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A hierarchy of noncanonical Hamiltonian systems: circulation laws in an extended phase space

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Abstract

The dynamics of an ideal fluid or plasma is constrained by topological invariants such as the circulation of (canonical) momentum or, equivalently, the flux of the vorticity or magnetic fields. In the Hamiltonian formalism, topological invariants restrict the orbits to submanifolds of the phase space. While the coadjoint orbits have a natural symplectic structure, the global geometry of the degenerate (constrained) Poisson manifold can be very complex. Some invariants are represented by the center of the Poisson algebra (i.e., the Casimir elements such as the helicities), and then, the global structure of phase space is delineated by Casimir leaves. However, a general constraint is not necessarily *integrable*, which precludes the existence of an appropriate Casimir element; the circulation is an example of such an invariant. In this work, we formulate a systematic method to embed a Hamiltonian system in an extended phase space; we introduce *mock fields* and extend the Poisson algebra so that the mock fields are Lie-dragged by the flow vector. A mock field defines a new Casimir element, a *cross helicity*, which represents topological constraints including the circulation. Unearthing a Casimir element brings about immense advantage in the study of dynamics and equilibria—the so-called energy-Casimir method becomes readily available. Yet, a mock field does not a priori have a physical meaning. Here we proffer an interpretation of a Casimir element obtained, e.g., by such a construction as an *adiabatic invariant* associated with a hidden 'microscopic' angle variable, and in this way give the mock field a physical meaning. We proceed further and consider a perturbation of the Hamiltonian by a canonical pair, composed of the

Casimir element and the angle, that causes the topological constraint to be unfrozen. The theory is applied to the tearing modes of magneto-hydrodynamics.

1. Introduction

The theory of dynamics can be viewed as built from two elements: matter and space; the former is physically an energy, while the latter is mathematically a geometry. Hamiltonian mechanics formulates an energy as a Hamiltonian that is a function on a phase space X, and the geometry of the phase space is dictated by a Poisson bracket $[F, G]$ (F and G are functions on X), and is called a Poisson manifold. The most basic form of a Poisson manifold is realized by symplectic geometry, in which case the Hamiltonian mechanics is said to be *canonical*. A general Poisson manifold, however, may be far more complex than a symplectic manifold, and orbits may be constrained by complicated topological invariants that foliate the phase space into submanifolds (leaves). Locally, a submanifold can be regarded as a symplectic leaf (Lie–Darboux theorem); however, a Poisson operator may have singularities at which leaves bifurcate or intersect.

A nontrivial (non-constant) member C of the *center* of the Poisson algebra (i.e. $[F, C] = 0$ for every F) is called a *Casimir element*, which is a constant of motion ($\mathrm{d}C/\mathrm{d}t = [H, C] = 0$) for every Hamiltonian H. Contrary to usual constants of motion that pertain to symmetries of a specific Hamiltonian, there are *topological constraints* that are independent of the choice of a Hamiltonian and are due to the Poisson bracket alone. Among various topological constraints, Casimir elements have special importance. We call the level-set of a Casimir element a *Casimir leaf*, on which equilibrium points or statistical equilibrium distributions may have interesting bifurcated structures, even when the Hamiltonian is simple. Since the transformation of a Hamiltonian H to an *energy-Casimir function* $H_\mu = H - \mu C$ does not change the dynamics $\mathrm{d}F/\mathrm{d}t = [F, H] = [F, H_\mu]$ (Kruskal and Oberman 1958, Hazeltine *et al* 1984, Morrison 1998, Arnold and Khesin 1998), the equilibrium points (the critical points of H_μ) may bifurcate when we change μ as a parameter (or, when we seek equilibria on different Casimir leaves) (Yoshida and Dewar 2012). Similarly, the Gibbs distribution on a Casimir leaf is given by $e^{\beta(H-\mu C)}$, which can be regarded as a grand-canonical distribution function (μ is a chemical potential) (Yoshida *et al* 2013). We note that the equilibrium or the Gibbs distribution function of a canonical Hamiltonian system can be nontrivial only when the Hamiltonian is a bumpy function, but this is not the case for a weakly coupled system like a usual fluid or a plasma.

In the context of the present study, we highlight another distinction of Casimir elements among topological constraints. In Yoshida and Morrison (2014b), we proffered an interpretation of a Casimir element as an *adiabatic invariant* associated with a hidden 'microscopic' angle variable. Adding the angle variable to the phase space, we 'alchemized' the Casimir element into an action variable, which together with the angle variable forms a canonical pair. Then, perturbing the Hamiltonian by the new canonical variables, we unfroze the Casimir element. By this theory, we extended the scope of ideal Hamiltonian mechanics to see what happens when the orbit is allowed to deviate from the leaves of the Poisson manifold. A finite dissipation may break the ideal constraints and free the orbit to move among different leaves when a very small dissipation that does not destroy the basic structure of the Poisson manifold is considered (as opposed to large dissipation that diminishes the 'dimension' of the dynamics). Thus, ideal constraints can be removed, giving rise to some instabilities.

In this work, we formulate a general systematic method for embedding a Poisson manifold into a higher-dimensional phase space and, in doing so, express the topological constraints (restricting important instabilities omitted in the ideal model) in terms of Casimir elements of the embedded system. This idea is motivated by early work (Morrison and Hazeltine 1984) in which it was observed that adding additional variables to a noncanonical Hamiltonian theory enriched the Casimir structure and made available more general equilibria for the energy-Casimir method. The idea was later used explicitly in the Vlasov–Poisson context (Morrison 1987), and our method of embedding is a special case of the general theory of extensions given in Thiffeault and Morrison (1998), and Thiffeault and Morrison (2000). Specifically, we introduce a *mock field* by which a local topological constraint (which cannot be elucidated by the original Casimir elements) is represented as a Casimir element, a *cross helicity* pertinent to the mock field (the reader is referred to Fukumoto (2008), and Fukumoto and Sakuma (2013) for the original idea of unifying topological invariants as cross helicities). Then, the mock field is the target to be perturbed when one wishes to break topological constraints.

We put the theory to the test by analyzing the equations of ideal magnetohydrodynamics (MHD), which was first shown in Morrison and Greene (1980) to have noncanonical Hamiltonian form on an infinite-dimensional phase space of Eulerian variables. Alfvén's law, that the local magnetic flux on every co-moving surface is a topological invariant, prevents any change in the linkage of magnetic field lines. Alternatively this law can be viewed as a rephrasing of Kelvin's circulation law with the magnetic field replacing the vorticity. Therefore, tearing modes, which grow by creating magnetic islands, are forbidden in an ideal plasma (Furth 1963a; Furth *et al* 1963b; White 1983). Here we show that the magnetic flux on a co-moving surface is the cross helicity pertinent to a Lie-dragged *pure-state* (Yoshida *et al* 2014c) mock field. Hence, upon unfreezing this cross helicity the local (resonant) magnetic flux can give rise to tearing modes (Yoshida and Dewar 2012, Yoshida and Morrison 2014b).

2. A hierarchy of noncanonical Hamiltonian systems

2.1. Noncanonical Hamiltonian systems and degenerate Poisson manifolds

A general Hamiltonian system may be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}u = \mathcal{J}(u)\,\partial_u H(u), \tag{2.1}$$

where u is the state vector, a member of the phase space X (here a Hilbert space endowed with an inner product $\langle\ ,\ \rangle$), $H(u)$ is the Hamiltonian (here a real-valued functional on X), and $\mathcal{J}$ is the Poisson operator (or cosymplectic bivector). We allow $\mathcal{J}$ to be a function of u on X, and write it as $\mathcal{J}(u)$. We assume that the Poisson bracket, the bilinear product,

$$[F, G] = \langle \partial_u F(u), \mathcal{J}\partial_u G(u) \rangle$$

is antisymmetric and satisfies the Jacobi identity.

A *canonical* Hamiltonian system is endowed with a *symplectic* Poisson operator where

$$\mathcal{J}_c = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

However, our interest is in *noncanonical* systems endowed with Poisson operators $\mathcal{J}$ that are inhomogeneous and degenerate (i.e., $\operatorname{Ker}(\mathcal{J}(u))$ contains nonzero elements, and its dimension may change depending on the position in X). Since $\mathcal{J}$ is antisymmetric, $\operatorname{Ker}(\mathcal{J}(u)) = \operatorname{Coker}(\mathcal{J}(u))$, and hence, every orbit is topologically constrained on the orthogonal complement of $\operatorname{Ker}(\mathcal{J}(u))$.

A functional $C(u)$ such that $[C, G] = 0$ for all G is called a Casimir element (or an element of the center of the Poisson algebra). If $\operatorname{Ker}[\mathcal{J}] = \{0\}$, the case for a canonical Hamiltonian system, then there is only a trivial element C = constant in the center. Evidently, a Casimir element $C(u)$ is a solution to the differential equation

$$\mathcal{J}(u)\,\partial_u C(u) = 0. \tag{2.2}$$

When the phase space X has a finite dimension, (2.2) is a first-order partial differential equation. If $\operatorname{Ker}(\mathcal{J}(u))$ has a constant dimension ν in an open set $X_\nu \subseteq X$, we can integrate (2.2) in X_ν to obtain ν independent solutions, i.e., $\operatorname{Ker}(\mathcal{J}(u))$ is locally spanned by the gradients of ν Casimir elements (Lie–Darboux theorem). The intersection of all Casimir leaves (the level-sets of Casimir elements) is the effective phase space, on which $\mathcal{J}(u)$ reduces to a symplectic Poisson operator.

However, the general (global) integrability of (2.2) is a mathematical challenge; the point where the rank of $\mathcal{J}(u)$ changes is a singularity of (2.2), from which singular (hyperfunction) solutions are generated. Moreover, because models of a fluids and plasmas are formulated on an infinite-dimensional phase space, for these systems (2.2) is a functional differential equation. The reader is referred to Yoshida *et al* (2014a) for an example of a singular Casimir element generated by singularities in a function space.

Our strategy of improving the integrability of (2.2) and extending the set of topological constraints expressible in terms of Casimir elements is to embed the Poisson manifold in higher-dimensional spaces. For an element $v \in \operatorname{Ker}(\mathcal{J}(u))$, (2.2) demands a solution in terms of a *gradient* (exterior derivative) of a scalar potential (0-form) $C(u)$. Such a solution is possible only when v is an exact 1-from, or at least v must be a closed 1-form for the local integrability. Our idea is to add extra components to v and make it exact in a higher-dimension space. Although this description is a finite-dimensional story, we will develop an infinite-dimensional theory. In the next subsection, we see how Casimir elements change as the phase space is extended.

2.2. Example of two-dimensional vortex dynamics

In table 1 we compare well-known examples of two-dimensional vortex dynamics systems, the Hamiltonian structures of which were given in Morrison (1982), Morrison and Hazeltine (1984), Marsden and Morrison (1984). We denote by $\omega = -\Delta\varphi$ the vorticity with Δ being the Laplacian and $\varphi \in H_0^1(\Omega) \cap H^2(\Omega)$ for the two-dimensional Eulerian velocity field $\boldsymbol{V} = {}^t(\partial_y\varphi, -\partial_x\varphi)$. Given a Hamiltonian

$$H_{\mathrm{E}}(\omega) = -\frac{1}{2}\int \mathrm{d}^2x\;\omega\,\Delta^{-1}\omega,$$

Table 1. Hierarchy of two-dimensional vortex systems. Here $\{a, b\} = \partial_y a \partial_x b - \partial_x a \partial_y b$, and $\circ$ implies insertion of the function to the right of the operator.

System	State vector	Poisson operator	Casimir elements
(I)	ω	$\{\omega, \circ\}$	$C_0 = \int \mathrm{d}^2x\, f(\omega)$
(II)	${}^t(\omega, \psi)$	$\begin{pmatrix} \{\omega, \circ\} & \{\psi, \circ\} \\ \{\psi, \circ\} & 0 \end{pmatrix}$	$C_1 = \int \mathrm{d}^2x\, \omega g(\psi)$ $C_2 = \int \mathrm{d}^2x\, f(\psi)$
(III)	${}^t(\omega, \psi, \check{\psi})$	$\begin{pmatrix} \{\omega, \circ\} & \{\psi, \circ\} & \{\check{\psi}, \circ\} \\ \{\psi, \circ\} & 0 & 0 \\ \{\check{\psi}, \circ\} & 0 & 0 \end{pmatrix}$	$C_2 = \int \mathrm{d}^2x\, f(\psi)$ $C_3 = \int \mathrm{d}^2x\, h(\psi\check{\psi})$ $C_4 = \int \mathrm{d}^2x\, \check{f}(\check{\psi})$

the system (I) is the vorticity equation for Eulerian flow,

$$\partial_t \omega + \boldsymbol{V} \cdot \nabla \omega = 0.$$

In table 1 we show the Poisson operator and Casimir elements for this system.

If ψ is the Gauss potential of a magnetic field, i.e., $\boldsymbol{B} = {}^t\left(\partial_y \psi, -\partial_x \psi\right)$, and the Hamiltonian is

$$H_{\mathrm{RMHD}}(\omega, \psi) = -\frac{1}{2} \int \mathrm{d}^2x \left[\omega \Delta^{-1} \omega + \psi \Delta \psi\right],$$

the system (II) is the reduced MHD equation,

$$\partial_t \omega + \boldsymbol{V} \cdot \nabla \omega = \boldsymbol{J} \times \boldsymbol{B},$$
$$\partial_t \psi + \boldsymbol{V} \cdot \nabla \psi = 0.$$

In the system (II), C_0 is no longer a constant of motion, being replaced by C_1 and C_2 of table 1.

However, if the Hamiltonian H is independent of ψ, both ω and ψ obey the same evolution equation. Then, ψ is a *mock field* which can be chosen arbitrarily without changing the dynamics of the actual field ω. At the special choice of $\psi = \omega$, both C_1 and C_2 evaluate as C_0, i.e., C_0 is 'subsumed' by C_1 and C_2 as their special value (indeed, C_1 and C_2 carry a larger amount of information of the system (I); in section 2.3, we will see how the mock field probes topological constraints). The constancy of C_0 is, then, due to the symmetry $\partial_\psi H = 0$. A modification of the Hamiltonian to involve ψ destroys the constancy of C_0; the electromagnetic interaction is a physical example of such a modification.

We can extend the phase space further to obtain a system (III) by adding another field $\check{\psi}$ that obeys the same evolution equation as ψ. In the reduced MHD system, $\check{\psi}$ is a mock field, i.e., it does not have a direct physical meaning; however, in the original RMHD context such a field physically corresponds to the pressure in the high-beta MHD model (Morrison and Hazeltine 1984, Thiffeault and Morrison 1998). For this further extended system we obtain the additional Casimir elements C_3 and C_4 of table 1, as first shown in Morrison and Hazeltine (1984).

2.3. Integrability of topological constraints

An interesting consequence of extending the system from (I) to (II) is found in the *integrability* of the $\mathrm{Ker}(\mathcal{J})$, or the topological constraints. In (I),

$$\mathrm{Ker}\left(\mathcal{J}(\omega)\right) = \left\{\psi;\ \{\omega, \psi\} = 0\right\},$$

which implies that ψ and ω are related, invoking a certain scalar $\zeta(x, y)$, by

$$\psi = \eta(\zeta), \quad \omega = \xi(\zeta). \tag{2.3}$$

As far as ξ is a monotonic function, we may write $\psi = \eta\left(\xi^{-1}(\omega)\right)$, which we can integrate to obtain the Casimir element $C_0(\omega)$ with $f(\omega)$ such that $f'(\omega) = \eta\left(\xi^{-1}(\omega)\right)$. Other elements of $\mathrm{Ker}\left(\mathcal{J}(\omega)\right)$ that are given by nonmonotonic ξ are not integrable to define Casimir elements. Yet, we can integrate such elements as $C_1(\omega, \psi)$ in the extended space of (II). In fact, every member of $\mathrm{Ker}\left(\mathcal{J}(\omega)\right)$ can be represented as $\partial_\omega C_1 = g(\psi)$ by choosing ψ in $\mathrm{Ker}\left(\mathcal{J}(\omega)\right)$.

Similarly, in the system (II), we encounter the deficit of the Casimir element $C_2 = \int \mathrm{d}^2x f(\psi)$ in covering all elements ${}^t(0, \chi) \in \mathrm{Ker}\left(\mathcal{J}(\omega, \psi)\right)$ such that $\{\psi, \chi\} = 0$. By the help of a mock field $\check{\psi}$, we can integrate every element of $\mathrm{Ker}\left(\mathcal{J}(\omega, \psi)\right)$ as C_3.

2.4. Minimum canonization invoking Casimir elements

If a topological constraint on a noncanonical system is represented by a Casimir element, we can define a canonical pair by adding an angle variable; then, the Casimir element morphs into an action variable (Yoshida and Morrison 2014b).

Here we consider a finite-dimensional model (which may be regarded as a relevant degenerate part of an infinite-dimensional system). Let J be a Poisson operator (matrix) on an n-dimensional phase space $X = \mathbb{R}^n$ parameterized by $z = (z_1, \cdots, z_n)$. We assume that $\mathrm{Ker}(J)$ has a dimension ν and n—ν is an even number. We first canonize J on $X/\mathrm{Ker}(J)$. Let

$$z' = \left(\zeta_1, \cdots, \zeta_{n-\nu}, C_1, \cdots, C_\nu\right) \in \mathbb{R}^n,$$

by which J is transformed into a Darboux standard form:

$$J' = \left(\begin{array}{c|c} J_c & \\ \hline & 0_\nu \end{array}\right). \tag{2.4}$$

We can extend J' to an $\tilde{n} \times \tilde{n}$ canonical matrix such that

$$J_{ex} = \left(\begin{array}{c|cc} J_c & & \\ \hline & 0_\nu & -I_\nu \\ & I_\nu & 0_\nu \end{array}\right). \tag{2.5}$$

The corresponding variables are denoted by

$$z_{ex} = \left(\zeta_1, \cdots, \zeta_{n-\nu}, C_1, \cdots, C_\nu, \vartheta_1, \cdots, \vartheta_\nu\right) \in \mathbb{R}^{\tilde{n}}.$$

This extended Poisson matrix J_{ex} is *symplectic*, i.e., the extended system is *canonized*, which is in marked contrast to the noncanonical extension discussed in section 2.2. The noncanonical extension is the first step for representing topological constraints by Casimir elements. Next, we extend the phase space further to canonize the Casimir elements. By perturbing the Hamiltonian with the added angle variables, we can unfreeze the Casimir elements. This perturbation brings about an increase in the number of degrees of freedom of the system, and is an example of a *singular perturbation*.

3. Topological constraints in ideal magnetohydrodynamics

Hereafter, we consider the example of a noncanonical Hamiltonian system provided by a three-dimensional ideal MHD system. The dynamics is strongly constrained by the magnetic flux conservation on every co-moving surface. Local magnetic fluxes are, however, not always Casimir elements (in two-dimensional dynamics, some are implied by the Casimir elements C_2; see section 2.2). Applying the method of the previous section, we extend the system to write local fluxes, which are loop integrals, as Casimir elements. In this section, we review the basic formulation, boundary conditions, and the magnetic flux conservation law.

3.1. Magnetohydrodynamics

Denoting by ρ the mass density, $\boldsymbol{V}$ the fluid velocity, $\boldsymbol{B}$ the magnetic field, and h the specific enthalpy, the governing equations of magnetohydrodynamics (MHD) are

$$\partial_t \rho = -\nabla \cdot \left(\boldsymbol{V}\rho\right), \tag{3.1}$$

$$\partial_t \boldsymbol{V} = -\left(\nabla \times \boldsymbol{V}\right) \times \boldsymbol{V} - \nabla\left(h + V^2/2\right) + \rho^{-1}\left(\nabla \times \boldsymbol{B}\right) \times \boldsymbol{B}, \tag{3.2}$$

$$\partial_t \boldsymbol{B} = \nabla \times \left(\boldsymbol{V} \times \boldsymbol{B}\right). \tag{3.3}$$

Here we assume a barotropic relation to write the enthalpy $h = h(\rho)$ (which is related to the thermal energy $\mathcal{E}$ by $h = \partial\left(\rho\mathcal{E}\right)/\partial\rho$). The variables are normalized in standard Alfvén units with energy densities (thermal $\rho\mathcal{E}$, kinetic $\rho_0 V^2$ and magnetic $B^2/2\mu_0$) normalized by a representative magnetic energy density B_0^2/μ_0. Evidently, the state vector for this system is $u = {}^t\left(\rho, \boldsymbol{V}, \boldsymbol{B}\right)$.

We consider a bounded domain $\Omega \subset \mathbb{R}^3$ on which the Hamiltonian (energy) has a finite value. Here we start with a simply connected Ω (a multiply connected domain will be discussed in section 3.2). We denote by $\partial\Omega$ the boundary of Ω, which is a smooth two-dimensional manifold consisting of a finite number of connected components. Denoting by ν the unit normal vector on the boundary $\partial\Omega$, and by $f\big|_{\partial\Omega}$ the restriction of f onto the boundary $\partial\Omega$ (which is defined by a continuous trace operator), we assume the following standard boundary conditions on the flow velocity $\boldsymbol{V}$ and the magnetic field $\boldsymbol{B}$:

$$\nu \cdot \boldsymbol{V}\big|_{\partial\Omega} = 0, \tag{3.4}$$

$$\nu \cdot \boldsymbol{B}\big|_{\partial\Omega} = 0. \tag{3.5}$$

Physically, (3.4) means that the fluid (plasma) is confined in the domain and cannot cross the boundary. The magnetic field is also confined in the domain; (3.5) is a consequence of (in fact, a little more stronger than) a perfectly conducting boundary condition isolating Ω electromagnetically from the complementary space, which demands that the tangential component of the electric field $\boldsymbol{E}$ vanishes on $\partial\Omega$, i.e.

$$\nu \times \boldsymbol{E}\big|_{\partial\Omega} = 0. \tag{3.6}$$

Writing $\boldsymbol{E} = -\partial_t \boldsymbol{A} - \nabla\phi$ with a scalar potential ϕ, we observe, for every disk $S \subset \partial\Omega$ (where ∂S is the boundary of the disk S and τ is the unit tangent vector along ∂S),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}^2x\ \nu \cdot \boldsymbol{B} &= \int_S \mathrm{d}^2x\ \nu \cdot \left(\partial_t \boldsymbol{B}\right)\\ &= \oint_{\partial S} \mathrm{d}x\ \tau \cdot \left(\partial_t \boldsymbol{A}\right)\\ &= -\oint_{\partial S} \mathrm{d}x\ \tau \cdot \left(\boldsymbol{E} + \nabla\phi\right) = 0,\end{aligned} \tag{3.7}$$

since $\tau \cdot \boldsymbol{E} = 0$ by (3.6), and $\nabla\phi$ is an exact differential. Assuming that $\nu \cdot \boldsymbol{B} = 0$ at $t = 0$, we obtain the homogeneous boundary condition (3.5).

3.2. Total flux conservation: cohomology constraint

When the domain Ω is multiply connected, the boundary conditions (3.4) and (3.5) are insufficient to determine a unique solution; we have to specify the 'magnetic flux' on each *cut* Σ_ℓ of the handle of Ω. Here, we make Ω into a simply connected domain Ω_0 by inserting cuts Σ_ℓ across each handle of Ω: $\Omega_0 := \Omega \backslash \left(\bigcup_{\ell=1}^m \Sigma_\ell\right)$, where m is the *genus* of Ω (see appendix A).

Hereafter, we assume $m \geqslant 1$. The *fluxes* of $\boldsymbol{B}$, given by

$$\Phi_\ell\left(\boldsymbol{B}\right) = \int_{\Sigma_\ell} \mathrm{d}^2x\ \nu \cdot \boldsymbol{B} \quad \left(\ell = 1, \cdots, m\right), \tag{3.8}$$

are the constants of motion (ν is the unit normal vector of Σ_ℓ) when we assume the perfectly conducting boundary condition (3.6). In fact, replacing S by Σ_ℓ in (3.7), we obtain $\mathrm{d}\Phi_\ell/\mathrm{d}t = 0$, since the boundary $\partial\Sigma_\ell$ of Σ_ℓ is a cycle on $\partial\Omega$ where the tangential electric field vanishes.

The flux conditions $\Phi_\ell\left(\boldsymbol{B}\right) =$ constant ($\ell = 1, \cdots, m$) mean that the cohomology class of 2-forms ($\boldsymbol{B}_H$ such that $\nabla \times \boldsymbol{B}_H = 0$, $\nabla \cdot \boldsymbol{B}_H = 0$, $\nu \cdot \boldsymbol{B}_H\big|_{\partial\Omega} = 0$) included in $\boldsymbol{B}$ are fixed constants (see appendix A).

3.3. Local flux conservation and circulation theorem

Whereas the aforementioned magnetic flux constraints pertain to the cohomology of the fixed domain Ω (which restrict a finite number m degrees of freedom), every local magnetic flux on an arbitrary co-moving surface σ is also constrained, i.e., the magnetic flux (or, equivalently, the circulation of the vector potential along the boundary $\partial\sigma$ of the disk σ)

$$\Phi_\sigma(t) = \int_{\sigma(t)} \mathrm{d}^2x\ \nu \cdot \boldsymbol{B} = \oint_{\partial\sigma(t)} \mathrm{d}x\ \tau \cdot \boldsymbol{A}$$

is a constant of motion. This conservation law (often called Alfvén's theorem in the MHD context, but equivalent to Kelvin's circulation theorem) is a direct consequence of the magnetic induction equation (3.3), which implies that the 2-form $\boldsymbol{B}$ is Lie-dragged by the flow $\boldsymbol{V}$. Because of this infinite set of conservation laws, the magnetic field lines are forbidden to change their topology.

In the next section, we will study the meaning of these total and local flux conservation laws from the perspective of Hamiltonian mechanics.

4. Hamiltonian structure of magnetohydrodynamics

4.1. Noncanonical Poisson bracket and Casimir elements

The preceding MHD equations possess the noncanonical Hamiltonian form first given in Morrison and Greene (1980) where the phase space X contains the state vector $u = {}^t(\rho, \boldsymbol{V}, \boldsymbol{B})$, and the Hamiltonian and Poisson operator are given as follows:

$$H = \int_\Omega \mathrm{d}^3x \left\{ \rho \left[\frac{V^2}{2} + \mathcal{E}(\rho) \right] + \frac{B^2}{2} \right\}, \tag{4.1}$$

$$\mathcal{J} = \begin{pmatrix} 0 & -\nabla\cdot & 0 \\ -\nabla & -\rho^{-1}(\nabla\times\boldsymbol{V})\times & \rho^{-1}(\nabla\times\circ)\times\boldsymbol{B} \\ 0 & \nabla\times\left(\circ\times\rho^{-1}\boldsymbol{B}\right) & 0 \end{pmatrix}. \tag{4.2}$$

Here $\circ$ implies insertion of the function to the right of the operator. We formally endow the phase space X with the standard L^2 norm. The Poisson operator $\mathcal{J}$ is a differential operator with variable coefficients, and the domain of $\mathcal{J}$ is a subspace of X (hence, we assume boundary conditions (3.4) and (3.5) when we operate with $\mathcal{J}$). There are subtleties associated with the mathematical identification of $D(\mathcal{J})$ and we will address the minimum amount needed for our purposes here.

It is easily verified that a Poisson bracket $[F, G] = \langle \partial_u F, \mathcal{J}\partial_u G\rangle$ is antisymmetric and using the techniques of Morrison (1982) it was verified that it satisfies Jacobi's identity. When the specific enthalpy $h(\rho) = \partial(\rho\mathcal{E}(\rho))/\partial\rho$ is a continuous function, $H(\rho, \boldsymbol{V}, \boldsymbol{B})$ is a C^1-class functional of the state vector $u = {}^t(\rho, \boldsymbol{V}, \boldsymbol{B})$, and the functional gradient $\partial_u H(u)$ is evaluated in the classical sense. With this structure, the Hamilton form of (2.1) reproduces the MHD equations (3.1)–(3.3).

The Poisson operator $\mathcal{J}$ has well-known Casimir elements (Morrison 1982, Padhye and Morrison 1996, Hameiri 2004, Andreussi *et al* 2012):

$$C_1 = \int_\Omega \mathrm{d}^3x\, \rho, \tag{4.3}$$

$$C_2 = \frac{1}{2}\int_\Omega \mathrm{d}^3x\, \boldsymbol{A}\cdot\boldsymbol{B}, \tag{4.4}$$

$$C_3 = \int_\Omega \mathrm{d}^3x\, \boldsymbol{V}\cdot\boldsymbol{B}, \tag{4.5}$$

where $\boldsymbol{A}$ is the vector potential ($\boldsymbol{A} = \text{curl}^{-1}\boldsymbol{B}$), which is evaluated with a fixed gauge and boundary conditions. The Casimir C_1 is the total mass, C_2 the magnetic helicity, and C_3 the cross helicity.

4.2. A Casimir element representing the total fluxes

The total flux pertinent to the cohomology of the domain Ω can be regarded as a *singular* Casimir element of the MHD system. We may formally write

$$\Phi_\ell\left(\boldsymbol{B}\right) = \int_{\Sigma_\ell} \mathrm{d}^2x\ \boldsymbol{n}\cdot\boldsymbol{B} = \int_\Omega \mathrm{d}^3x\ \boldsymbol{\sigma}_\ell\cdot\boldsymbol{B}$$

with a singular 1-form such that

$$\boldsymbol{\sigma}_\ell = \nabla[\![\bar{\theta}_\ell]\!],$$

where $\bar{\theta}_\ell = \theta_\ell/(2\pi)$ with θ_ℓ the angle measured from Σ_ℓ going around the handle ℓ, and $[\![\alpha]\!]$ is Gauss's symbol for the maximum integer smaller than $\alpha \in \mathbb{R}$, i.e., $[\![\bar{\theta}_\ell]\!] = [\![\theta_\ell/(2\pi)]\!]$ is the 'winding number' of the angle θ_ℓ, which steps by unity at Σ_ℓ (see appendix A). Formally, we calculate $\partial_u\Phi_\ell\left(\boldsymbol{B}\right) = \left(0,\ 0,\ \nabla[\![\bar{\theta}_\ell]\!]\right)$, and $\nabla\times\left(\nabla[\![\bar{\theta}_\ell]\!]\right) = 0$, hence, $\Phi_\ell\left(\boldsymbol{B}\right)$ is a Casimir element.

Remark 1. (Separation of cohomology.) If the domain Ω is multiply connected (i.e., the genus $m \geqslant 1$) and the magnetic flux $\Phi_\ell\left(\boldsymbol{B}\right)$ on each handle ($\ell = 1, \cdots, m$) is constrained by the boundary condition (3.6), only the internal magnetic field $\boldsymbol{B}_\Sigma = \boldsymbol{B} - \boldsymbol{B}_H$ is the dynamical variable (see appendix A). We may replace the total $\boldsymbol{B}$ by $\boldsymbol{B}_\Sigma$ in defining the state vector u. Then, the Casimir elements $\Phi_1\left(\boldsymbol{B}_\Sigma\right), \cdots, \Phi_m\left(\boldsymbol{B}_\Sigma\right)$ trivialize, and we define the magnetic helicity as

$$C'_2 = \frac{1}{2}\int_\Omega \mathrm{d}^3x\boldsymbol{A}_\Sigma\cdot\boldsymbol{B}_\Sigma + \int_\Omega \mathrm{d}^3x\boldsymbol{A}_H\cdot\boldsymbol{B}_\Sigma, \tag{4.6}$$

where $\nabla\times\boldsymbol{A}_\Sigma = \boldsymbol{B}_\Sigma$ and $\nabla\times\boldsymbol{A}_H = \boldsymbol{B}_H$ (see remark 1 of Yoshida and Dewar 2012).

4.3. Extension of the phase space

To formulate the local magnetic flux as a Casimir element, we extend the phase space as in section 2.2 in order to include topological index information in the set of dynamical variables. Adding a 2-form $\check{\boldsymbol{B}}$, which we call a *mock field*, to the MHD variables, gives the extended phase space state vector

$$\tilde{u} = {}^t\left(\rho,\ \boldsymbol{V},\ \boldsymbol{B},\ \check{\boldsymbol{B}}\right), \tag{4.7}$$

on which we define a degenerate Poisson manifold by

$$\tilde{\mathcal{J}} = \begin{pmatrix} 0 & -\nabla\cdot & 0 & 0 \\ -\nabla & -\rho^{-1}\left(\nabla\times\boldsymbol{V}\right)\times & \rho^{-1}\left(\nabla\times\circ\right)\times\boldsymbol{B} & \rho^{-1}\left(\nabla\times\circ\right)\times\check{\boldsymbol{B}} \\ 0 & \nabla\times\left(\circ\times\rho^{-1}\boldsymbol{B}\right) & 0 & 0 \\ 0 & \nabla\times\left(\circ\times\rho^{-1}\check{\boldsymbol{B}}\right) & 0 & 0 \end{pmatrix}. \tag{4.8}$$

We assume that $\check{\boldsymbol{B}}$ obeys the same boundary condition as $\boldsymbol{B}$,

$$\boldsymbol{n} \cdot \check{\boldsymbol{B}}\big|_{\partial\Omega} = 0. \tag{4.9}$$

Using the same Hamiltonian (4.1), we obtain an extended dynamics governed by exactly the same equations (3.1)–(3.3) together with an additional equation

$$\partial_t \check{\boldsymbol{B}} = \nabla \times \left(\boldsymbol{V} \times \check{\boldsymbol{B}} \right). \tag{4.10}$$

The projection of the orbit onto the original phase space reproduces the same dynamics; the mock field $\check{\boldsymbol{B}}$ is just a passive 2-covector moved by the flow $\boldsymbol{V}$ of the original system.

The extended Poisson operator (4.8) has the set of Casimir elements composed of C_1, C_2, and a new *cross helicity*

$$C_4 = \int_\Omega \mathrm{d}^3x\, \boldsymbol{A} \cdot \check{\boldsymbol{B}}, \tag{4.11}$$

as well as a mock magnetic helicity

$$C_5 = \frac{1}{2} \int_\Omega \mathrm{d}^3x\, \check{\boldsymbol{A}} \cdot \check{\boldsymbol{B}}. \tag{4.12}$$

Interestingly, the original (standard) cross helicity $C_3 = \int_\Omega \mathrm{d}^3x\, \boldsymbol{V} \cdot \boldsymbol{B}$ is no longer a Casimir element of the extended system, although it is still a constant of motion. The constancy of C_3 is now due to the 'symmetry' of a Hamiltonian with ignorable dependence on the mock field $\check{\boldsymbol{B}}$; for *every* Hamiltonian $H\left(\rho, \boldsymbol{V}, \boldsymbol{B}\right)$, which is not necessarily the MHD Hamiltonian (4.1), we find, denoting $\rho^\dagger = \partial_\rho H$, $\boldsymbol{V}^\dagger = \partial_{\boldsymbol{V}} H$, $\boldsymbol{B}^\dagger = \partial_{\boldsymbol{B}} H$, and noticing $\partial_{\check{\boldsymbol{B}}} H = 0$ (while H may be an arbitrary C^1-class functional of u, we must assume $\partial_u H = {}^t\left(\rho^\dagger, \boldsymbol{V}^\dagger, \boldsymbol{B}^\dagger\right) \in D\left(\mathcal{J}\right)$),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} C_3 &= \int_\Omega \mathrm{d}^3x \left\{ \left(\partial_t \boldsymbol{V}\right) \cdot \boldsymbol{B} + \boldsymbol{V} \cdot \left(\partial_t \boldsymbol{B}\right) \right\} \\
&= \int_\Omega \mathrm{d}^3x \left\{ \left[-\nabla\rho^\dagger - \rho^{-1}\left(\nabla \times \boldsymbol{V}\right) \times \boldsymbol{V}^\dagger + \rho^{-1}\left(\nabla \times \boldsymbol{B}^\dagger\right) \times \boldsymbol{B} \right] \cdot \boldsymbol{B} \right. \\
&\quad \left. + \boldsymbol{V} \cdot \left[\nabla \times \left(\rho^{-1} \boldsymbol{V}^\dagger \times \boldsymbol{B}\right) \right] \right\} \\
&= 0.
\end{aligned}$$

Here we have used the boundary condition $\nu \cdot \boldsymbol{V}^\dagger = 0$, which is guaranteed for $\partial_u H \in D\left(\mathcal{J}\right)$.

4.4. Local flux (circulation) as a Casimir element

Here we show that the cross helicity C_4 evaluates the *circulation* of $\boldsymbol{A}$ when we choose a 'pure-state 2-form' as the mock field $\check{\boldsymbol{B}}$. We can consider a *filamentary* $\check{\boldsymbol{B}}$ supported on a co-moving loop $L(t)$ such that, for every disk σ,

$$\int_\sigma \mathrm{d}^2x\, \nu \cdot \check{\boldsymbol{B}} = \mathcal{L}\left(L(t), \partial\sigma\right), \tag{4.13}$$

where $\mathcal{L}\left(L_1, L_2\right)$ denotes the linking number of two loops L_1 and L_2 (the exact definition will be given in section 5). Formally, the filamentary $\check{\boldsymbol{B}}$ is a delta-measure on a co-moving loop $L(t)$ carrying a unit mock flux. Inserting such $\check{\boldsymbol{B}}$ into the cross helicity C_4, we obtain

$$C_4 = \int_\Omega \mathrm{d}^3x\, \boldsymbol{A} \cdot \check{\boldsymbol{B}} = \oint_{L(t)} \mathrm{d}x\, \tau \cdot \boldsymbol{A}. \tag{4.14}$$

Hence, the conservation of the cross helicity C_4 implies the conservation of the circulation, or equivalently, the local magnetic flux conservation on every disk bounded by $L(t)$.

We note that the Casimir element C_4 integrates the infinite number of constraints on all local fluxes (or circulations), because it is a function of the mock field $\check{\boldsymbol{B}}$ (actually, the initial value $\check{\boldsymbol{B}}(0)$) which can be chosen arbitrarily without changing the dynamics of the actual field $u = {}^t(\rho, \boldsymbol{V}, \boldsymbol{B})$. The point is that the flow map (which transports the actual field u, as well as the mock field $\check{\boldsymbol{B}}$) is independent of $\check{\boldsymbol{B}}$. The same dynamics of the actual field u is 'simultaneously' constrained by any C_4 evaluated for arbitrary $\check{\boldsymbol{B}}(0)$. In fact, these invariants are the reflection of the topological constraints on the original system of the actual field; they are represented, by the help of the co-moving mock fields, as the cross helicity C_4. In principle, the cross helicity may be regarded as the translation of the Lagrangian label into the language of the Hamiltonian formalism; see Fukumoto (2008).

Remark 2. (Two-dimensional MHD system.) In the two-dimensional system of section 2.2, the cross helicity C_4 parallels the Casimir element $\int \mathrm{d}^2x\, f(\psi)$ of the reduced MHD system (Morrison 1982, Morrison and Hazeltine 1984, Marsden and Morrison 1984, Morrison 1998). To see this consider a cylindrical domain $\Omega = \Sigma \times [0, 1]$ ($\Sigma \subset \mathbb{R}^2$) and two-dimensional vectors $\boldsymbol{V} = {}^t(\partial_y\varphi, -\partial_x\varphi)$ and $\boldsymbol{B} = {}^t(\partial_y\psi, -\partial_x\psi)$, which satisfy periodic boundary conditions at $z = 0$ and 1. We may assume $\boldsymbol{A} = \psi\boldsymbol{e}_z$. With a constant ρ, $u = {}^t(\rho, \boldsymbol{V}, \boldsymbol{B})$ may satisfy the MHD equations (3.1)–(3.3) in Ω, as well as the reduced MHD equations, the system (II) of section 2.2, in Σ. Let $\xi(t)$ be a co-moving point in Σ, and $\check{\boldsymbol{B}} = \delta(\boldsymbol{x} - \xi(t))\boldsymbol{e}_z$. Then,

$$C_4 = \int \mathrm{d}^2z\, \delta(\boldsymbol{x} - \xi(t))\psi(\boldsymbol{x}) = \psi(\xi(t)).$$

Integrating C_4 over all points $\xi(0) \in \Sigma$ with a weight function f yields $\int \mathrm{d}^2x\, f(\psi)$.

In the next section, we shall identify the unit-flux filament as a pure state of a Banach algebra, and show that the co-moving filament is a singular solution of (4.10).

5. Dynamics of loops: Poincaré dual of local flux

5.1. Pure state of Banach algebra

A unit-flux filament is identified as a *pure-state* 2-form (physically a vorticity or a magnetic field, which, however, is a mock field); see Yoshida *et al* (2014c). Naturally, a 2-form is in the Poincaré-dual relation with a 2-chain (two-dimensional surface), and a pure-state 2-form is a two-dimensional surface measure. The filamentary $\check{\boldsymbol{B}}$ is, then, the temporal cross-section of a 2-chain in the space-time.

Definition 1. (Pure state.) Let M be a smooth manifold of dimension n, and $\Omega \subset M$ be a p-dimensional connected null-boundary submanifold of class C^1. Each Ω can be regarded as an equivalent of a *pure-state functional* η_Ω on the space $\wedge^p T^*M$ of continuous p-forms:

$$\eta_\Omega\colon\ \omega \mapsto \int_\Omega \omega,$$

which can be represented as

$$\eta_\Omega(\omega) = \int_M \mathfrak{J}(\Omega) \wedge \omega = \int_\Omega \omega$$

with an $(n-p)$-dimensional δ-measure $\mathfrak{J}(\Omega) = \wedge^{n-p}\, \delta\left(x^\mu - \xi^\mu\right) \mathrm{d}x^\mu$, where x^μ are local coordinates, and

$$\operatorname{supp} \mathfrak{J}(\Omega) = \Omega = \left\{ \boldsymbol{x} \in \mathbb{R}^n;\ x^\mu = \xi^\mu\left(\mu = 1, \cdots, n-p\right) \right\}.$$

We call $\mathfrak{J}(\Omega)$ a *pure state* $(n-p)$-form, which is a member of the Hodge-dual space of $\wedge^p T^*M$.

5.2. Orbit of a filament

Here we show that the co-moving pure-state filament is a (singular) solution of (4.10). For the convenience of formulation, we rewrite the determining equation (4.10) of the mock field $\check{\boldsymbol{B}}$ in the four-dimensional Galilei space-time M_G; we draw heavily on the theory of relativistic helicity in Minkowski space-time developed in Yoshida *et al* (2014c). Normalizing the speed of light so that $c = 1$, we denote the four-dimensional coordinates as $\left(x^0, x^1, x^2. x^3\right) = (t, x, y, z) \in M_G$. The (nonrelativistic) four-vector is $U = U^\mu \partial_\mu = \left(\mathrm{d}x^\mu / \mathrm{d}t\right) \partial_\mu \in TM_G$, which has four components $U = \left(1, \boldsymbol{V}\right)$. We may identify the mock field $\check{\boldsymbol{B}}$ as the three-vector part of a 2-form: we define $F = F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu / 2$ with a 'Faraday tensor'

$$F_{\mu\nu} = \begin{pmatrix} 0 & \check{E}_1 & \check{E}_2 & \check{E}_3 \\ -\check{E}_1 & 0 & -\check{B}_3 & \check{B}_2 \\ -\check{E}_2 & \check{B}_3 & 0 & -\check{B}_1 \\ -\check{E}_3 & -\check{B}_2 & \check{B}_1 & 0 \end{pmatrix}, \tag{5.1}$$

where $\check{\boldsymbol{E}}$ is a certain three-vector satisfying Faraday's law

$$\nabla \times \check{\boldsymbol{E}} = -\partial_t \check{\boldsymbol{B}}. \tag{5.2}$$

Invoking these notations, the 'vorticity equation' (4.10) reads

$$\mathrm{d}i_U F = 0. \tag{5.3}$$

By (5.2) together with $\nabla \cdot \check{\boldsymbol{B}} = 0$, F is a closed 2-form ($\mathrm{d}F = 0$), thus we may rewrite (5.3) as

$$L_U F = 0, \tag{5.4}$$

where $L_U = \mathrm{d}i_U + i_U \mathrm{d}$ is the Lie derivative. Notice that (5.3) consists of six independent equations; three of which are (4.10), and the remaining are the energy equation

$$\partial_t \left(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}\right) - \nabla\left(\boldsymbol{E} \cdot \boldsymbol{V}\right) = 0,$$

which is solved by a potential energy ϕ such that $\boldsymbol{E} \cdot \boldsymbol{V} = -\partial_t \phi$ and $\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = -\nabla\phi$.

Let $\mathfrak{J}(\Gamma_0)$ be a pure-state 3-form (vortex filament) supported on a loop Γ_0 in M_G, which we may write

$$\mathfrak{J}\left(\Gamma_0\right) = \delta_{\Gamma_0} b = \delta_{\Gamma_0}\Big(b_1 \mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - b_2\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 + b_3 \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\Big). \tag{5.5}$$

We denote by $\mathcal{T}_U(t)$ the diffeomorphism generated by the vector U (i.e. $\mathrm{d}\mathcal{T}_U(t)/\mathrm{d}t = U$). The orbit of $\Gamma(t) = \mathcal{T}_U(t)\Gamma_0$ defines a surface (2-chain)

$$\Sigma = \bigcup_{t\in\mathbb{R}} \Gamma(t), \tag{5.6}$$

and its Poincaré-dual is written as

$$\mathfrak{J}\left(\Sigma\right) = -\delta_\Sigma i_U b = \delta_\Sigma \frac{1}{2}\mathcal{F}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu, \tag{5.7}$$

where

$$\mathcal{F}_{\mu\nu} = \begin{pmatrix} 0 & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ -\varepsilon_1 & 0 & -b_3 & b_2 \\ -\varepsilon_2 & b_3 & 0 & -b_1 \\ -\varepsilon_3 & -b_2 & b_1 & 0 \end{pmatrix} \tag{5.8}$$

with $\varepsilon = -\boldsymbol{V} \times \boldsymbol{b}$. Evidently,

$$i_U \mathfrak{J}\left(\Sigma\right) = -\delta_\Sigma i_U i_U b = 0;$$

hence, $F = \mathfrak{J}\left(\Sigma\right)$ satisfies (5.3). We note that $\mathfrak{J}\left(\Sigma\right)$ is a singular function defined as the dual of the Banach algebra of differential forms (definition 1), hence the vorticity equation (5.3) is interpreted in a weak sense; see Yoshida *et al* (2014c) for the mathematical justification. The t-plane projection of $\mathfrak{J}\left(\Sigma\right)$ yields (denoting $\Xi(t) = \left\{x;\ x^0 = t\right\}$)

$$-\delta_{\Xi(t)} \mathrm{d}x^0 \wedge \mathfrak{J}\left(\Sigma\right) = \delta_{\Gamma(t)} b(t), \tag{5.9}$$

which is a pure-state filament on a co-moving loop $\Gamma(t)$. Now we have

Theorem 1. *Suppose that an initial mock field $\check{\boldsymbol{B}}(0)$ is given as a pure-state $\mathfrak{J}\left(\Gamma(0)\right)$ on a loop $\Gamma(0)$ bounding a disk. Then, the orbit $\Sigma = \cup_{t\in\mathbb{R}} \Gamma(t)$ defines a pure-state 2-from $\mathfrak{J}\left(\Sigma\right)$ that satisfies the vorticity equation (5.3). The t-plane projection of $\mathfrak{J}\left(\Sigma\right)$ is a pure-state filament $\check{\boldsymbol{B}}(t) = \mathfrak{J}\left(\Gamma(t)\right)$ on a co-moving loop $\Gamma(t)$.*

6. Singular Casimir element: application to tearing modes

In this section, we study a different type of singular Casimir element, the cross helicity of extended MHD, which controls bifurcation of topologically different equilibria. The theory is applied to the tearing modes that are bifurcated equilibria on Casimir leaves (Yoshida and Dewar 2012); as long as the Casimir element is constrained, each tearing mode is stationary. However, by a singular perturbation that unfreezes the Casimir element, some tearing modes that have lower energies can be excited by changing the cross helicity.

6.1. Equilibrium points of energy-Casimir functional

We start by reviewing the equilibria of standard energy-Casimir functionals. When we have a Casimir element $C(u)$ in a noncanonical Hamiltonian system, a transformation of the Hamiltonian $H(u)$ such as

$$H(u) \mapsto H_\mu(u) = H(u) - \mu C(u) \tag{6.1}$$

(with an arbitrary real constant μ) does not change the dynamics. In fact, the Hamilton form is invariant under this transformation. We call the transformed Hamiltonian $H_\mu(u)$ an *energy-Casimir* function (Kruskal and Oberman 1958, Hazeltine *et al* 1984, Morrison and Eliezer 1986, Morrison 1998, Arnold and Khesin 1998).

Interpreting the parameter μ as a Lagrange multiplier of the equilibrium variational principle, $H_\mu(u)$ is the effective Hamiltonian with the constraint that restricts the Casimir element $C(u)$ to be a given value (since $C(u)$ is a constant of motion, its value is fixed by its initial value). As we will see in some examples, Hamiltonians are rather simple, often being 'norms' on the phase space. However, an energy-Casimir functional may have a nontrivial structure. Geometrically, $H_\mu(u)$ is the distribution of $H(u)$ on a Casimir leaf. If Casimir leaves are distorted with respect to the energy norm, the effective Hamiltonian may have a complex distribution on the leaf, which is, in fact, the origin of various interesting structures in noncanonical Hamiltonian systems.

Applying this energy-Casimir method to the MHD system, we obtain the Beltrami–Bernoulli equilibria. Constraining the Casimir elements (4.3)–(4.5) on the Hamiltonian (4.1), we consider

$$\partial_u H_{\mu_1,\mu_2,\mu_3} = 0, \quad H_{\mu_1,\mu_2,\mu_3} = H - \mu_1 C_1 - \mu_2 C_2 - \mu_3 C_3, \tag{6.2}$$

which reads as

$$V^2/2 + h - \mu_1 = 0, \tag{6.3}$$

$$\rho \boldsymbol{V} - \mu_3 \boldsymbol{B} = 0, \tag{6.4}$$

$$\nabla \times \boldsymbol{B} - \mu_2 \boldsymbol{B} - \mu_3 \nabla \times \boldsymbol{V} = 0. \tag{6.5}$$

In deriving (6.5), we have applied the curl operator. Putting $\mu_3 = 0$ simplifies the solutions to be the Beltrami fields such that $\nabla \times \boldsymbol{B} = \mu_2 \boldsymbol{B}$, $\boldsymbol{V} = 0$, and $h = \mu_1$.

In the next subsection, we apply the energy-Casimir method to the extended MHD system with a mock field $\check{\boldsymbol{B}}$, and show that an interesting bifurcation occurs at a 'singularity' in the phase space.

6.2. Singular Casimir element

Let us recall the determining equation of the cross helicity; a functional $C\left(\boldsymbol{B}, \check{\boldsymbol{B}}\right)$ is a Casimir element if

$$\tilde{\mathcal{J}} \partial_{\tilde{u}} C\left(\boldsymbol{B}, \check{\boldsymbol{B}}\right) = {}^t\left(0, \rho^{-1}\left[\left(\nabla \times \partial_{\boldsymbol{B}} C\right) \times \boldsymbol{B} + \left(\nabla \times \partial_{\check{\boldsymbol{B}}} C\right) \times \check{\boldsymbol{B}}\right], 0, 0\right) \tag{6.6}$$

vanishes. Evidently, $C_4 = \int \mathrm{d}^3x \boldsymbol{A} \cdot \check{\boldsymbol{B}}$ (with arbitrary $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ and $\check{\boldsymbol{B}} = \nabla \times \check{\boldsymbol{A}}$) satisfies (6.6). Here, we are interested in the *singularity* at which the rank of $\tilde{\mathcal{J}}$ drops; there is a pair of $\boldsymbol{B}^* = \nabla \times \boldsymbol{A}^*$ and $\check{\boldsymbol{B}}^* = \nabla \times \check{\boldsymbol{A}}^*$ such that the two terms on the right-hand side of (6.6) vanish separately, i.e.

$$\boldsymbol{B}^* \times \left(\nabla \times \check{\boldsymbol{A}}^*\right) = \left(\nabla \times \boldsymbol{A}^*\right) \times \check{\boldsymbol{B}}^* = 0. \tag{6.7}$$

We let $C_4^* := C_4\left(\boldsymbol{B}^*, \check{\boldsymbol{B}}^*\right)$ and call it a *singular cross helicity*. A significance of C_4^* is (in addition to $\tilde{\mathcal{J}}\partial_{\tilde{u}}C_4^* = 0$)

$$\mathcal{J}\partial_u C_4^* = 0. \tag{6.8}$$

When $\check{\boldsymbol{B}}^*$ is a mock field (i.e., the Hamiltonian H is independent of $\check{\boldsymbol{B}}$), we may regard C_4^* as a Casimir element of the original MHD system.

A trivial solution of (6.7) is $\boldsymbol{A}^* = \check{\boldsymbol{A}}^*$, by which C_4^* coincides with C_2. Apparently $\mathrm{Rank}\left(\tilde{\mathcal{J}}\right)$ changes at $\boldsymbol{A}^* = \check{\boldsymbol{A}}^*$, where the lost Casimir element, the conventional cross helicity C_3 resides as a *singular Casimir element*.

Interestingly, we may find nontrivial, hyperfunction solutions emerging from the *resonance singularity* of the differential equation (6.7). Here we solve (6.7) for $\check{\boldsymbol{A}}^*$ by giving $\boldsymbol{B}^*$. The determining equation can be rewritten as

$$\nabla \times \check{\boldsymbol{A}}^* = \eta \boldsymbol{B}^* \tag{6.9}$$

with some scalar function η, which, however, is not a free function; the divergence of both sides of (6.9) yields

$$\boldsymbol{B}^* \cdot \nabla\eta = 0, \tag{6.10}$$

which implies that η is constant along the magnetic field lines. For the integrability of η, the magnetic field $\boldsymbol{B}^*$ must have integrable field lines; a continuous spatial symmetry guarantees this. Here we consider a *slab geometry*, in which we may write $\boldsymbol{B}^* = {}^t\left(0, B_y^*(x), B_z^*(x)\right)$. Let us consider

$$\check{\boldsymbol{A}}^* = {}^t\left(0, \check{A}_y^*(x), \check{A}_z^*(x)\right)\mathrm{e}^{\mathrm{i}\left(k_y y + k_z z\right)}. \tag{6.11}$$

Putting $\check{A}_y^*(x) = \mathrm{i}k_y\vartheta(x)$ and $\check{A}_z^*(x) = \mathrm{i}k_z\vartheta(x)$, (6.7) reduces to

$$\left[B_y^*(x)k_y + B_z^*(x)k_z\right]\partial_x\vartheta(x) = 0, \tag{6.12}$$

which yields

$$\vartheta(x) = c_0 + c_1 Y(x - x_r), \tag{6.13}$$

where c_0, c_1 are complex constants, and k_y, k_z and x_r (real constants) are chosen to satisfy the *resonance condition*

$$B_y^*(x_r)k_y + B_z^*(x_r)k_z = 0. \tag{6.14}$$

Then, $\eta = \mathrm{i}\left(k_y/B_z^*\right)\mathrm{e}^{\mathrm{i}\left(k_y y + k_z z\right)}\delta(x - x_r)$.

Remark 3. (Linear theory.) In the forgoing derivation, the singular Casimir element C_4^* is essentially the same as the formulation of the *resonant helical flux Casimir element* C_b given in Yoshida and Dewar (2012), which was used to construct tearing modes. It is remarkable, however, that the present argument is totally nonlinear, while C_b was formulated for a linearized Poisson operator (i.e., $\mathcal{J}(u)$ evaluated at an equilibrium point $u = u_0$). These quantities are compared as follows:

- The singular cross helicity C_4^* is a bilinear form combining the physical field $\boldsymbol{A}^*$ and the mock field $\breve{\boldsymbol{B}}^*$.
- The resonant helical flux is a linear form of a 'perturbation field' $\tilde{\boldsymbol{B}}$ multiplied by a kernel element $\breve{\boldsymbol{A}}^*$ (which is denoted by $\boldsymbol{b}$ in Yoshida and Dewar (2012)) of $\mathcal{J}(u_0)$,

$$C_b\left(\tilde{\boldsymbol{B}}\right) = \int \mathrm{d}^3x \tilde{\boldsymbol{B}} \cdot \breve{\boldsymbol{A}}^*.$$

In the determining equation (6.7) of $\breve{\boldsymbol{A}}^*$, $\boldsymbol{B}^*$ may be regarded as an 'equilibrium field'. By separating a perturbation $\tilde{\boldsymbol{B}}$ and an equilibrium $\boldsymbol{B}^*$, C_b is defined as a linear form on the space of perturbations. However, C_4^* is a special value of C_4 evaluated at the singularity $\boldsymbol{B} = \boldsymbol{B}^* = \nabla \times \boldsymbol{A}^*$ and $\breve{\boldsymbol{B}} = \breve{\boldsymbol{B}}^*$ in the phase space of total fields.

6.3. Tearing mode

Because of the similarity between C_4^* and the resonant helical flux Casimir element (see Remark 3), the formulation of tearing modes goes almost parallel to that of Yoshida and Dewar (2012). Here, we describe only the essence of the theory.

We begin with an energy-Casimir functional on the extended phase space

$$H_\mu = H - \mu_1 C_1 - \mu_2 C_2 - \mu_4 C_4 - \mu_5 C_5. \tag{6.15}$$

and consider a stationary point of

$$\partial_u H_\mu = 0. \tag{6.16}$$

Notice that we are not demanding $\partial_{\tilde{u}} H_\mu = 0$; hence, the solution of (6.16) is not necessarily an equilibrium point. However, if we evaluate (6.16) at the *singularity* $\boldsymbol{B} = \boldsymbol{B}^*$ and $\breve{\boldsymbol{B}} = \breve{\boldsymbol{B}}^*$, we obtain

$$\partial_u H_\mu \Big|_{\boldsymbol{B}^*, \breve{\boldsymbol{B}}^*} = \begin{pmatrix} V^2/2 + h - \mu_1 \\ \rho \boldsymbol{V} \\ \boldsymbol{B}^* - \mu_2 \boldsymbol{A}^* - \mu_4 \breve{\boldsymbol{A}}^* \end{pmatrix} = 0. \tag{6.17}$$

Let M^* denote the solution of (6.17), then by (6.7), we obtain

$$\tilde{\mathcal{J}} \partial_{\tilde{u}} H_\mu \Big|_{M^*} = \tilde{\mathcal{J}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\mu_4 \boldsymbol{A}^* - \mu_5 \breve{\boldsymbol{A}}^* \end{pmatrix} = 0.$$

Hence, M^* is an equilibrium of the extended MHD, which *bifurcates* from the singularity $\boldsymbol{B} = \boldsymbol{B}^*$ and $\breve{\boldsymbol{B}} = \breve{\boldsymbol{B}}^*$. However, M^* is not a stationary point of the energy-Casimir functional H_μ; the present 'energy-Casimir method' (6.16) is different from the conventional one in that only the partial derivative with respect to u (the actual field) is evaluated. Yet, we may view (6.16) as the energy-Casimir method on the original MHD phase space, if C_4 is a Casimir element pertinent to $\mathcal{J}$ (we regard the mock field $\breve{\boldsymbol{B}}$ as a parameter, and then, C_5 is just a constant). In fact it is, when C_4 evaluates as C_4^* and satisfies (6.8).

By the determining equation (6.17), the equilibrium has zero velocity ($\boldsymbol{V} = 0$), constant enthalpy ($h = \mu_1$), and a magnetic field satisfying

$$\nabla \times \boldsymbol{B}^* - \mu_2 \boldsymbol{B}^* - \mu_4 \check{\boldsymbol{B}}^* = 0, \tag{6.18}$$

where $\check{\boldsymbol{B}}^*$ is the hyperfunction stemming from the resonant singularity. Noticing that $\nabla \times \boldsymbol{B}^*$ is the current, and writing (6.18) as $\nabla \times \boldsymbol{B}^* = \mu_2 \boldsymbol{B}^* + \mu_4 \check{\boldsymbol{B}}^*$, the first term on the right-hand side is the 'force-free current' in the aforementioned Beltrami equilibrium, and the second term is the singular sheet current that is characteristic of the tearing-mode equilibrium.

Note, as discussed above, we can unfreeze the singular cross helicity C_4^* by making a canonical pair with an angle variable (see section 2.4).

7. Summary and conclusions

By embedding a Poisson manifold of a noncanonical Hamiltonian system into a higher-dimensional phase space, we can delineate topological structures within a simpler picture. For example, we showed that the topological constraint on magnetic field lines in an ideal plasma, Alfvén's magnetic frozen-in law (or vortex lines in a neutral fluid, Kelvin's circulation law), is not described by a foliation of the Poisson manifold because these constraints are not *integrable* to define Casimir leaves. However, by introducing a *mock field* and embedding the MHD system in a higher-dimensional phase space, we found that the local magnetic fluxes are represented as Casimir elements (cross helicities coupling the magnetic field and the mock field). We have also elucidated the underlying Banach algebra describing the Poincaré duality of the mock field and chains determining the local flux (or circulation).

The representation of a topological constraint by a Casimir element has an immense advantage in studying the global structure of the Poisson manifold. This is especially true for studying *singularities* on the manifold (where the rank of the Poisson operator changes) where different leaves intersect or new leaf bifurcations take place. For a Poisson operator that is a differential operator, the singularity in the phase space (function space) is related to the singularity in the base space of the operator, which yields singular (hyperfunction) solutions as kernel elements. In the example of ideal MHD, the *resonance singularity* yielded a current-sheet solution, and its integral defines a singular Casimir element, by which a *tearing-mode* equilibrium bifurcates (in the picture of ideal MHD, a tearing mode is stationary because of the flux constraint).

The mock field was lifted into a physical field by a Hamiltonian that includes it, while it is initially a mathematical artifact introduced to describe the Poisson manifold in the higher-dimensional space. Interpreting a Casimir element as an *adiabatic invariant* associated with a hidden 'microscopic' angle variable, we extended the phase space by adding the angle variable to the original noncanonical system. Then, the constancy of the Casimir element was no longer due to the 'topological defect' (non-trivial kernel) of the Poisson operator, but due to the symmetry of the Hamiltonian (the newly added angle variable is, of course, ignorable in the Hamiltonian). We then unfroze the Casimir element and allowed it to be dynamic by perturbing the Hamiltonian with a term dependent on the added angle variable.

As an explicit application of the formalism, consider the following picture of the tearing-mode instability in a plasma. A tearing mode can be formulated as an equilibrium point on a helical-flux Casimir leaf (Yoshida and Dewar 2012). As long as the helical-flux is constrained, the tearing-mode cannot grow. Upon introducing a perturbation to change the helical flux, as well as to 'dissipate' the energy, an unstable tearing mode can, then, be formulated as a negative-energy perturbation that can grow by diminishing the energy. In this picture, the negative energy is absorbed by an 'external system' through the pathway introduced by the new angle variable; the extended phase space of the canonized Hamiltonian system includes this 'external system' so that the total energy remains conserved.

We envision that many dissipation driven instabilities in fluids and plasma systems can be cast into this basic geometric picture.

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Appendix A. Harmonic field and cohomology

We denote by $L^2(\Omega)$ the Hilbert space of Lebesgue-measurable, square-integrable real vector functions on Ω, which is endowed with the standard inner product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle = \int_\Omega \mathrm{d}^3x\, \boldsymbol{a} \cdot \boldsymbol{b}$ and norm $\| \boldsymbol{a} \| = \langle \boldsymbol{a}, \boldsymbol{a} \rangle^{1/2}$. We use the same notation for the L^2-norm and inner product, regardless of the dimensions of independent and dependent variables, and we also use the standard notation for Sobolev spaces.

To separate the fixed degrees of freedom pertinent to the fluxes, we invoke the Hodge–Kodaira decomposition, with the definitions

$$L^2_\sigma(\Omega) = \left\{ \boldsymbol{u} \in L^2(\Omega);\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \right\}, \tag{1.1a}$$

$$L^2_\Sigma(\Omega) = \left\{ \boldsymbol{u} \in L^2(\Omega);\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0,\ \Phi_\ell(\boldsymbol{u}) = 0 (\ \forall \ell) \right\}, \tag{1.1b}$$

$$L^2_{\mathrm{H}}(\Omega) = \left\{ \boldsymbol{u} \in L^2(\Omega);\ \nabla \times \boldsymbol{u} = 0,\ \nabla \cdot \boldsymbol{u} = 0,\ \boldsymbol{n} \cdot \boldsymbol{u} = 0 \right\}. \tag{1.1c}$$

The dimension of $L^2_{\mathrm{H}}(\Omega)$, the space of *harmonic fields* (or De Rham cohomologies), is equal to the genus m of Ω and $L^2_{\mathrm{H}}(\Omega)$ is spanned by gradients of *angle variables* θ_ℓ ($\ell = 1, \cdots, m$) such that

$$\nabla\theta_\ell \in L^2_{\mathrm{H}}(\Omega), \quad [\![\theta_\ell]\!]_{\Sigma_\ell} = \theta_\ell\Big|_{\Sigma_\ell^+} - \theta_\ell\Big|_{\Sigma_\ell^-} = 1, \tag{1.2}$$

where $\Sigma_\ell^\pm$ denote both sides of Σ_ℓ. For calculational convenience we have normalized the angle by 2π.

We can now state the orthogonal *Hodge–Kodaira decomposition*:

$$L^2_\sigma(\Omega) = L^2_\Sigma(\Omega) \oplus L^2_{\mathrm{H}}(\Omega). \tag{1.3}$$

If $\boldsymbol{B} \in L^2_\sigma(\Omega)$ is a magnetic field, it can be decomposed into the fixed 'vacuum' field $\boldsymbol{B}_{\mathrm{H}} \in L^2_{\mathrm{H}}(\Omega)$ (which carries the given fluxes $\Phi_1, \cdots, \Phi_m$) and a residual component $\boldsymbol{B}_\Sigma \in L^2_\Sigma(\Omega)$ driven by currents within the volume Ω.

The components $\boldsymbol{B}_\Sigma$ and $\boldsymbol{B}_{\mathrm{H}}$ can be represented uniquely, up to arbitrary constants, respectively, by a vector potential $\boldsymbol{A}_\Sigma \in L^2_\Sigma(\Omega)$ and a (multi-valued) scalar potential $\sum_\ell j_\ell \theta_\ell$, where the 'periods' j_ℓ give the loop integrals of $\boldsymbol{B}_{\mathrm{H}}$ through the handles of Ω, so that, by Ampère's law, the periods are proportional to currents external to Ω.

References

Andreussi T, Morrison P J and Pegoraro F 2012 Hamiltonian magnetohydrodynamics: symmetric formulation, Casimir invariants, and equilibrium variational principles *Phys. Plasmas* **19** 052102

Arnold V I and Khesin B A 1998 *Topological Methods in Hydrodynamics* (New York: Springer)

Fukumoto Y 2008 A unified view of topological invariants of fluid flows *Topologica* **1** 003

Fukumoto Y and Sakuma H 2013 A unified view of topological invariants of barotropic and baroclinic fluids and their application to formal stability analysis of three-dimensional ideal gas flows *Procedia IUTAM* **7** 213–22

Furth H P 1963 Hydromagnetic instabilities due to finite resistivity *Propagation and Instabilities in Plasmas* ed W T Futterman (Stanford, CA: Stanford University Press) pp 87–102

Furth H P, Killeen J and Rosenbluth M N 1963 Finite resistivity instabilities of a sheet pinch *Phys. Fluids* **6** 459–84

Hameiri E 2004 The complete set of Casimir constants of the motion in magnetohydrodynamics *Phys. Plasmas* **11** 3423–31

Hazeltine R D, Holm D D, Marsden J E and Morrison P J 1984 Generalized Poisson brackets and nonlinear liapunov stability—application to reduced MHD *Int. Conf. on Plasma Physics Proc. 1* ed M Q Tran and M L Sawley (Lausanne: Ecole Polytechnique Federale de Lausanne) p 203

Kruskal M D and Oberman C 1958 On the stability of plasma in static equilibrium *Phys. Fluids* **1** 275–80

Marsden J E and Morrison P J 1984 Noncanonical Hamiltonian field theory and reduced MHD *Contemp. Math.* **28** 133–50

Morrison P J and Greene J M 1980 Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics *Phys. Rev. Lett.* **45** 790–4

Morrison P J and Greene J M 1982 Noncanonical Hamiltonian density formulation of hydrodynamics and ideal magnetohydrodynamics *Phys. Rev. Lett.* **48** 7569

Morrison P J 1982 Poisson brackets for fluids and plasmas *AIP Conf. Proc.* **88** 13–46

Morrison P J and Hazeltine R D 1984 Hamiltonian formulation of reduced magnetohydrodynamics *Phys. Fluids* **27** 886–97

Morrison P J and Eliezer S 1986 Spontaneous symmetry breaking and neutral stability in the noncanonical Hamiltonian formalism *Phys. Rev.* A **33** 4205–14

Morrison P J 1987 Variational principle and stability of nonmonotonic Vlasov–Poisson equilibria *Z. Naturforsch.* A **42** 1115–23

Morrison P J 1998 Hamiltonian description of the ideal fluid *Rev. Mod. Phys.* **70** 467–521

Padhye N and Morrison P J 1996 Fluid element relabeling symmetry *Phys. Lett.* A **219** 287–92

Thiffeault J-L and Morrison P J 1998 Invariants and labels in Lie–Poisson systems *Ann. New York Acad. Sci.* **867** 109–19

Thiffeault J-L and Morrison P J 2000 Classification of Casimir invariants of Lie–Poisson brackets *Physica* D **136** 205–44

White R B 1983 Resistive instabilities and field-line reconnection *Basic Plasma Physics 1* ed A A Galeev and R N Sudan (Amsterdam: North-Holland) pp 611–76

Yoshida Z and Dewar D 2012 Helical bifurcation and tearing mode in a plasma—a description based on Casimir foliation *J. Phys. A: Math. Theor.* **45** 365503

Yoshida Z, Saitoh H, Yano Y, Mikami H, Kasaoka N, Sakamoto W, Morikawa J, Furukawa M and Mahajan S M 2013 Self-organized confinement by magnetic dipole: recent results from RT-1 and theoretical modeling *Plasma Phys. Control. Fusion* **55** 014018

Yoshida Z, Morrison P and Dobarro F 2014a Singular Casimir elements of the Euler equation and equilibrium points *J. Math. Fluid Mech.* **16** 41–57

Yoshida Z and Morrison P J 2014b Unfreezing Casimir invariants—singular perturbation giving rise to forbidden instabilities *Nonlinear Physical Systems—Spectral Analysis, Stability and Bifurcations* (London: ISTE) pp 401–25

Yoshida Z, Kawazura Y and Yokoyama T 2014c Relativistic helicity and link in Minkowski space-time *J. Math. Phys.* **55** 043101