phi}{r} + \frac{B_\phi}{r}\nabla\left(rC_{B_\phi}\right) - C_\psi\nabla\psi$$
$$+ (\Omega/\rho)\nabla A - \boldsymbol{M}_\perp\left[A, \frac{1}{\rho}\right] + \nabla\left(\frac{1}{\rho}\boldsymbol{M}_\perp\cdot\nabla A\times\nabla\phi\right) = 0. \tag{54}$$

From these equations, with $A \equiv 0$ we directly obtain the following Casimir invariants

$$C_{1\sigma} = \int_V \rho\mathcal{K}\left(\frac{\sigma}{\rho}\right)\mathrm{d}^3r, \tag{55}$$

$$C_{1\psi} = \int_V \rho\mathcal{J}(\psi)\,\mathrm{d}^3r, \tag{56}$$

$$C_2 = \int_V \frac{B_\phi}{r}\mathcal{H}(\psi)\,\mathrm{d}^3r, \tag{57}$$

$$C_3 = \int_V rM_\phi\mathcal{G}(\psi)\,\mathrm{d}^3r, \tag{58}$$

where the Casimir invariant $C_{1\sigma}$ is to be expected, since it is a Casimir invariant also for the general, nonaxisymmetric MHD bracket (first shown in [27, 34, 35]).

Now, stepping back to the general form of the Casimir conditions ($A \neq 0$), the two equations (50) and (53) imply an overspecification of the function A. Thus, unless the constraint

$$\left[\psi, \frac{\sigma}{\rho}\right] = 0 \tag{59}$$

is assumed, no other solution of the system is possible. This feature of MHD was already pointed out in [27], where Casimir invariants were first sought, and again in [34, 35] in the context of MHD relabelling symmetry. However, if the constraint (59) holds (which is the case needed for the derivation of the GE equations) one more Casimir invariant exists. Whence (51) and (52) can be rewritten as

$$0 = \left[A'\frac{rM_\phi}{\rho} - rC_{B_\phi}, \psi\right] \qquad \text{and} \qquad 0 = \left[A'\frac{B_\phi}{r\rho} - \frac{C_{M_\phi}}{r}, \psi\right], \tag{60}$$

which imply

$$rC_{B_\phi} = A'\frac{rM_\phi}{\rho} + f_1 \qquad \text{and} \qquad \frac{C_{M_\phi}}{r} = A'\frac{B_\phi}{r\rho} + f_2, \tag{61}$$

where f_1 and f_2 are two arbitrary functions of ψ. By integrating conditions (49) and (61), we obtain

$$C_4 = \int_V\left(\frac{M_\phi B_\phi}{\rho}A' + \frac{|\nabla\phi|^2}{\rho}\nabla A\cdot\nabla\chi + \frac{[\Upsilon, A]}{\rho}\right)\mathrm{d}^3r = \int_V \boldsymbol{v}\cdot\boldsymbol{B}\,A'\,\mathrm{d}^3r, \tag{62}$$

where A is a function of ψ or σ/ρ. Note that because f_1 and f_2 give rise to additional terms of the form of (57) and (58), they can be neglected. It is straightforward to prove that C_4 also satisfies (54).

6. Axisymmetric variational principle

Now we proceed as described in section 1 to construct the energy-Casimir variational principle for axisymmetric MHD. With the knowledge that extrema of the energy-Casimir functional correspond to equilibria, we consider

$$\mathfrak{F}_{\mathrm{AS}} = H_{\mathrm{AS}} - \int_V \rho\mathcal{J}\,\mathrm{d}^3r - \int_V \frac{B_\phi}{r}\mathcal{H}\,\mathrm{d}^3r - \int_V rM_\phi\mathcal{G}\,\mathrm{d}^3r - \int_V \boldsymbol{v}\cdot\boldsymbol{B}\,\mathcal{F}\,\mathrm{d}^3r, \tag{63}$$

where H_{AS} is given by (34) and $\mathcal{F} \equiv A'$, $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ are four arbitrary functions of ψ. Note from (59) the entropy is also a function of the magnetic flux, $s = \mathcal{I}(\psi)$, except possibly for regions where $\boldsymbol{\nabla}\psi = 0$. Using this, U in H_{AS} obtains ψ dependence through its dependence on s.

Setting the first variation of $\mathfrak{F}_{\mathrm{AS}}$ equal to zero yields the following set of equations:

$$\frac{\delta\mathfrak{F}_{\mathrm{AS}}}{\delta\Upsilon} = -\boldsymbol{\nabla}\cdot\left(\frac{\nabla\Upsilon}{\rho}\right) - \left[\frac{1}{\rho},\chi\right] + \mathcal{F}\left[\frac{1}{\rho},\psi\right] = 0, \tag{64}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{AS}}}{\delta\chi} = -\boldsymbol{\nabla}\cdot\left(\frac{\nabla\chi}{\rho r^2}\right) + \left[\frac{1}{\rho},\Upsilon\right] + \boldsymbol{\nabla}\cdot\left(\frac{\mathcal{F}}{\rho}\frac{\boldsymbol{\nabla}\psi}{r^2}\right) = 0, \tag{65}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{AS}}}{\delta M_\phi} = \frac{M_\phi}{\rho} - r\mathcal{G} - \frac{\mathcal{F}}{\rho}B_\phi = 0, \tag{66}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{AS}}}{\delta B_\phi} = \frac{B_\phi}{4\pi} - \frac{\mathcal{H}}{r} - \frac{\mathcal{F}}{\rho}M_\phi = 0, \tag{67}$$

$$\begin{aligned}\frac{\delta\mathfrak{F}_{\mathrm{AS}}}{\delta\psi} &= -\frac{\mathcal{L}\psi}{4\pi} + \rho U_s\mathcal{I}' - \rho\mathcal{J}' - \frac{B_\phi}{r}\mathcal{H}' - rM_\phi\mathcal{G}' \\ &\quad + \left(\boldsymbol{\nabla}\cdot\left(\frac{\nabla\chi}{\rho r^2}\right) - \left[\frac{1}{\rho},\Upsilon\right]\right)\mathcal{F} - \frac{M_\phi B_\phi}{\rho}\mathcal{F}' = 0,\end{aligned} \tag{68}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{AS}}}{\delta\rho} = -\frac{M^2}{2\rho^2} + (\rho U)_\rho - \mathcal{J} + \frac{M_\phi B_\phi}{\rho^2}\mathcal{F} + \frac{\mathcal{F}}{\rho^2 r^2}\boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\chi + \frac{\mathcal{F}}{\rho^2}[\Upsilon,\psi] = 0, \tag{69}$$

Equations (64)–(69) can be seen to be equivalent to the GE equations previously obtained in [9, 11, 19–21]. To this end observe that (64) and (65) can be rewritten as

$$\boldsymbol{\nabla}\times\boldsymbol{W} = 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\cdot\boldsymbol{W} = 0,$$

where

$$\boldsymbol{W} := \frac{1}{\rho r^2}\left(\boldsymbol{\nabla}\chi - r^2\boldsymbol{\nabla}\Upsilon\times\boldsymbol{\nabla}\phi - \mathcal{F}\boldsymbol{\nabla}\psi\right).$$

Thus, except for the gradient of a harmonic function, we obtain

$$\mathcal{F}\boldsymbol{\nabla}\psi = \boldsymbol{\nabla}\chi - r^2\boldsymbol{\nabla}\Upsilon\times\boldsymbol{\nabla}\phi \tag{70}$$

and, consequently, $\Delta\Upsilon = D = \nabla\cdot M = 0$.

Upon making use of (70) with $\Upsilon \equiv 0$ and some manipulation and rearrangement of (64)–(69), we can summarize as follows:

$$\mathcal{F}(\psi) = 4\pi\chi', \tag{71}$$

$$\mathcal{G}(\psi) = \frac{v_\phi}{r} - \frac{\mathcal{F}B_\phi}{4\pi\rho r}, \tag{72}$$

$$\mathcal{H}(\psi) = rB_\phi - r\mathcal{F}v_\phi, \tag{73}$$

$$\mathcal{I}(\psi) = s, \tag{74}$$

$$\mathcal{J}(\psi) = U + \rho U_\rho + \frac{v^2}{2} - rv_\phi\mathcal{G}, \tag{75}$$

where in (71) $\chi = \chi(\psi)$ with the prime indicating differentiation with respect to ψ, (74) is necessary for the cross helicity Casimir and (75) is a Bernoulli-like equation. These five functions must be assigned in order to find a specific equilibrium configuration, and in turn these functions are determined by the specification of the Casimir invariants. In addition to the above we have the partial differential equation

$$\nabla\cdot\left[\left(1-\frac{\mathcal{F}^2}{4\pi\rho}\right)\frac{\nabla\psi}{r^2}\right] + \frac{\mathcal{F}\mathcal{F}'}{4\pi\rho}\frac{|\nabla\psi|^2}{r^2} = -4\pi\rho\left(\mathcal{J}' + rv_\phi\mathcal{G}'\right) - \frac{1}{r^2}(\mathcal{H} + rv_\phi\mathcal{F})(\mathcal{H}' + rv_\phi\mathcal{F}') + 4\pi U_s\mathcal{I}', \tag{76}$$

which is the sought after GE equation.

7. Axisymmetric HD

The above formulation for axisymmetric MHD can be effectively adapted to limiting cases. For example, from it we can directly obtain the axisymmetric formulations of ideal HD or MHD with zero poloidal flow by neglecting the appropriate terms in both the Poisson bracket and the Hamiltonian. In the hydrodynamic limit, i.e. $\psi \equiv 0$ and $B_\phi \equiv 0$, we obtain for the Poisson bracket

$$\begin{aligned}\{F, G\}_{\mathrm{ASHD}} = -\int_V \Big\{&\rho\left([G_\rho, F_\Omega] - [F_\rho, G_\Omega]\right) - \rho\left(\nabla F_D\cdot\nabla G_\rho - \nabla G_D\cdot\nabla F_\rho\right)\\ &+ rM_\phi\left(\left[\frac{G_{M_\phi}}{r}, F_\Omega\right] - \left[\frac{F_{M_\phi}}{r}, G_\Omega\right]\right)\\ &- rM_\phi\left(\nabla F_D\cdot\nabla\frac{G_{M_\phi}}{r} - \nabla G_D\cdot\nabla\frac{F_{M_\phi}}{r}\right)\\ &+\Omega[G_\Omega, F_\Omega] - \Omega\left(\nabla G_\Omega\cdot\nabla F_D - \nabla F_\Omega\cdot\nabla G_D\right) + \frac{\Omega}{|\nabla\phi|^2}[G_D, F_D]\\ &+\Upsilon\left([G_\Upsilon, F_\Omega] - [F_\Upsilon, G_\Omega]\right) - \Upsilon\left(\nabla F_D\cdot\nabla G_\Upsilon - \nabla G_D\cdot\nabla F_\Upsilon\right)\\ &-|\nabla\phi|^2\nabla\chi\cdot\left(\nabla F_\Omega G_\Upsilon - \nabla G_\Omega F_\Upsilon\right) - \chi\left([F_D, G_\Upsilon] - [G_D, F_\Upsilon]\right)\\ &+\sigma\left([G_\sigma, F_\Omega] - [F_\sigma, G_\Omega]\right) - \sigma\left(\nabla F_D\cdot\nabla G_\sigma - \nabla G_D\cdot\nabla F_\sigma\right)\Big\}\mathrm{d}^3r,\end{aligned} \tag{77}$$

while the Hamiltonian becomes

$$H_{\mathrm{ASHD}}\left[\rho, \chi, \Upsilon, M_\phi, \sigma\right] = \int_V\left(\frac{M_\phi^2}{2\rho} + \frac{|\nabla\chi|^2}{2\rho r^2} + \frac{|\nabla\Upsilon|^2}{2\rho} + \frac{[\Upsilon, \chi]}{\rho} + \rho U\right)\mathrm{d}^3r. \tag{78}$$

The Casimir condition (49) remains unchanged, while conditions (50)–(53) reduce to

$$0 = \left[\frac{\sigma}{\rho}, A\right], \tag{79}$$

$$0 = \left[\frac{rM_\phi}{\rho}, A\right], \tag{80}$$

and (54) becomes

$$\rho\nabla C_\rho + \sigma\nabla C_\sigma + rM_\phi\nabla\frac{C_{M_\phi}}{r} + \frac{\Omega}{\rho}\nabla A - \boldsymbol{M}_\perp\left[A, \frac{1}{\rho}\right] + \nabla\left(\frac{1}{\rho}\boldsymbol{M}_\perp\cdot\nabla A\times\nabla\phi\right) = 0. \tag{81}$$

These conditions produce the Casimirs

$$C_{1\sigma} = \int_V \rho\mathcal{K}\left(\frac{\sigma}{\rho}\right)\mathrm{d}^3r, \tag{82}$$

$$C_{1\phi} = \int_V \rho\mathcal{M}\left(\frac{rM_\phi}{\rho}\right)\mathrm{d}^3r, \tag{83}$$

whereas the Casimirs (56) and (58) reduce to less general forms of $C_{1\sigma}$ and $C_{1\phi}$,

$$C_{1\psi} = \int_V \rho\mathcal{J}_0\,\mathrm{d}^3r \qquad \text{and} \qquad C_3 = \int_V rM_\phi\mathcal{G}_0\,\mathrm{d}^3r$$

where $\mathcal{J}_0$ and $\mathcal{G}_0$ are constants.

For the HD bracket, (79) and (80) yield

$$\left[\frac{rM_\phi}{\rho}, \frac{\sigma}{\rho}\right] = 0, \tag{84}$$

i.e. rM_ϕ/ρ and σ/ρ should be functionally dependent or the problem is over-constrained. If this constraint is assumed, then $C_{1\sigma} = C_{1\phi}$. Moreover, the functional obtained by integrating (49)

$$C_4 = \int_V\left(\frac{|\nabla\phi|^2}{\rho}\nabla A\cdot\nabla\chi + \frac{[\Upsilon, A]}{\rho}\right)\mathrm{d}^3r = \int_V \frac{A'}{2}\boldsymbol{v}\cdot\nabla\times\boldsymbol{v}\,\mathrm{d}^3r \tag{85}$$

is a Casimir, where A is a function of rM_ϕ/ρ or σ/ρ (this can be easily proven by substituting (85) into (81)).

The energy-Casimir functional for HD equilibria is then given by

$$\mathfrak{F}_{\mathrm{ASHD}} = H_{\mathrm{ASHD}} - \int_V \rho\mathcal{M}\,\mathrm{d}^3r - \int_V \frac{\mathcal{N}}{2}\boldsymbol{v}\cdot\nabla\times\boldsymbol{v}\,\mathrm{d}^3r,$$

where $\mathcal{N} = A'$. Using (78) we obtain

$$\mathfrak{F}_{\mathrm{ASHD}} = \int_V\left(\frac{M_\phi^2}{2\rho} + \frac{|\nabla\chi|^2}{2\rho r^2} + \frac{|\nabla\Upsilon|^2}{2\rho} + \frac{[\Upsilon,\chi]}{\rho} + \rho U - \rho\mathcal{M} - \frac{\mathcal{N}}{2}\boldsymbol{v}\cdot\nabla\times\boldsymbol{v}\right)\mathrm{d}^3r \tag{86}$$

and setting the first variation of $\mathfrak{F}_{\mathrm{ASHD}}$ equal to zero yields the following:

$$\frac{\delta\mathfrak{F}_{\mathrm{ASHD}}}{\delta\Upsilon} = -\nabla\cdot\left(\frac{\nabla\Upsilon}{\rho}\right) - \left[\frac{1}{\rho}, \chi\right] + \mathcal{N}\left[\frac{1}{\rho}, rM_\phi/\rho\right] = 0, \tag{87}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{ASHD}}}{\delta\chi} = -\nabla\cdot\left(\frac{\nabla\chi}{\rho r^2}\right) + \left[\frac{1}{\rho}, \Upsilon\right] + \nabla\cdot\left(\frac{\mathcal{N}}{\rho}\frac{\nabla\left(rM_\phi/\rho\right)}{r^2}\right) = 0, \tag{88}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{ASHD}}}{\delta M_\phi} = \frac{M_\phi}{\rho} + rU_s\mathcal{I}' - r\mathcal{M}' + \frac{r}{\rho}\left(\nabla\cdot\left(\frac{\nabla\chi}{\rho r^2}\right) - \left[\frac{1}{\rho}, \Upsilon\right]\right)\mathcal{N} = 0, \tag{89}$$

$$\frac{\delta\mathfrak{F}_{\mathrm{ASHD}}}{\delta\rho} = \frac{M^2}{2\rho^2} + (\rho U)_\rho - \mathcal{M} = 0, \tag{90}$$

where the last equation has been obtained by using (89) and the relation

$$\mathcal{N}\nabla\left(rM_\phi/\rho\right) = \nabla\chi - r^2\nabla\Upsilon\times\nabla\phi, \tag{91}$$

which follows from (87) and (88).

From the above, the GS (or more accurately the Bragg–Hawthorne) equation can be easily obtained. If we assume ρ is constant and $\Upsilon\equiv 0$, consistent with (91), then upon multiplying (89) by r and dividing by $\mathcal{N}$, in order to obtain functional dependences in terms of χ, we see that the χ term of (89) is the GS operator. Since Υ is zero, $r^2\mathcal{M}'/\mathcal{N}$ corresponds to a Bernoulli-like term and thus is a function of χ defined by (90) and the term $(rM_\phi/\rho)/\mathcal{N}$ is equivalent to the $C\,\mathrm{d}C/\mathrm{d}\psi$ term of, e.g., [7]. One of the reasons we introduced the variable χ was to facilitate this derivation.

8. Translation symmetry

Although the above calculations were for axial symmetry, it is clear how to proceed for any symmetry that reduces the problem to two-dimensions. Here we treat the case of translational symmetry along one of the Cartesian axes. Thus we introduce the Cartesian coordinates (x, y, z) and assume z is the ignorable coordinate, i.e. $\partial/\partial z = 0$. In analogy with (15) and (16), the magnetic field and momentum density can be represented as

$$\boldsymbol{B} = B_z\hat{\boldsymbol{z}} + \nabla\psi\times\hat{\boldsymbol{z}}, \tag{92}$$

$$\boldsymbol{M} = M_z\hat{\boldsymbol{z}} + \nabla\chi\times\hat{\boldsymbol{z}} + \nabla\Upsilon, \tag{93}$$

where $\hat{\boldsymbol{z}}$ is the unit vector in the z-direction.

The functional derivative with respect to $\boldsymbol{M}$ results

$$F_{\boldsymbol{M}} = F_{M_z}\hat{\boldsymbol{z}} + \nabla F_\Omega\times\hat{\boldsymbol{z}} - \nabla F_D, \tag{94}$$

where

$$\Delta F_D = F_\Upsilon = -\nabla\cdot F_{\boldsymbol{M}}, \qquad \Delta F_\Omega = -F_\chi = \hat{\boldsymbol{z}}\cdot\nabla\times F_{\boldsymbol{M}} \tag{95}$$

and

$$F_{M_z} = F_{\boldsymbol{M}}\cdot\hat{\boldsymbol{z}}. \tag{96}$$

In a similar way, we can prove that the functional derivative with respect to B_z and ψ results

$$F_{B_z} = F_{\boldsymbol{B}}\cdot\hat{\boldsymbol{z}} \qquad \text{and} \qquad F_\psi = \hat{\boldsymbol{z}}\cdot\nabla\times F_{\boldsymbol{B}}. \tag{97}$$

As to be expected, in this case the Poisson bracket has a form significantly similar to the axisymmetric one,

$$\begin{aligned}
\{F, G\}_{\mathrm{TS}} = -\int_V \Big\{&\rho\left(\left[G_\rho, F_\Omega\right] - \left[F_\rho, G_\Omega\right]\right) - \rho\left(\nabla F_D\cdot\nabla G_\rho - \nabla G_D\cdot\nabla F_\rho\right)\\
&+ M_z\left(\left[G_{M_z}, F_\Omega\right] - \left[F_{M_z}, G_\Omega\right]\right) - M_z\left(\nabla F_D\cdot\nabla G_{M_z} - \nabla G_D\cdot\nabla F_{M_z}\right)\\
&+ \Omega\left[G_\Omega, F_\Omega\right] - \Omega\left(\nabla F_D\cdot\nabla G_\Omega - \nabla G_D\cdot\nabla F_\Omega\right) + \Omega\left[G_D, F_D\right]\\
&+ \Upsilon\left(\left[G_\Upsilon, F_\Omega\right] - \left[F_\Upsilon, G_\Omega\right]\right) - \Upsilon\left(\nabla F_D\cdot\nabla G_\Upsilon - \nabla G_D\cdot\nabla F_\Upsilon\right)\\
&- \nabla\chi\cdot\left(\nabla F_\Omega G_\Upsilon - \nabla G_\Omega F_\Upsilon\right) - \chi\left(\left[F_D, G_\Upsilon\right] - \left[G_D, F_\Upsilon\right]\right)\\
&+ \sigma\left(\left[G_\sigma, F_\Omega\right] - \left[F_\sigma, G_\Omega\right]\right) - \sigma\left(\nabla F_D\cdot\nabla G_\sigma - \nabla G_D\cdot\nabla F_\sigma\right)\\
&+ B_z\left(\left[G_{B_z}, F_\Omega\right] - \left[F_{B_z}, G_\Omega\right]\right) - B_z\left(\nabla F_D\cdot\nabla G_{B_z} - \nabla G_D\cdot\nabla F_{B_z}\right)\\
&+ \psi\left(\left[G_\psi, F_\Omega\right] - \left[F_\psi, G_\Omega\right]\right) - \psi\left(\nabla F_D\cdot\nabla G_\psi - \nabla G_D\cdot\nabla F_\psi\right)\\
&+ \psi\left(\left[G_{B_z}, F_{M_z}\right] - \left[F_{B_z}, G_{M_z}\right]\right) + \psi\left(G_\Upsilon F_\psi - F_\Upsilon G_\psi\right)\Big\}\,\mathrm{d}^3r,
\end{aligned} \tag{98}$$

where

$$[A, B] = \hat{z} \cdot (\nabla A \times \nabla B) = \nabla \cdot \left(\hat{z} \times B \nabla A\right). \tag{99}$$

All the differences from (32) lie in the metric elements used. In terms of the set of variables $\zeta_{\mathrm{TS}} = (\rho, \chi, \Upsilon, M_z, \sigma, \psi, B_z)$, the Hamiltonian becomes

$$\begin{aligned} H_{\mathrm{TS}}[\zeta_{\mathrm{TS}}] &= \frac{1}{2}\int_V \left(\frac{M^2}{\rho} + \frac{B^2}{4\pi} + 2\rho U\right) \mathrm{d}^3 r \\ &= \int_V \left(\frac{M_z^2}{2\rho} + \frac{|\nabla\chi|^2}{2\rho} + \frac{|\nabla\Upsilon|^2}{2\rho} + \frac{[\Upsilon, \chi]}{\rho} + \frac{|\nabla\psi|^2}{8\pi} + \frac{B_z^2}{8\pi} + \rho U\right) \mathrm{d}^3 r. \end{aligned} \tag{100}$$

The same holds for the Casimir conditions and thus for the Casimirs themselves. Thus, replacing rM_ϕ by M_z and B_ϕ/r by B_z, the Casimirs result

$$C_{1\sigma} = \int_V \rho \mathcal{J}\left(\frac{\sigma}{\rho}\right) \mathrm{d}^3 r, \tag{101}$$

$$C_{1\psi} = \int_V \rho \mathcal{K}(\psi)\, \mathrm{d}^3 r, \tag{102}$$

$$C_2 = \int_V B_z \mathcal{H}(\psi)\, \mathrm{d}^3 r, \tag{103}$$

$$C_3 = \int_V M_z \mathcal{G}(\psi)\, \mathrm{d}^3 r, \tag{104}$$

and, if the condition $[\psi, \sigma/\rho] = 0$ is assumed,

$$C_4 = \int_V \left(\frac{M_z B_z}{\rho} A' + \frac{1}{\rho}\nabla A \cdot \nabla\chi + \frac{[\Upsilon, A]}{\rho}\right) \mathrm{d}^3 r = \int_V \boldsymbol{v} \cdot \boldsymbol{B} A' \mathrm{d}^3 r. \tag{105}$$

Translational symmetry is also interesting for the HD problem. As described in section 7, the Poisson bracket for the HD limit can be deduced from (98) simply neglecting the terms with ψ and B_z. In this case, we obtain

$$\begin{aligned} \{F, G\}_{\mathrm{TSHD}} = -\int_V \Big\{ &\rho\left(\left[G_\rho, F_\Omega\right] - \left[F_\rho, G_\Omega\right]\right) - \rho\left(\nabla F_D \cdot \nabla G_\rho - \nabla G_D \cdot \nabla F_\rho\right) \\ &+ M_z\left(\left[G_{M_z}, F_\Omega\right] - \left[F_{M_z}, G_\Omega\right]\right) - M_z\left(\nabla F_D \cdot \nabla G_{M_z} - \nabla G_D \cdot \nabla F_{M_z}\right) \\ &+ \Omega\left[G_\Omega, F_\Omega\right] - \Omega\left(\nabla F_D \cdot \nabla G_\Omega - \nabla G_D \cdot \nabla F_\Omega\right) + \Omega\left[G_D, F_D\right] \\ &+ \Upsilon\left(\left[G_\Upsilon, F_\Omega\right] - \left[F_\Upsilon, G_\Omega\right]\right) - \Upsilon\left(\nabla F_D \cdot \nabla G_\Upsilon - \nabla G_D \cdot \nabla F_\Upsilon\right) \\ &- \nabla\chi \cdot \left(\nabla F_\Omega G_\Upsilon - \nabla G_\Omega F_\Upsilon\right) - \chi\left(\left[F_D, G_\Upsilon\right] - \left[G_D, F_\Upsilon\right]\right) \\ &+ \sigma\left(\left[G_\sigma, F_\Omega\right] - \left[F_\sigma, G_\Omega\right]\right) - \sigma\left(\nabla F_D \cdot \nabla G_\sigma - \nabla G_D \cdot \nabla F_\sigma\right)\Big\} \mathrm{d}^3 r, \end{aligned} \tag{106}$$

and the Hamiltonian becomes

$$H_{\mathrm{TSHD}}\left[\rho, \chi, \Upsilon, M_z, \sigma\right] = \int_V \left(\frac{M_z^2}{2\rho} + \frac{|\nabla\chi|^2}{2\rho} + \frac{|\nabla\Upsilon|^2}{2\rho} + \frac{[\Upsilon, \chi]}{\rho} + \rho U\right) \mathrm{d}^3 r. \tag{107}$$

The HD Casimirs are

$$C_{1\sigma} = \int_V \rho \mathcal{J}\left(\frac{\sigma}{\rho}\right) \mathrm{d}^3 r, \tag{108}$$

$$C_{1z} = \int_V \rho \mathcal{M}\left(\frac{M_z}{\rho}\right) \mathrm{d}^3 r, \tag{109}$$

$$C_4 = \int_V \frac{1}{\rho}\nabla A \cdot \nabla\chi + \frac{1}{\rho}[\Upsilon, A]\, \mathrm{d}^3 r = \int_V \frac{A'}{2} \boldsymbol{v} \cdot \nabla \times \boldsymbol{v}\, \mathrm{d}^3 r, \tag{110}$$

where C_4 exists if the additional assumption $\left[M_z/\rho, \sigma/\rho\right] = 0$ holds.

Therefore, variational energy-Casimirs formulations for translation symmetric configurations can be written for both the MHD and HD problems.

9. Conclusions

In summary, we have described how to use the chain rule for functionals to reduce the noncanonical Poisson bracket for MHD to one for axisymmetric and translationally symmetric MHD and HD. We have described a procedure for obtaining Casimir invariants from noncanonical Poisson brackets and used the procedure for obtaining the Casimir invariants for the considered symmetrical theories.

We have described in general why extrema of the energy plus Casimir invariants correspond to equilibria, thereby giving an explanation for the *ad hoc* variational principles that have existed in plasma physics, and we have explicitly obtained such variational principles for general equilibria. Thus, we have clarified that the Lagrangian functional of previous work is actually a Hamiltonian.

In the search of extrema we have either considered variations that vanish at the domain boundaries or have assumed that natural conditions hold, thus neglecting surface terms. Boundary conditions are application specific and there are many possibilities. For example, in a free boundary problem one must include the dynamical variables that describe the motion of the boundary and this is one means to further complicate the situation. Clearly when applying this procedure to a specific geometrical plasma configuration, the shape and position of the boundaries must also be specified and the physics of the situation will determine how to do so.

The procedures we have described are quite general. Although we have explicitly considered axial symmetry and translational symmetry, with slight modification the results can be adapted to helical symmetry, and it is clear how the procedure works for any dimension reducing symmetry. Through the years, the procedure has been applied to theories more general than MHD, such as the Hamiltonian four-field model of [24] that includes gyroviscosity and more recently the reconnection model of [31, 49] and the electromagnetic gyrofluid model of [32]. We emphasize the point that it is important to show that invariants are Casimir invariants because if one changes the Hamiltonian they remain invariant. In [49] this idea was used to add external forcing, for possible application to resonant magnetic perturbations. In future work we hope to explore such perturbations. In any event we are poised to address stability with or without assuming dynamically accessibility.

Appendix A. Axisymmetric Poisson bracket derivation

Here we show the details of the transformation of the noncanonical Poisson bracket of (13) to that of (32) in terms of the variables $\zeta_{\mathrm{AS}} = \left(\rho, \chi, \Upsilon, M_\phi, \sigma, \psi, B_\phi\right)$, defined by (15)–(16) and with the assumption $\partial/\partial\phi = 0$. For convenience we label the terms of (13) as follows:

$$\begin{aligned}\{F, G\} = -\int_V & \underbrace{\left\{\rho\left(F_M \cdot \nabla G_\rho - G_M \cdot \nabla F_\rho\right)\right.}_{A} + \underbrace{\boldsymbol{M} \cdot \left[\left(F_M \cdot \nabla\right) G_M - \left(G_M \cdot \nabla\right) F_M\right]}_{B} \\ & + \underbrace{\sigma\left(F_M \cdot \nabla G_\sigma - G_M \cdot \nabla F_\sigma\right)}_{C} + \underbrace{\boldsymbol{B}\left[\left(F_M \cdot \nabla\right) G_B - \left(G_M \cdot \nabla\right) F_B\right]}_{D} \\ & + \underbrace{\left.\boldsymbol{B} \cdot \left(\nabla F_M \cdot G_B - \nabla G_M \cdot F_B\right)\right\}}_{E} \mathrm{d}^3 r. \end{aligned} \tag{A.1}$$

The term A can be written as follows by exploiting (29):

$$\rho\left(F_M\cdot\nabla G_\rho - G_M\cdot\nabla F_\rho\right) = \rho\left(\left[G_\rho, F_\Omega\right] - \left[F_\rho, G_\Omega\right]\right) - \rho\left(\nabla F_D\cdot\nabla G_\rho - \nabla G_D\cdot\nabla F_\rho\right), \tag{A.2}$$

where we have introduced the notation

$$(\nabla A\times\nabla B)\cdot\nabla\phi = [A, B]. \tag{A.3}$$

The term B can be decomposed as

$$M\cdot[(F_M\cdot\nabla)\,G_M - (G_M\cdot\nabla)\,F_M] = \underbrace{M_\phi\hat{\phi}\cdot[(F_M\cdot\nabla)\,G_M - (G_M\cdot\nabla)\,F_M]}_{B_1} + \underbrace{M_p\cdot[(F_M\cdot\nabla)\,G_M - (G_M\cdot\nabla)\,F_M]}_{B_2}. \tag{A.4}$$

The first term on the right, labelled $B1$, can be rewritten as

$$\begin{aligned} M_\phi\hat{\phi}\cdot[(F_M\cdot\nabla)\,G_M - (G_M\cdot\nabla)\,F_M] &= rM_\phi\left(F_{M_p}\cdot\nabla\frac{G_{M_\phi}}{r} - G_{M_p}\cdot\nabla\frac{F_{M_\phi}}{r}\right)\\ &= rM_\phi\left(\left[\frac{G_{M_\phi}}{r}, F_\Omega\right] - \left[\frac{F_{M_\phi}}{r}, G_\Omega\right]\right)\\ &\quad -rM_\phi\left(\nabla F_D\cdot\nabla\frac{G_{M_\phi}}{r} - \nabla G_D\cdot\nabla\frac{F_{M_\phi}}{r}\right) \end{aligned} \tag{A.5}$$

where in the last step we have again used expression (29). By using vector identities and considering that $F_\Upsilon = \nabla\cdot F_M$, the second term of (A.4) can be further decomposed as

$$M_p\cdot[(F_M\cdot\nabla)\,G_M - (G_M\cdot\nabla)\,F_M] = \underbrace{M_p\cdot\nabla\times(G_M\times F_M)}_{B_{2a}} - \underbrace{M_p\cdot(F_M G_\Upsilon - G_M F_\Upsilon)}_{B_{2b}}. \tag{A.6}$$

The term labelled B_{2a} yields

$$M_p\cdot\nabla\times(G_M\times F_M) = \left(\nabla\times M_p\right)\cdot(G_M\times F_M) = \frac{\Omega}{|\nabla\phi|^2}\nabla\phi\cdot\left(G_{M_p}\times F_{M_p}\right),$$

which upon substituting the expressions for G_{M_p} and F_{M_p} of (29) and (31), respectively, we obtain

$$M_p\cdot\nabla\times(G_M\times F_M) = \Omega\,[G_\Omega, F_\Omega] - \Omega\,(\nabla G_\Omega\cdot\nabla F_D - \nabla F_\Omega\cdot\nabla G_D) + \frac{\Omega}{|\nabla\phi|^2}\,[G_D, F_D]. \tag{A.7}$$

The second term on the right side of (A.6), i.e. the term labelled B_{2b}, can be rewritten as

$$\begin{aligned} M_p\cdot(F_M G_\Upsilon - G_M F_\Upsilon) &= (\nabla\chi\times\nabla\phi + \nabla\Upsilon)\cdot(F_M G_\Upsilon - G_M F_\Upsilon)\\ &= (\nabla\chi\times\nabla\phi)\cdot(F_M G_\Upsilon - G_M F_\Upsilon) + \nabla\Upsilon\cdot(F_M G_\Upsilon - G_M F_\Upsilon). \end{aligned} \tag{A.8}$$

The first part of this expression results in

$$(\nabla\chi\times\nabla\phi)\cdot\left(F_{M_p}G_\Upsilon - G_{M_p}F_\Upsilon\right) = -|\nabla\phi|^2\,\nabla\chi\cdot(\nabla F_\Omega G_\Upsilon - \nabla G_\Omega F_\Upsilon) - \chi\,([F_D, G_\Upsilon] - [G_D, F_\Upsilon]) \tag{A.9}$$

and integrating the last term by parts, we obtain

$$\nabla\Upsilon\cdot\left(F_M G_\Upsilon - G_M F_\Upsilon\right) = -\Upsilon\left(F_M\cdot\nabla G_\Upsilon - G_M\cdot\nabla F_\Upsilon\right), \tag{A.10}$$

since $\nabla\cdot F_M = F_\Upsilon$ and $\nabla\cdot G_M = G_\Upsilon$. It is straightforward to reduce expression (A.10) to the forms of (A.2) and (A.5),

$$\begin{aligned}-\Upsilon\left(F_M\cdot\nabla G_\Upsilon - G_M\cdot\nabla F_\Upsilon\right) &= -\Upsilon\left([G_\Upsilon, F_\Omega] - [F_\Upsilon, G_\Omega]\right)\\ &\quad + \Upsilon\left(\nabla F_D\cdot\nabla G_\Upsilon - \nabla F_D\cdot\nabla G_\Upsilon\right).\end{aligned} \tag{A.11}$$

In the same way, the term C results in

$$\begin{aligned}\sigma\left(F_M\cdot\nabla G_\sigma - G_M\cdot\nabla F_\sigma\right) &= \sigma\left([G_\sigma, F_\Omega] - [F_\sigma, G_\Omega]\right)\\ &\quad - \sigma\left(\nabla F_D\cdot\nabla G_\sigma - \nabla G_D\cdot\nabla F_\sigma\right).\end{aligned} \tag{A.12}$$

The part of the Poisson bracket which depends on the magnetic field, i.e. terms D and E, can be rewritten as

$$\begin{aligned}&\boldsymbol{B}\left(\left(F_M\cdot\nabla\right)G_B - \left(G_M\cdot\nabla\right)F_B\right) + \boldsymbol{B}\cdot\left(\nabla F_M\cdot G_B - \nabla G_M\cdot F_B\right)\\ &\qquad = \boldsymbol{B}\cdot\left(G_B\times\left(\nabla\times F_M\right) - F_M\times\left(\nabla\times G_B\right)\right),\end{aligned} \tag{A.13}$$

where we have used vector identities and $\nabla\cdot\boldsymbol{B} = 0$. The term on the right side of (A.13) can be decomposed into three parts:

$$\begin{aligned}&\boldsymbol{B}\cdot\left(G_B\times\left(\nabla\times F_M\right) - F_M\times\left(\nabla\times G_B\right)\right)\\ &\quad = \underbrace{B_\phi\hat{\phi}\cdot\left(G_{M_p}\times\left(\nabla\times F_{B_\phi}\hat{\phi}\right) - F_{M_p}\times\left(\nabla\times G_{B_\phi}\hat{\phi}\right)\right)}_{D_1}\\ &\qquad + \underbrace{\boldsymbol{B}_p\cdot\left(G_{M_\phi}\hat{\phi}\times\left(\nabla\times F_{B_\phi}\hat{\phi}\right) - F_{M_\phi}\hat{\phi}\times\left(\nabla\times G_{B_\phi}\hat{\phi}\right)\right)}_{D_2}\\ &\qquad + \underbrace{\boldsymbol{B}_p\cdot\left(G_{M_p}\times\left(\nabla\times F_{B_p}\right) - F_{M_p}\times\left(\nabla\times G_{B_p}\right)\right)}_{D_3}.\end{aligned} \tag{A.14}$$

The term D_1 becomes

$$\begin{aligned}B_\phi\hat{\phi}\cdot\left(G_{M_p}\times\left(\nabla\times F_{B_\phi}\hat{\phi}\right) - F_{M_p}\times\left(\nabla\times G_{B_\phi}\hat{\phi}\right)\right) &= \frac{B_\phi}{r}\left(\left[rG_{B_\phi}, F_\Omega\right] - \left[rF_{B_\phi}, G_\Omega\right]\right)\\ &\quad - \frac{B_\phi}{r}\left(\nabla\left(rG_{B_\phi}\right)\cdot\nabla F_D - \nabla\left(rF_{B_\phi}\right)\cdot\nabla G_D\right),\end{aligned} \tag{A.15}$$

while the term D_2 can be rewritten as

$$\begin{aligned}&\boldsymbol{B}_p\cdot\left(G_{M_\phi}\hat{\phi}\times\left(\nabla\times F_{B_\phi}\hat{\phi}\right) - F_{M_\phi}\hat{\phi}\times\left(\nabla\times G_{B_\phi}\hat{\phi}\right)\right)\\ &\qquad = \psi\left(\left[rG_{B_\phi}, \frac{F_{M_\phi}}{r}\right] - \left[rF_{B_\phi}, \frac{G_{M_\phi}}{r}\right]\right),\end{aligned} \tag{A.16}$$

where we have used the expression $\boldsymbol{B}_p = \nabla\psi\times\nabla\phi$ and integrated by parts. Then, the last term of (A.14) yields

$$\boldsymbol{B}_p\cdot\left(G_{M_p}\times\left(\nabla\times F_{B_p}\right) - F_{M_p}\times\left(\nabla\times G_{B_p}\right)\right) = -\nabla\psi\cdot\left(F_{M_p}G_\psi - G_{M_p}F_\psi\right), \tag{A.17}$$

which can be rewritten by integrating by parts and exploiting expression (29) as

$$\begin{aligned}-\nabla\psi\cdot\left(F_{M_p}G_\psi - G_{M_p}F_\psi\right) &= \psi\left(F_{M_p}\cdot\nabla G_\psi - G_{M_p}\cdot\nabla F_\psi\right) - \psi\left(F_\Upsilon G_\psi - G_\Upsilon F_\psi\right)\\ &= \psi\left(\left[G_\psi, F_\Omega\right] - \left[F_\psi, G_\Omega\right]\right) - \psi\left(\nabla F_D\cdot\nabla G_\psi - \nabla G_D\cdot\nabla F_\psi\right)\\ &\quad + \psi\left(G_\Upsilon F_\psi - F_\Upsilon G_\psi\right).\end{aligned} \tag{A.18}$$

Finally, by substituting all of the expressions $A - E$ derived above into (A.1), we obtain the axisymmetric MHD Poisson bracket of (32).

Appendix B. Axisymmetric equations of motion

Here we explicitly calculate the set of equations implied by (35) and thereby obtain the axisymmetric equations of motion.

The first equation of the set is the mass conservation equation

$$\frac{\partial \rho}{\partial t} = \{\rho, H_{\mathrm{AS}}\}_{\mathrm{AS}} = [H_\Omega, \rho] + \nabla \cdot (\rho \nabla H_D) = -\nabla \cdot (\rho \nabla H_\Omega \times \nabla \phi - \rho \nabla H_D) \tag{B.1}$$

where in the second equality on the right we drop the subscript AS on H_{AS} to lessen notational clutter. We do this in the rest of this appendix. Because

$$\nabla H_\Omega \times \nabla \phi - \nabla H_D = H_{M_p} = \frac{\boldsymbol{M}_p}{\rho}, \tag{B.2}$$

we obtain from (B.1) the mass conservation equation in the form

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{M}_p = -\Delta \Upsilon. \tag{B.3}$$

Conservation of poloidal momentum is described in terms of the time derivatives of χ and Υ. These become

$$\begin{aligned}\frac{\partial \chi}{\partial t} = \{\chi, H_{\mathrm{AS}}\}_{\mathrm{AS}} = -\mathcal{L}^{-1}\Bigg(&\left[\rho, H_\rho\right] + \left[r M_\phi, \frac{H_{M_\phi}}{r}\right] + [\sigma, H_\sigma] + \left[\frac{B_\phi}{r}, r H_{B_\phi}\right] \\ &+ \left[\psi, H_\psi\right] + [\Omega, H_\Omega] - \nabla \cdot (\Omega \nabla H_D) + [\Upsilon, H_\Upsilon] + \nabla \cdot \left(H_\Upsilon |\nabla \phi|^2 \nabla \chi\right)\Bigg)\end{aligned} \tag{B.4}$$

and

$$\begin{aligned}\frac{\partial \Upsilon}{\partial t} = \{\Upsilon, H_{\mathrm{AS}}\}_{\mathrm{AS}} = -\Delta^{-1}\Bigg(&\nabla \cdot \left(\rho \nabla H_\rho\right) + \nabla \cdot \left(r M_\phi \nabla \frac{H_{M_\phi}}{r}\right) + \nabla \cdot (\sigma \nabla H_\sigma) \\ &+ \nabla \cdot \left(\frac{B_\phi}{r} \nabla r H_{B_\phi}\right) + \nabla \cdot \left(\psi \nabla H_\psi\right) + \nabla \cdot (\Omega \nabla H_\Omega) + \left[r^2 \Omega, H_D\right] \\ &+ \nabla \cdot (\Upsilon \nabla H_\Upsilon) + [\chi, H_\Upsilon] + \triangle ([\Upsilon, H_\Omega] - \nabla \cdot (\Upsilon \nabla H_D) + [\chi, H_D] \\ &+ |\nabla \phi|^2 \nabla \chi \cdot \nabla H_\Omega - \psi H_\psi)\Bigg).\end{aligned} \tag{B.5}$$

Using the relations $D = \nabla \cdot \boldsymbol{M} = \Delta \Upsilon$ and $\Omega = -\nabla \cdot \left(|\nabla \phi|^2 \nabla \chi\right) = \mathcal{L}\chi$, the poloidal components of the momentum equation can then be represented in terms of the curl and divergence of $\boldsymbol{M}_p$,

$$\begin{aligned}\frac{\partial \Omega}{\partial t} = \nabla \phi \cdot \nabla \times \frac{\partial \boldsymbol{M}_p}{\partial t} = -\nabla \phi \cdot \nabla \times \Bigg(&\rho \nabla H_\rho + r M_\phi \nabla \frac{H_{M_\phi}}{r} + \sigma \nabla H_\sigma + \frac{B_\phi}{r} \nabla r H_{B_\phi} \\ &+ \psi \nabla H_\psi + \Omega \nabla H_\Omega - \Omega r^2 \nabla \phi \times \nabla H_D + \Upsilon \nabla H_\Upsilon + H_\Upsilon \nabla \phi \times \nabla \chi\Bigg)\end{aligned} \tag{B.6}$$

and

$$\begin{aligned}\frac{\partial D}{\partial t} = \nabla \cdot \frac{\partial \boldsymbol{M}_p}{\partial t} = -\nabla \cdot \Bigg(&\rho \nabla H_\rho + r M_\phi \nabla \frac{H_{M_\phi}}{r} + \sigma \nabla H_\sigma + \frac{B_\phi}{r} \nabla r H_{B_\phi} \\ &+ \psi \nabla H_\psi + \Omega \nabla H_\Omega - \Omega r^2 \nabla \phi \times \nabla H_D + \Upsilon \nabla H_\Upsilon + H_\Upsilon \nabla \phi \times \nabla \chi \\ &+ \nabla \left(-\Upsilon H_\Upsilon + \boldsymbol{M}_p \cdot H_{M_p} - \psi H_\psi\right)\Bigg).\end{aligned} \tag{B.7}$$

Thus, combining these two equations we obtain

$$\frac{\partial \boldsymbol{M}_p}{\partial t} = -\left(\rho \nabla H_\rho + r M_\phi \nabla \frac{H_{M_\phi}}{r} + \sigma \nabla H_\sigma + \frac{B_\phi}{r} \nabla r H_{B_\phi} + \nabla \psi H_\psi + \Omega r^2 \nabla \phi \times H_{M_p} - H_\Upsilon \boldsymbol{M}_p + \nabla \left(\boldsymbol{M}_p \cdot H_{M_p}\right)\right). \tag{B.8}$$

The functional derivatives of the Hamiltonian H_{AS} of (34) with respect to the variables ζ_{AS} are given by the following:

$$H_\chi = -\nabla \cdot \left(\frac{1}{\rho}\frac{\nabla \chi}{r^2}\right) + \left[\frac{1}{\rho}, \Upsilon\right] = -\nabla \cdot \left(\frac{\nabla \phi \times \boldsymbol{M}_p}{\rho}\right) \tag{B.9}$$

$$H_\Upsilon = -\nabla \cdot \left(\frac{1}{\rho}\nabla \Upsilon\right) + \left[\chi, \frac{1}{\rho}\right] = -\nabla \cdot \left(\frac{\boldsymbol{M}_p}{\rho}\right) \tag{B.10}$$

and

$$H_\rho = -\frac{1}{2}\left(\frac{|\boldsymbol{M}|}{\rho}\right)^2 + U + \rho U_\rho, \qquad H_\sigma = U_s,$$
$$H_{M_\phi} = \frac{M_\phi}{\rho}, \qquad H_{B_\phi} = \frac{B_\phi}{4\pi}, \qquad H_\psi = -\frac{1}{4\pi}\mathcal{L}\psi. \tag{B.11}$$

Using these expressions it is possible to rewrite (B.8) as

$$\frac{\partial \boldsymbol{M}_p}{\partial t} = -\rho \nabla \left[\frac{1}{2}\left(\frac{|\boldsymbol{M}|}{\rho}\right)^2 + U + \rho U_\rho\right] + \frac{M_\phi}{r}\nabla\left(\frac{r M_\phi}{\rho}\right) - \sigma \nabla U_s - \frac{B_\phi}{r}\nabla\left(r\frac{B_\phi}{4\pi}\right) + \frac{1}{4\pi}\mathcal{L}\psi \nabla \psi - \mathcal{L}\chi \frac{r^2 \nabla \phi \times \boldsymbol{M}_p}{\rho} - \boldsymbol{M}_p \nabla \cdot \left(\frac{\boldsymbol{M}_p}{\rho}\right) + \left|\boldsymbol{M}_p\right|^2 \nabla \frac{1}{\rho}. \tag{B.12}$$

Upon substituting (B.12) into (B.6) and (B.7) yields,

$$\frac{\partial \Omega}{\partial t} = \nabla \phi \cdot \nabla \times \left\{-\rho \nabla \left[\frac{1}{2}\left(\frac{|\boldsymbol{M}|}{\rho}\right)^2 + U + \rho U_\rho\right] + \frac{M_\phi}{r}\nabla\left(\frac{r M_\phi}{\rho}\right) - \sigma \nabla U_s - \frac{B_\phi}{r}\nabla\left(r\frac{B_\phi}{4\pi}\right) + \frac{1}{4\pi}\mathcal{L}\psi \nabla \psi - \mathcal{L}\chi \frac{r^2 \nabla \phi \times \boldsymbol{M}_p}{\rho} - \boldsymbol{M}_p \nabla \cdot \left(\frac{\boldsymbol{M}_p}{\rho}\right) + \left|\boldsymbol{M}_p\right|^2 \nabla \frac{1}{\rho}\right\} \tag{B.13}$$

and

$$\frac{\partial D}{\partial t} = \nabla \cdot \left\{-\rho \nabla \left[\frac{1}{2}\left(\frac{|\boldsymbol{M}|}{\rho}\right)^2 + U + \rho U_\rho\right] + \frac{M_\phi}{r}\nabla\left(\frac{r M_\phi}{\rho}\right) - \sigma \nabla U_s - \frac{B_\phi}{r}\nabla\left(r\frac{B_\phi}{4\pi}\right) + \frac{1}{4\pi}\mathcal{L}\psi \nabla \psi - \mathcal{L}\chi \frac{r^2 \nabla \phi \times \boldsymbol{M}_p}{\rho} - \boldsymbol{M}_p \nabla \cdot \left(\frac{\boldsymbol{M}_p}{\rho}\right) + \left|\boldsymbol{M}_p\right|^2 \nabla \frac{1}{\rho}\right\}, \tag{B.14}$$

respectively.

The time derivative of the magnetic flux function is given by

$$\frac{\partial \psi}{\partial t} = \{\psi, H_{\mathrm{AS}}\}_{\mathrm{AS}} = [H_\Omega, \psi] + \nabla \cdot (\psi \nabla H_D) - \psi H_\Upsilon = -\frac{\boldsymbol{M}_p}{\rho} \cdot \nabla \psi, \tag{B.15}$$

which expresses the fact that ψ is a Lagrangian invariant. Moreover, the time derivative of the azimuthal component of the magnetic field is given by

$$\frac{\partial B_\phi}{\partial t} = \left\{B_\phi, H_{\mathrm{AS}}\right\}_{\mathrm{AS}} = r\left(\left[\frac{B_\phi}{r}, H_\Omega\right] + \nabla \cdot \left(\frac{B_\phi}{r}\nabla H_D\right) - \left[\psi, \frac{H_{M_\phi}}{r}\right]\right)$$
$$= -r\nabla \cdot \left(\frac{B_\phi}{r\rho}\boldsymbol{M}_p - \frac{M_\phi}{\rho r}\boldsymbol{B}_p\right). \tag{B.16}$$

Now, the azimuthal momentum density is easily seen to satisfy

$$\frac{\partial M_\phi}{\partial t} = \{M_\phi, H_{\mathrm{AS}}\}_{\mathrm{AS}} = \frac{1}{r}\left([H_\Omega, rM_\phi] + \nabla\cdot\left(rM_\phi \nabla H_D\right) - [\psi, rH_{B_\phi}]\right)$$
$$= -\frac{1}{r}\nabla\cdot\left(\frac{rM_\phi}{\rho}\boldsymbol{M}_p - \frac{rB_\phi}{4\pi}\boldsymbol{B}_p\right). \tag{B.17}$$

Finally, paralleling the manipulations used for the mass conservation equation, the entropy equation is seen to be

$$\frac{\partial \sigma}{\partial t} = \{\sigma, H_{\mathrm{AS}}\}_{\mathrm{AS}} = [H_\Omega, \sigma] + \nabla\cdot(\sigma\nabla H_D) = -\nabla\cdot\left(\frac{\sigma}{\rho}\boldsymbol{M}_p\right). \tag{B.18}$$

For time-independent configurations, (B.15)–(B.18) imply conditions (71)–(74) and thus the system of GE equations.

References

[1] Grad H and Rubin H 1958 *Int. Atomic Energy Agency Conf. Proc. vol 31 (Geneva, Switzerland)* pp 190–7
[2] Shafranov V D 1966 *Rev. Plasma Phys.* **2** 103–51
[3] Lüst R and Schlüter A 1954 *Z. Astrophys.* **34** 263–82
[4] Bragg S L and Hawthorne W R 1950 *J. Aerosol Sci.* **17** 243–9
[5] Long R R 1953 *J. Met.* **10** 197–203
[6] Squire H B 1956 Rotating fluids *Surveys in Mechanics* ed G K Batchelor and R M Davies (Cambridge: Cambridge University Press) pp 139–61
[7] Elcrat A R and Miller K G 2003 *Diff. Integral Eqns* **16** 949–68
[8] Scott H A and Lovelace R V E 1982 *Astrophys. J.* **252** 765–74
[9] Heinemann M and Olbert S 1978 *J. Geophys. Res.* **83** 2457–60
[10] Tsinganos K C 1982 *Astrophys. J.* **252** 775–90
[11] Lovelace R V E, Mehanian C, Mobarry C M and Sulkanen M E 1986 *Astrophys. J. Suppl. Ser.* **62** 1–37
[12] Goedbloed J P and Lifschitz A 1997 *Phys. Plasmas* **4** 3544–64
[13] Woltjer L 1958 *Proc. Natl Acad. Sci. USA* **44** 833–41
[14] Woltjer L 1959 *Proc. Natl Acad. Sci. USA* **45** 769–71
[15] Woltjer L 1959 *Astrophys J.* **130** 400–4
[16] Woltjer L 1959 *Astrophys J.* **130** 405–13
[17] Taylor J B 1974 *Phys. Rev. Lett.* **33** 1139–41
[18] Taylor J B 1986 *Rev. Mod. Phys.* **58** 741–63
[19] Hameiri E 1998 *Phys. Plasmas* **5** 3270–81
[20] Goedbloed J P 2004 *Phys. Plasmas* **11** L81–3
[21] Andreussi T and Pegoraro F 2008 *Phys. Plasmas* **15** 092108
[22] Morrison P J and Greene J M 1980 *Phys. Rev. Lett.* **45** 790–4
[23] Morrison P J and Greene J M 1982 *Phys. Rev. Lett.* **48** 569
[24] Hazeltine R D, Hsu C T and Morrison P J 1987 *Phys. Fluids* **30** 3204–11
[25] Marsden J E and Morrison P J 1984 *Contemp. Math.* **28** 133–50
[26] Morrison P J 2005 *Phys. Plasmas* **12** 058102
[27] Morrison P J 1982 *AIP Conf. Proc.* **88** 13–45
[28] Morrison P J 1998 *Rev. Mod. Phys.* **70** 467–521
[29] Salmon R 1988 *Ann. Rev. Fluid Mech.* **20** 225–56
[30] Marsden J E, Morrison P J and Weinstein A 1984 *Contemp. Math.* **28** 115–24
[31] Tassi E, Morrison P J, Waelbroeck F L and Grasso D 2008 *Plasma Phys. Control. Fusion* **50** 085014
[32] Waelbroeck F L, Hazeltine R D and Morrison P J 2009 *Phys. Plasmas* **16** 032109
[33] Tassi E, Chandre C and Morrison P J 2009 *Phys. Plasmas* **16** 082301
[34] Padhye N and Morrison P J 1996 *Phys. Lett.* A **219** 287–92
[35] Padhye N and Morrison P J 1996 *Plasma Phys. Rep.* **22** 869–77
[36] Kruskal M D and Oberman C 1958 *Phys. Fluids* **1** 275–80
[37] Hazeltine R D, Holm D D, Marsden J E and Morrison P J 1984 *Proc. Int. Conf. on Plasma Physics (Lausanne, Switzerland)* ed M Q Tran and M L Sawley pp 1–9
[38] Holm D D, Marsden J E, Ratiu R and Weinstein A 1985 *Phys. Rep.* **123** 1–116

[39] Morrison P J and Eliezer S 1986 *Phys. Rev.* A **33** 4205–14
[40] Morrison P J and Hazeltine R D 1984 *Phys. Fluids* **27** 886–97
[41] Khater A H and Moawad S M 2003 *Plasma Phys. Control. Fusion* **45** 265–74
[42] Hameiri E 2004 *Phys. Plasmas* **11** 3423–31
[43] Ilgisonis V I and Khalzov I V 2005 *JETP Lett.* **82** 570–4
[44] Morrison P J and Kotschenreuther M 1990 The free energy principle, negative energy modes and stability *Nonlinear World: IV Int. Workshop on Nonlinear and Turbulent Processes in Physics (Kiev, Russia)* ed V G Bar'yakhtar *et al* (Singapore: World Scientific)
[45] Morrison P J and Pfirsch D 1989 *Phys. Rev.* A **40** 3898–910
[46] Hirota M, Yoshida Z and Hameiri E 2006 *Phys. Plasmas* **13** 022107
[47] Newcomb W A 1962 *Nucl. Fusion Suppl.* (part 2) 451–63
[48] Steinhauer L C and Ishida A 1998 *Phys. Plasmas* **5** 2609–22
[49] Tassi E, Morrison P J, Grasso D and Pegoraro F 2009 *Nucl. Fusion* **49** 034007