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Variational necessary and sufficient stability conditions for inviscid shear flow

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A necessary and sufficient condition for linear stability of inviscid parallel shear flow is formulated by developing a novel variational principle, where the velocity profile is assumed to be monotonic and analytic. It is shown that unstable eigenvalues of Rayleigh's equation (which is a non-self-adjoint eigenvalue problem) can be associated with positive eigenvalues of a certain self-adjoint operator. The stability is therefore determined by maximizing a quadratic form, which is theoretically and numerically more tractable than directly solving Rayleigh's equation. This variational stability criterion is based on the understanding of Kreǐn signature for continuous spectra and is applicable to other stability problems of infinite-dimensional Hamiltonian systems.

1. Introduction

In this paper, ideas from the theory of Hamiltonian systems are used to obtain both necessary and sufficient stability conditions by a variational procedure. The proposed procedure is of general utility, but the treatment here will be confined to plane parallel inviscid shear flow (e.g. [1]). In this section, we give an overview of the underlying basis for the procedure in terms of a finite-dimensional Hamiltonian framework, and then place the present contribution in the context of the many previous results for shear flow.

For some Hamiltonian systems, the sign of the curvature of the potential energy function provides a necessary and sufficient condition for stability. This is referred to as Lagrange's theorem, which is the crux of many fluid and plasma stability results including the

'δW' energy principle of ideal magnetohydrodynamics (MHD) [2]. For a more general class of Hamiltonian systems, definiteness of the Hamiltonian Hessian matrix evaluated at the equilibrium point of interest provides only a sufficient condition for stability. This is sometimes referred to as Dirichlet's theorem, which is the crux of many sufficient conditions for stability in the fluid and plasma literature. The essential reason that Dirichlet's theorem does not provide a necessary condition for stability is the possible existence of negative energy modes. Negative energy modes are modes of undamped oscillation, for which the Hamiltonian decreases when the mode is excited, i.e. the second variation of the Hamiltonian evaluated on the mode is negative. For stable (non-degenerate) Hamiltonian systems of n degrees-of-freedom the linear dynamics can be brought by a canonical transformation into the following normal form:

$$H = \sum_{\alpha=1}^{n} \frac{\sigma_\alpha \omega_\alpha}{2}(p_\alpha^2 + q_\alpha^2), \tag{1.1}$$

where $(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n)$ are the canonically conjugate coordinates, ω_α are positive real numbers representing the mode frequencies, and $\sigma_\alpha \in \{\pm 1\}$ are the signatures of the mode, often called Kreǐn signature, with $+1$ and -1 corresponding to positive and negative energy modes, respectively. Evidently, systems with both positive and negative energy modes are linearly stable but do not have a definite Hessian matrix. (See [3] for review.)

An advantage afforded by Lagrange's criterion over Dirichlet's is the powerful Rayleigh–Ritz variational method [4], which underlies the MHD and other energy principles. With this method, one needs to only produce a trial function that makes the Rayleigh quotient negative in order to show instability and, also in this way, threshold parameter values for the transition to instability can be determined. When this method is applicable, it is of great utility because linear stability conditions for interesting fluid and plasma systems are generally difficult to derive theoretically. However, it only applies to a restricted class of Hamiltonian systems for which the eigenvalue problem is self-adjoint, i.e. systems with steady-state bifurcations to instability through zero frequency that have pure exponential growth upon transition, and it is known that systems with shear flow are not self-adjoint and have instead Kreǐn bifurcations [5] to overstability, i.e. unstable eigenvalues with both non-zero real and imaginary parts. Such bifurcations are often called Hamiltonian–Hopf bifurcations [5–8] and can be viewed as a resonance between positive and negative energy modes leading to instability.

Thus, we are led to re-examine Dirichlet's principle and seek an alteration that affords the utility of the Rayleigh–Ritz variational method for investigation of Kreǐn bifurcations. For finite-dimensional Hamiltonian system written in the normal form of (1.1), it is evident that $I_\alpha = p_\alpha^2 + q_\alpha^2$ is a constant of motion for each α. From these constants of motion, one can construct a constant of motion with definite Hessian simply by flipping the signs of the signature in the normal form Hamiltonian. Unfortunately, *a priori* knowledge of the existence of such a definite constant of motion is generally not at hand, and one must actually solve the eigenvalue problem in order to be informed of its existence. This is the essential reason Dirichlet's theorem does not give both necessary and sufficient conditions for stability and a Rayleigh-like variational method is not at hand.

However, there are two related discoveries that we can exploit to circumvent this deficiency for problems with continuous spectra, such as those related to the Vlasov equation, MHD and the plane shear flow problem considered here. The first is the infinite sequence of constants of motion discovered in [9], which were elaborated on in [10] and used in the present context in [11]. The second is the discovery of a Kreǐn-like signature for the continuous spectra of Vlasov–Poisson equilibria in [12,13], which was applied to plane shear flow in [14] and extended to a large class of systems in [15–17]. These constants of motion in conjunction with the definition of signature allow the construction of a quadratic form, which we will call Q, the variation of which can be used in a manner akin Rayleigh's principle for ascertaining necessary and sufficient conditions for stability. A version of the quadratic form Q was previously given in [11], but it was not used to obtain sufficient conditions for instability. We note in passing that the discovery of signature for the continuous spectrum has also led to rigorous Kreĭn-like theorems [18–20], where discrete eigenvalues emerge from the continuous spectrum, termed Continuum Hamiltonian Hopf bifurcations. (See also [21,22] on infinite-dimensional Hamiltonian systems and many other contributions in the recent book [23].)

There have been many significant contributions to the classical plane parallel shear flow problem; thus, it is important to put our contribution into perspective, which we do here. The most famous condition is Rayleigh's criterion [24] that stipulates the existence of an inflection point in the velocity profile is necessary for instability, a criterion that was improved by Fjørtoft [25]. These criteria were obtained by direct manipulation of Rayleigh's equation (see equation (2.4)), which governs linear disturbance about a base shear flow. The first allusion to Hamiltonian structure for this system appears in the works of Arnold [26–28], who obtained more general sufficient conditions for stability by making use of additional invariants. This idea was anticipated in the plasma physics literature [29], where the additional invariants were referred to as generalized entropies; today, the generalized entropies are referred to as Casimir invariants and the general procedure is called the energy-Casimir method (e.g. [3,30–32]). In [14], it was shown explicitly that all of the above criteria amount to a version of Dirichlet's theorem for this infinite-dimensional Hamiltonian system, where it was also shown how to explicitly map the system into the infinite-dimensional version of the normal form of (1.1). In this way, a signature for the continuous spectrum was first defined for this system, by paralleling the analogous procedure for the Vlasov–Poisson system [12,13]. Arnold also introduced an important kind of constrained variation he termed isovortical perturbations (e.g. [33]), which are a special case of the dynamically accessible variations of [34,35] that are generated by Poisson brackets [3,36]. Isovortical perturbations together with a more general Dirichlet-like sufficient condition for shear flow due to Barston [11] play important roles for obtaining the results of the present paper. The general idea of these earlier works is to obtain improved sufficient conditions for stability by constructing a suitable quadratic form (or the Lyapunov function) with the aid of additional invariants. This idea is powerful especially for integrable systems (e.g. [37,38]), in which abundant constants of motion are available not only for linearized but also for nonlinear dynamics.

We also note that prior to our variational approach, necessary and sufficient stability conditions were obtained for certain classes of shear flows using two other non-Hamiltonian approaches. One is the perturbation expansion around a neutrally stable eigenmode, which was pioneered by Tollmien [39] and developed by many authors [40–43], and the other is analysis based on the Nyquist method [44,45] (applied to MHD in [46]). These two approaches, however, require detailed probing of Rayleigh's equation to obtain information under specific conditions. Our variational approach is not only consistent with these earlier results, but also advantageous in that we do not have to solve Rayleigh's equation in a rigorous manner. Namely, we can prove the instability by simply finding a test function (in the appropriate function space) that makes our quadratic form Q positive. This is useful because explicit solutions for Rayleigh's equation are generally not available for a given velocity profile. Moreover, in numerical calculation, one can obtain stability boundaries more efficiently from this variational problem (i.e. the maximization of Q), compared to solving Rayleigh's equation. We emphasize that our approach is not limited to shear flow with rather simple velocity profiles, but the same idea is applicable to various fluid and plasma stability problems to which it can be of practical use.

Our paper is organized as follows. In §2, Rayleigh's equation is first introduced, and in §3, the notion of isovortical variation is reviewed. Here, we describe the quadratic form Q, which provides the necessary and sufficient conditions if the velocity profile satisfies the assumptions of analyticity and monotonicity. In particular, we present the main theorem of this work (theorem 3.1), in which the quadratic form Q is given explicitly. Then, in §4, the proof of the main theorem is given. Here, we first focus on restricting perturbations to the appropriate function space, and then perform the spectral decomposition in a rigorous manner, which largely reproduces the well-known spectral properties of Rayleigh's equation (e.g. [47]). Next, we calculate the signature of Q by applying techniques [14,16,17] for the action–angle representation of continuous spectrum, where the positive signature of Q indeed predicts the existence of unstable eigenvalues. Finally, in §4, we show that the function space (the search space on which Q should be maximized) can be extended to a larger one, which is actually beneficial for solving the variational problem more effectively. Section 5 contains a demonstration that our variational criterion reproduces the earlier results of the Nyquist method [44,45] and the perturbation analysis of the neutral modes [39–43], while §6, contains several numerical examples that demonstrate of our theorem. We summarize in §7.

2. Rayleigh equation

We consider the linear stability of inviscid parallel shear flow $\boldsymbol{U} = (0, U(x))$ on a domain $(x, y) \in [-L, L] \times [-\infty, \infty]$ bounded by two walls at $x = \pm L$, where the flow is assumed to be incompressible and two-dimensional. By introducing the z-component of the vorticity disturbance as $w(x,t)\,\mathrm{e}^{\mathrm{i}ky} + \mathrm{c.c.}$ for a wavenumber $k > 0$, the linearized vorticity equation is written as follows:

$$\begin{aligned} \mathrm{i}\partial_t w &= kUw + kU''\mathcal{G}w \\ &=: k\mathcal{L}w, \end{aligned} \tag{2.1}$$

where the prime ($'$) indicates the x derivative, and the convolution operator $\mathcal{G}$ is defined by

$$(\mathcal{G}w)(x,t) = \int_{-L}^{L} g(x,s)w(s,t)\,\mathrm{d}s, \tag{2.2}$$

with a kernel,

$$g(x,s) = \begin{cases} -\dfrac{\sinh k(s-L)\sinh k(x+L)}{k\sinh 2kL} & x < s \\ -\dfrac{\sinh k(s+L)\sinh k(x-L)}{k\sinh 2kL} & s < x. \end{cases} \tag{2.3}$$

The stream function $\phi(x,t)$ is therefore given by $\phi = \mathcal{G}w$ or $w = \mathcal{G}^{-1}\phi = -\phi'' + k^2\phi$. By assuming an exponential behaviour $\phi(x,t) = \hat{\phi}(x)\,\mathrm{e}^{-\mathrm{i}\omega t}$ with a complex frequency $\omega \in \mathbb{C}$, the eigenvalue problem for (2.1) is known as Rayleigh's equation [24]

$$(c-U)(\hat{\phi}'' - k^2\hat{\phi}) + U''\hat{\phi} = 0 \tag{2.4}$$

and

$$\hat{\phi}(-L) = \hat{\phi}(L) = 0, \tag{2.5}$$

where $c = \omega/k$ is a complex phase speed. If this equation has a non-trivial solution $\hat{\phi}$ for c with a positive imaginary part, $\operatorname{Im} c > 0$, the linearized system (2.1) is spectrally unstable due to an exponentially growing eigenmode.

3. Variational stability criterion

Hamiltonian structure of the linearized vorticity equation (2.1) is highly related to its adjoint equation for $\xi(x,t)$ [14,16,33]

$$\begin{aligned} \mathrm{i}\partial_t \xi &= kU\xi + k\mathcal{G}(U''\xi) \\ &=: k\mathcal{L}^*\xi, \end{aligned} \tag{3.1}$$

where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$ with respect to the inner product,

$$\langle \bar{\xi}, \eta\rangle = \int_{-L}^{L} \overline{\xi(x)}\eta(x)\,\mathrm{d}x \quad \text{for } \xi,\ \eta \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2. \tag{3.2}$$

Here, $\bar{\xi}$ denotes the complex conjugate of ξ, and we consider the function space for disturbances w and ξ to be the complex Hilbert space $\mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ on $[-L, L]$ that is defined by the norm $\|\xi\|^2_{\mathbf{L}^2+\mathrm{i}\mathbf{L}^2} = \langle \bar{\xi}, \xi\rangle$.

Since the relation $U''\mathcal{L}^* = \mathcal{L}U''$ holds, $w = -U''\xi$ is found to be a solution of (2.1) if ξ is a solution of (3.1). This solution must vanish at positions where $U'' = 0$, and hence it does not span the whole space $\mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ when $U'' = 0$ somewhere. This class of perturbations belonging to the range of U'' is said to be isovortical because the vorticity disturbance (w) is induced by a displacement (ξ) of the fluid while preserving the conservation law of circulation [33] (see also appendix A). In this manner, Arnold [33] derived a constant of motion

$$\delta^2 H = \int_{-L}^{L} \bar{\xi} U''[U\xi + \mathcal{G}(U''\xi)]\,\mathrm{d}x. \tag{3.3}$$

Arnold showed that this is the second variation of the energy with respect to the isovortical variation, while in [14] it was shown that this quantity is in fact the Hamiltonian for the linear Hamiltonian dynamics and there the diagonalizing transformation to action–angle variables was first obtained. In a frame moving at a velocity U_*, Arnold replaced U by $U - U_*$ in $\delta^2 H$ to obtain

$$\begin{aligned} \delta^2 H_* &= \int_{-L}^{L} \bar{\xi} U''[(U - U_*)\xi + \mathcal{G}(U''\xi)]\,\mathrm{d}x \\ &= \int_{-L}^{L} \bar{w}\left(\frac{U - U_*}{U''} + \mathcal{G}\right) w\,\mathrm{d}x, \end{aligned} \tag{3.4}$$

which is also a constant of motion, while the last expression may not be well-defined when U'' becomes zero and w is not isovortical. In [14], it was shown explicitly that $\delta^2 H_*$ is in fact the second variation of the full Hamiltonian in the inertial frame boosted by velocity U_* by adding the appropriate total momentum.

Thus, we have a version of Dirichlet's theorem, where the shear flow $U(x)$ is stable in the sense of Lyapunov, if there exists $U_* \in \mathbb{R}$ such that the quadratic form $\delta^2 H_*$ is either positive or negative definite, i.e. $\exists \epsilon > 0$ such that $\delta^2 H_* \geq \epsilon \langle \bar{w}, w\rangle$ or $-\delta^2 H_* \geq \epsilon \langle \bar{w}, w\rangle$. For example, when $U'' \neq 0$ everywhere, one can make $\delta^2 H_*$ positive definite by choosing U_* such that $(U - U_*)/U'' > 0$ holds everywhere, which reproduces the Rayleigh criterion [24]. When $U(x)$ has only one inflection point x_{I} (i.e. $U''(x_{\mathrm{I}}) = 0$), the choice $U_* = U_{\mathrm{I}} := U(x_{\mathrm{I}})$ is made by Arnold. Then, $\delta^2 H_*$ is positive definite if $(U - U_{\mathrm{I}})/U'' > 0$ holds everywhere, in agreement with the Fjørtoft criterion [25]. These facts imply that this variational criterion of Arnold [26–28] applies to a larger class of flow profiles than Rayleigh–Fjørtoft's stability theorem. However, all these criteria, including a generalization by Barston [11], are still sufficient conditions for stability and, hence, are indeterminate when $\delta^2 H_*$ is indefinite, as discussed in §1 there could be negative energy modes.

In this work, we obtain an improved variational criterion, but this requires introducing the following assumptions on $U(x)$.

Assumption.

(A 1) *$U(x)$ is an analytic (i.e. regular) and bounded function on* $[-L, L]$.
(A 2) *$U(x)$ is strictly monotonic* [*i.e. $U'(x) \neq 0$ for all x*] *and, if $U''(x_{\mathrm{I}}) = 0$ at $x = x_{\mathrm{I}}$, then $U'''(x_{\mathrm{I}}) \neq 0$.*

The last statement implies that the inflection point x_{I} must be a simple zero of $U''(x)$ and the sign of $U''(x)$ must change at $x = x_{\mathrm{I}}$. We expect that it is not difficult to relax these restrictions on $U(x)$ *except for* the monotonicity. To simplify our mathematical arguments, we will not pursue generalization in the present work, but we do remark upon this point in our summary of §7.

Our main result is that a necessary and sufficient condition for spectral stability is attained by the following variational criterion.

Theorem 3.1. *Let $U(x)$ satisfy* (A 1) *and* (A 2). *Denote the inflection points of U by $x_{\mathrm{I}n}$, $n = 1, 2, \ldots, N$, and define a quadratic form $Q = \langle \xi, \mathcal{H}\xi\rangle$ as*

$$Q = \nu \int_{-L}^{L} \xi \prod_{n=1}^{N} [U - U_{\mathrm{I}n} + U''\mathcal{G}] U''\xi\,\mathrm{d}x, \tag{3.5}$$

where $U_{\mathrm{In}} = U(x_{\mathrm{In}})$ and either $\nu = 1$ or $\nu = -1$ is chosen such that

$$\nu U'' \prod_{n=1}^{N} (U - U_{\mathrm{In}}) \leq 0 \tag{3.6}$$

holds for all $x \in [-L, L]$. Equation (2.1) is spectrally stable if and only if

$$Q = \langle \xi, \mathcal{H}\xi \rangle \leq 0 \quad \textit{for all } \xi \in \mathbf{L}^2. \tag{3.7}$$

In this theorem, we have introduced Q on the real Hilbert space $\mathbf{L}^2$ defined by the norm $\|\xi\|^2_{\mathbf{L}^2} = \langle \xi, \xi \rangle$, which indicates that in practice the search space of this variational criterion is the half of $\mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ since $\mathcal{H}$ is a real self-adjoint operator. Actually, the stability condition (3.7) can be replaced by $Q = \langle \bar{\xi}, \mathcal{H}\xi \rangle \leq 0$ for all $\xi \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$, and we will prove the latter by regarding $\xi \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ as a solution of (3.1).

This $Q = \langle \bar{\xi}, \mathcal{H}\xi \rangle$ is equivalent to the constant of motion derived by Barston [11] (except for the coefficient ν), and $Q = -\nu\delta^2 H_*$ with $U_* = U_{\mathrm{I}}$ for the case of single inflection point. Hence, our theorem claims that Arnold–Barston's stability criteria (Dirichlet sufficient stability conditions) are in fact necessary and sufficient when $U(x)$ satisfies (A 1) and (A 2).

We remark that $Q = \langle \bar{\xi}, \mathcal{H}\xi \rangle$ no longer represents the second variation of the energy for the case of multiple infection points. Actually, it belongs to the class of infinite number of constants of motion introduced in [9–11].

Proposition 3.2. *Let $f(c)$ be any real polynomial of $c \in \mathbb{R}$. Then,*

$$Q_f = \int_{-L}^{L} \bar{\xi} U'' f(\mathcal{L}^*)\xi \,\mathrm{d}x = \int_{-L}^{L} \bar{\xi} f(\mathcal{L})(U''\xi)\,\mathrm{d}x \in \mathbb{R} \tag{3.8}$$

is a constant of motion for equation (3.1).

Proof. Using $U''\mathcal{L}^* = \mathcal{L}U''$ and $\mathcal{L}^* f(\mathcal{L}^*) = f(\mathcal{L}^*)\mathcal{L}^*$, we can directly show that Q_f is real and $\mathrm{d}Q_f/\mathrm{d}t = 0$. ■

Therefore, we have specifically chosen $f(c) = \nu \prod_{n=1}^{N} (c - U_{\mathrm{In}})$ to generate Q of theorem 3.1.

The proof of theorem 3.1 is given in the next section and our strategy is as follows. First, we reduce the function space $\mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ to a smaller one that will be denoted by $X + \mathrm{i}X$, to which unstable eigenfunctions must belong. Then, we decompose the spectrum $\sigma \subset \mathbb{C}$ of the operator $k\mathcal{L}^*$ into the neutrally stable part $\sigma_{\mathrm{c}} \subset \mathbb{R}$ (which is mostly a continuous spectrum) and the remaining part $\sigma \backslash \sigma_{\mathrm{c}} \subset \mathbb{C} \backslash \mathbb{R}$ (which is a set of pairs of growing and damping eigenvalues). By proving that $Q \leq 0$ for all the neutrally stable disturbance ξ belonging to σ_{c}, we will claim that $Q > 0$ for some $\xi \in X$ indicates the existence of at least one unstable eigenvalue, $\omega \in \sigma \backslash \sigma_{\mathrm{c}}$ that has a growth rate $\mathrm{Im}\,\omega > 0$.

4. Proof of theorem 3.1

(a) Reduction to isovortical disturbance

For the purpose of seeking unstable eigenmodes, we restrict the function space of disturbance to $X + \mathrm{i}X$, where

$$X = \mathbf{H}^1_0 \cap \mathbf{H}^2. \tag{4.1}$$

As usual, we denote by $\mathbf{H}^n$ the real Sobolev space on $[-L, L]$;

$$\mathbf{H}^n = \left\{ \xi \in \mathbf{L}^2 \;\middle|\; \sum_{j \leq n} \|\partial_x^j \xi\|_{\mathbf{L}^2} < \infty \right\} \tag{4.2}$$

and $\mathbf{H}^1_0$ denotes the subspace of $\mathbf{H}^1$ in which the boundary conditions, $\xi(-L) = \xi(L) = 0$, are imposed on $\xi \in \mathbf{H}^1_0$. The restriction from $\mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ to $X + \mathrm{i}X$ is feasible when $U(x)$ is a sufficiently smooth function. In this work, we simply assume (A 1) is sufficient for the following:

Proposition 4.1. *Let $U(x)$ satisfy (A 1). Given the initial condition $\xi(x,0)=\xi_0(x)\in X+\mathrm{i}X$, the solution $\xi(x,t)$ of (3.1) belongs to $X+\mathrm{i}X$ for all t. Moreover, $w=-U''\xi\in X+\mathrm{i}X$ is a solution of (2.1).*

Proof. By noting that $\mathcal{G}:\mathbf{L}^2\to X$ is one to one and onto, we find that $\mathcal{L}^*$ is a bounded operator on $X+\mathrm{i}X$ and, hence, the solution $\xi=\mathrm{e}^{-\mathrm{i}k\mathcal{L}^*t}\xi_0$ belongs to $X+\mathrm{i}X$ for all t. Using the property $U''\mathcal{L}^*=\mathcal{L}U''$, it is obvious that $w=-U''\xi$ is automatically a solution of (2.1) and also belongs to $X+\mathrm{i}X$. ■

When U'' vanishes somewhere on $[-L,L]$, the function space of $w=-U''\xi$ is further restricted to the range of U'' (i.e. the isovortical disturbance). We can find that all unstable eigenfunctions must belong to this space as follows.

Proposition 4.2. *Let $U(x)$ satisfy (A 1). Equation (2.1) is spectrally stable if and only if the adjoint equation (3.1) for $\xi\in X+\mathrm{i}X$ is spectrally stable.*

Proof. If $c\in\mathbb{C}$ and $\hat{w}=-\hat{\phi}''+k^2\hat{\phi}\in\mathbf{L}^2+\mathrm{i}\mathbf{L}^2$ satisfy Rayleigh's equation with a growth rate $\operatorname{Im}c>0$, then $(c-U)\neq 0$ holds everywhere and

$$\hat{\xi}=-\frac{1}{c-U}\mathcal{G}\hat{w}\in X+\mathrm{i}X \tag{4.3}$$

is found to be an eigenfunction of the adjoint equation (3.1) with the same eigenvalue c. Hence, the adjoint equation on $X+\mathrm{i}X$ is also spectrally unstable.

Conversely, if c and $\hat{\xi}\in X+\mathrm{i}X$ satisfy the adjoint eigenvalue problem with $\operatorname{Im}c>0$, then $U''\hat{\xi}$ is not identically zero and $\hat{w}=-U''\hat{\xi}$ satisfies the Rayleigh equation with the same c. ■

(b) Spectral decomposition

Next, we investigate the spectrum $\sigma\subset\mathbb{C}$ of the operator $k\mathcal{L}^*$. For a given initial condition $\xi(x,0)=\xi_0(x)\in X+\mathrm{i}X$, let $\Xi(x,\Omega)\in X+\mathrm{i}X$ be the solution of

$$(\Omega-k\mathcal{L}^*)\Xi(x,\Omega)=\xi_0(x), \tag{4.4}$$

for $\Omega\in\mathbb{C}\backslash\sigma$. Then, the solution of (3.1) is formally represented by the Dunford integral (or the inverse Laplace transform)

$$\xi(x,t)=\frac{1}{2\pi\mathrm{i}}\oint_{\Gamma(\sigma)}\Xi(x,\Omega)\,\mathrm{e}^{-\mathrm{i}\Omega t}\,\mathrm{d}\Omega, \tag{4.5}$$

where $\Gamma(\sigma)$ is a path of integration that encloses all the spectrum $\sigma\subset\mathbb{C}$ of $k\mathcal{L}^*$ counterclockwise (as shown in figure 1). In terms of $\Phi(x,\Omega)=-(\Omega/k-U)\Xi(x,\Omega)$, equation (4.4) is transformed into

$$\mathcal{E}(\Omega)\Phi(x,\Omega)=\frac{1}{k}(\xi_0''-k^2\xi_0) \tag{4.6}$$

and

$$\Phi(-L,\Omega)=\Phi(L,\Omega)=0, \tag{4.7}$$

where

$$\mathcal{E}(\Omega)=-\frac{\partial^2}{\partial x^2}+k^2-\frac{kU''}{\Omega-kU}. \tag{4.8}$$

Suppose that we have solved

$$\mathcal{E}(\Omega)\Phi_<(x,\Omega)=0,\quad \Phi_<(-L,\Omega)=0,\quad \Phi_<'(-L,\Omega)=1 \tag{4.9}$$

and

$$\mathcal{E}(\Omega)\Phi_>(x,\Omega)=0,\quad \Phi_>(L,\Omega)=0,\quad \Phi_>'(L,\Omega)=-1, \tag{4.10}$$

to obtain two linearly independent solutions $\Phi_<(x,\Omega)$ and $\Phi_>(x,\Omega)$. Then, by using the method of Green's function, the solution of (4.6) and (4.7) can be expressed as

$$\Phi(x,\Omega)=\frac{1}{W(\Omega)}\int_{-L}^{L}\Phi_{\mathrm{G}}(x,s,\Omega)\frac{1}{k}[\xi_0''(s)-k^2\xi_0(s)]\,\mathrm{d}s, \tag{4.11}$$

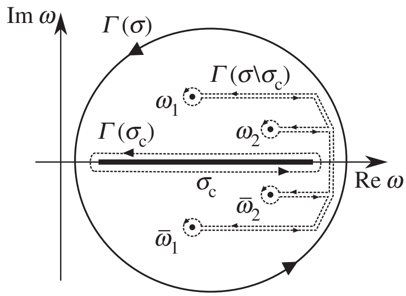

Figure 1. Schematic view of contour integral $\Gamma(\sigma) = \Gamma(\sigma_c) \cup \Gamma(\sigma \setminus \sigma_c)$ in the case of $\sigma \setminus \sigma_c = \{\omega_1, \omega_2, \bar{\omega}_1, \bar{\omega}_2\}$.

where

$$\Phi_G(x, s, \Omega) = \Phi_>(s, \Omega)\Phi_<(x, \Omega)Y(s - x) + \Phi_<(s, \Omega)\Phi_>(x, \Omega)Y(x - s), \tag{4.12}$$

with $Y(x)$ being the Heaviside function, and

$$\begin{aligned} W(\Omega) &= -\Phi_<(x, \Omega)\Phi'_>(x, \Omega) + \Phi'_<(x, \Omega)\Phi_>(x, \Omega) \\ &= \Phi_>(-L, \Omega) \\ &= \Phi_<(L, \Omega) \end{aligned} \tag{4.13}$$

is the Wronskian. When Ω avoids the range of kU, denoted by

$$\sigma_c = \{kU(x) \in \mathbb{R} \mid x \in [-L, L]\}, \tag{4.14}$$

the operator $\mathcal{E}(\Omega)$ is non-singular and both $\Phi_<$ and $\Phi_>$ are regular functions of Ω. Therefore, the spectrum σ of the operator $k\mathcal{L}^*$ is composed of a continuous spectrum σ_c [47,48] and some eigenvalues $\omega_j \in \mathbb{C}$, $j = 1, 2, \ldots$ (i.e. point spectra) that satisfy $W(\omega_j) = 0$. Owing to the property $W(\Omega) = \overline{W(\bar{\Omega})}$, the eigenvalues always exist as pairs of growing (ω_j) and damping ($\bar{\omega}_j$) ones when $\operatorname{Im} \omega_j > 0$, a spectral property of Hamiltonian systems (cf. [18–20]). To be more precise, σ_c is *mostly* the continuous spectrum because some real eigenvalues ($\omega_j \in \mathbb{R}$) may exist in it. In the remainder of this subsection, we will ascertain these spectral properties of $k\mathcal{L}^*$, which will be summarized explicitly in proposition 4.6.

For the purpose of showing the existence or non-existence of such eigenvalues, we will frequently use the following lemma.

Lemma 4.3. *Let $U(x)$ satisfy (A1) and (A2). For any $\omega \in \mathbb{R}$, the general solution $\Phi(x, \omega)$ of $\mathcal{E}(\omega)\Phi(x, \omega) = 0$ has at most one zero on each of the following intervals*:

(i) $[-L, L]$ *if there is no critical layer, i.e.* $\nexists \omega \in \mathbb{R}$ *that satisfies* $\omega = kU(x) \forall x \in [-L, L]$,
(ii) $[-L, x_c]$ *and* $[x_c, L]$ *if there is a critical layer* $x_c \in [-L, L]$ *that satisfies* $\omega = kU(x_c)$.

Proof. (i) A consequence of $\mathcal{E}(\omega)\Phi(x, \omega) = 0$ is the following identity (see the Appendix of [45]):

$$\left[\Phi\left(\Phi' - \frac{U'\Phi}{U - c}\right)\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} \left[\left(\Phi' - \frac{U'\Phi}{U - c}\right)^2 + k^2\Phi^2\right] dx, \tag{4.15}$$

which is valid for any solution Φ and subinterval $[x_1, x_2] \subseteq [-L, L]$. This identity follows directly from Rayleigh's equation by multiplying by Φ, manipulating and integrating. If there are two zeros (i.e. x_{z1} and x_{z2}) that satisfy $\Phi(x_{z1}, \omega) = \Phi(x_{z2}, \omega) = 0$, choosing $x_1 = x_{z1}$ and $x_2 = x_{z2}$ implies

$$\int_{x_{z1}}^{x_{z2}} \left[\left(\Phi' - \frac{U'\Phi}{U - c}\right)^2 + k^2\Phi^2\right] dx = 0, \tag{4.16}$$

which requires Φ to be the trivial solution $\Phi \equiv 0$. Thus, any (non-trivial) solution has at most one zero on $[-L, L]$.

(ii) Consider the interval $[x_c, L]$ (the same argument goes for $[-L, x_c]$). The identity (4.15) again holds for $x_c < x_1 < x_2 \le L$ and, hence, there is at most one zero on $(x_c, L]$. In the neighbourhood of x_c, the solution Φ is expressed by a linear combination of the Frobenius series solutions, in which $\Phi(x_c, \omega)$ is always bounded (see (4.21)). If $\Phi(x_c, \omega) \neq 0$, the lemma is automatically proved (although both sides of (4.15) go to infinity as $x_1 \to x_c$). If $\Phi(x_c, \omega) = 0$, the identity (4.15) is still valid for $x_1 = x_c$ and we can prove that there is no other zero (x_{z2}) except for $x_{z1} = x_c$ on $[x_c, L]$, in the same manner as (i). ■

It is well known from Tollmien's argument [39–41] that, as k varies, an unstable eigenvalue $\operatorname{Im} c > 0$ of Rayleigh's equation emerges through a neutrally stable eigenvalue $c = U(x_I) \in \mathbb{R}$ where $U''(x_I) = 0$. In other words, the neutrally stable eigenvalue $c \in \mathbb{R}$ can exist only if $U''(x_c) = 0$ at the corresponding critical layer $x_c = U^{-1}(c)$. Unfortunately, this argument is not always true especially for non-monotonic profiles of $U(x)$ (see [42,43,45] for mathematical justification in the shear flow context and [18,19] for a discussion of $k \neq 0$ bifurcations in the Vlasov context). In this work, we simply assume the monotonicity (A 2) and verify Tollmien's argument as follows.

Proposition 4.4. *Let $U(x)$ satisfy (A 1) and (A 2). Denote the inflection points of U by x_{In}, $n = 1, 2, \ldots, N$ and define $U_{\text{In}} := U(x_{\text{In}})$. Then, the function $W(\omega \pm \mathrm{i}0)$ of $\omega \in \mathbb{R}$ can vanish only at $\omega = kU_{\text{In}}$, $n = 1, 2, \ldots, N$, and moreover*

$$\lim_{\Omega \to \omega \pm \mathrm{i}0} \frac{\prod_{n=1}^{N} (\Omega - kU_{\text{In}})}{W(\Omega)} < \infty. \tag{4.17}$$

Proof. For $\omega \in \mathbb{R} \backslash \sigma_c$, there is no critical layer and, from lemma 4.3, the solution $\Phi_<(x, \omega)$ does not have zero on $[-L, L]$ except for $x = -L$. Hence, $W(\omega) = \Phi_<(L, \omega) \neq 0$.

For $\omega \in \sigma_c$, there is only one critical layer $x_c = U^{-1}(\omega/k)$. Since $U(x)$ is analytic, $\Phi_<(x, \Omega)$ can be expressed by a linear combination of the Frobenius series solutions (so-called Tollmien's inviscid solutions) around x_c. By taking account of the branch cut of the logarithmic function, it is written in the limit $\Omega \to \omega \pm \mathrm{i}0$ as

$$\begin{aligned} \Phi_<(x, \omega \pm \mathrm{i}0) &= C_r(\omega) \Phi_1(x, \omega) \\ &\quad + C_s(\omega) \left\{ \Phi_2(x, \omega) + \frac{U''(x_c)}{U'(x_c)} \Phi_1(x, \omega) [\log |x - x_c| \mp \pi \mathrm{i} Y(x - x_c)] \right\}, \end{aligned} \tag{4.18}$$

where $\Phi_1(x, \omega)$ and $\Phi_2(x, \omega)$ are real and regular functions with $\Phi_1(x_c, \omega) = \Phi_2'(x_c, \omega) = 0$ and $\Phi_1'(x_c, \omega) = \Phi_2(x_c, \omega) = 1$ [49]. From definition (4.9), the coefficients $C_r(\omega)$ and $C_s(\omega)$ are found to be real and so is $\Phi_<(x, \omega \pm \mathrm{i}0)$ on $[-L, x_c]$ since $Y(x - x_c) \equiv 0$ for $x < x_c$. Lemma 4.3 shows that $\Phi_<(x, \omega \pm \mathrm{i}0)$ has no zero on $[-L, x_c]$ other than $x = -L$ and hence $C_s(\omega) > 0$ (since $\Phi_1(x_c, \omega) = 0$).

If $\omega \neq kU_{\text{In}}$ (i.e. $x_c \neq x_{\text{In}}$), then $U''(x_c) \neq 0$ holds and $\Phi_<(x, \omega \pm \mathrm{i}0)$ possesses the imaginary part on $[x_c, L]$. Lemma 4.3 again shows that this imaginary part has no zero on $[x_c, L]$ other than $x = x_c$. Therefore, we conclude that $\operatorname{Im} W(\omega \pm \mathrm{i}0) = \operatorname{Im} \Phi_<(L, \omega \pm \mathrm{i}0) \neq 0$ for all $\omega \in \mathbb{R} \backslash \{kU_{\text{In}} | n = 1, 2, \ldots, N\}$.

If $\omega = kU_{\text{In}}$, then $\Phi_<(x, kU_{\text{In}})$ is a real and regular function on the whole domain $[-L, L]$ due to $U''(x_c) = 0$ and has at most one zero on $[x_c, L]$. Therefore, $W(kU_{\text{In}}) = 0$ may occur only when this zero corresponds to $x = L$, and as such the zero must be simple, namely, (4.17) holds. ■

Besides the singularity stemming from the zeros of $W(\Omega)$, $\Phi(x, \Omega)$ has also the following essential singularity along the continuous spectrum σ_c.

Proposition 4.5. *Let $U(x)$ satisfy (A 1) and (A 2). For all $\xi_0 \in X + \mathrm{i}X$ and $\omega \in \mathbb{R}$,*

$$\Psi(x, \omega \pm \mathrm{i}0) \in \mathbf{H}_0^1 + \mathrm{i}\mathbf{H}_0^1, \tag{4.19}$$

where

$$\Psi(x, \Omega) := \Phi(x, \Omega) \prod_{n=1}^{N} \left(\frac{\Omega}{k} - U_{\text{In}} \right). \tag{4.20}$$

Proof. For any fixed $s \in [-L, L]$, the function $\partial \Phi_G/\partial x(x, s, \omega \pm \mathrm{i}0)$ is regular almost everywhere except that it has a logarithmic singularity, $\log|x - x_c|$, and discontinuities, $Y(x - x_c)$ and $Y(x - s)$. Hence,

$$\int_{-L}^{L} \left| \frac{\partial \Phi_G}{\partial x}(x, s, \omega \pm \mathrm{i}0) \right| \mathrm{d}x < \infty. \tag{4.21}$$

Since $\xi_0'' - k^2 \xi_0 \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$, the following convolution integral is also square-integrable:

$$\lim_{\Omega=\omega\pm\mathrm{i}0} [W(\Omega)\Phi'(x, \Omega)] = \int_{-L}^{L} \frac{\partial \Phi_G}{\partial x}(x, s, \omega \pm \mathrm{i}0) \frac{1}{k} [\xi_0''(s) - k^2 \xi_0(s)]\, \mathrm{d}s \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2, \tag{4.22}$$

that is,

$$\lim_{\Omega=\omega\pm\mathrm{i}0} [W(\Omega)\Phi(x, \Omega)] \in \mathbf{H}^1 + \mathrm{i}\mathbf{H}^1. \tag{4.23}$$

In combination with (4.17) and the boundary condition $\Phi(\pm L, \Omega) = 0$, the proposition is proven. ■

In summary, the spectrum is decomposed as follows.

Proposition 4.6. *Let $U(x)$ satisfy (A 1) and (A 2). The spectrum $\sigma \subset \mathbb{C}$ of $k\mathcal{L}^*$ on $X + \mathrm{i}X$ is composed of*

(a) point spectra $\sigma \backslash \sigma_c = \{\omega_j, \bar{\omega}_j \in \mathbb{C} | \mathrm{Im}\, \omega_j \neq 0, W(\omega_j) = 0, j = 1, 2, \ldots, N_p\}$, where N_p is finite.
(b) point spectra $\sigma_I = \{kU_{In} \in \mathbb{R} | W(kU_{In} \pm \mathrm{i}0) = 0, n = 1, 2, \ldots, N\}$.
(c) continuous spectrum $\sigma_c \backslash \sigma_I$.

where (c) always exists while (a) and (b) can be empty. The residual spectrum is always empty.

Proof. The finiteness of N_p is proved in [47]. The existence of (b) follows from proposition 4.4. The proof of (c) and the absence of the residual spectrum are relegated to appendix B, since the subsequent discussion will not refer to these facts. ■

(c) Signature of Q

Now, let us substitute expression (4.5) into the quadratic form $Q = \langle \bar{\xi}, \mathcal{H}\xi \rangle$, which is a constant of motion for equation (3.1). By using the property of the resolvent $(\Omega - k\mathcal{L}^*)^{-1}$ [16,17], we obtain

$$Q = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma(\sigma)} h(\Omega)\, \mathrm{d}\Omega \tag{4.24}$$

where $h : \mathbb{C} \to \mathbb{C}$ is given by

$$h(\Omega) = \nu \int_{-L}^{L} \bar{\xi}_0 \left[-\frac{kU''}{\Omega - kU} \Psi(x, \Omega) \right] \mathrm{d}x. \tag{4.25}$$

Upon decomposing the spectrum $\sigma \subset \mathbb{C}$ into $\sigma_c \subset \mathbb{R}$ and others $\sigma \backslash \sigma_c \subset \mathbb{C} \backslash \mathbb{R}$ and, accordingly, deforming the contour $\Gamma(\sigma)$ into $\Gamma(\sigma_c)$ and $\Gamma(\sigma \backslash \sigma_c)$ (figure 1), we obtain $Q = Q|_{\sigma_c} + Q|_{\sigma \backslash \sigma_c}$ with

$$Q|_{\sigma_c} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma(\sigma_c)} h(\Omega)\, \mathrm{d}\Omega = \int_{\sigma_c} \hat{h}(\omega)\, \mathrm{d}\omega, \tag{4.26}$$

where

$$\hat{h}(\omega) = \frac{1}{2\pi \mathrm{i}} [-h(\omega + \mathrm{i}0) + h(\omega - \mathrm{i}0)]. \tag{4.27}$$

The existence of this limit for all $\omega \in \sigma_c$ is guaranteed by proposition 4.5 and $\xi_0 \in X + \mathrm{i}X$.

Then, we can prove the following inequality.

Proposition 4.7.

$$Q|_{\sigma_c} = \int_{\sigma_c} \hat{h}(\omega)\, \mathrm{d}\omega \leq 0 \tag{4.28}$$

for all solutions $\xi \in X + \mathrm{i}X$ of (3.1) with initial data $\xi_0 \in X + \mathrm{i}X$.

Proof. To observe the signature of the function $\hat{h}(\omega)$ more explicitly, we rewrite $h(\Omega)$ as

$$h(\Omega)=\frac{\nu k}{p(\Omega)}\int_{-L}^{L}\overline{\Psi(x,\bar{\Omega})}\mathcal{E}(\Omega)\Psi(x,\Omega)\,\mathrm{d}x-\nu\frac{p(\Omega)}{k}\int_{-L}^{L}(|\xi_0'|^2+k^2|\xi_0|^2)\,\mathrm{d}x, \tag{4.29}$$

where we have put $p(\Omega)=\prod_{n=1}^{N}(\Omega/k-U_{\mathrm{In}})$. Since $p(\Omega)$ is an regular function of Ω, we may neglect the second term on the right-hand side when calculating $\hat{h}(\omega)$. As shown in Proposition 10 of [16], the first term is further transformed into

$$\begin{aligned}k\int_{-L}^{L}&\overline{\Psi(x,\bar{\Omega})}\mathcal{E}(\Omega)\Psi(x,\Omega)\,\mathrm{d}x\\&=-k\langle\overline{F(x,\Omega)},\mathcal{E}(\Omega)F(x,\Omega)\rangle-k\langle\overline{G(x,\Omega)},\mathcal{E}(\Omega)G(x,\Omega)\rangle\\&\quad+p(\Omega)\langle\overline{G(x,\Omega)},\xi_0''-k^2\xi_0\rangle+p(\Omega)\langle\overline{\xi_0''-k^2\xi_0},G(x,\Omega)\rangle,\end{aligned} \tag{4.30}$$

where

$$F(x,\Omega)=\tfrac{1}{2}[\Psi(x,\Omega)-\Psi(x,\bar{\Omega})] \tag{4.31}$$

and

$$G(x,\Omega)=\tfrac{1}{2}[\Psi(x,\Omega)+\Psi(x,\bar{\Omega})]. \tag{4.32}$$

In the limit of $\Omega\to\omega\pm\mathrm{i}0$, the relations $F(\omega+\mathrm{i}0)=-F(\omega-\mathrm{i}0)$ and $G(\omega+\mathrm{i}0)=G(\omega-\mathrm{i}0)$ hold. Using the formula,

$$\begin{aligned}-\mathcal{E}(\omega+\mathrm{i}0)+\mathcal{E}(\omega-\mathrm{i}0)&=\frac{kU''(x)}{\omega+\mathrm{i}0-kU(x)}-\frac{kU''(x)}{\omega-\mathrm{i}0-kU(x)}\\&=-2\pi\mathrm{i}\frac{U''(x_{\mathrm{c}})}{|U'(x_{\mathrm{c}})|}\delta(x-x_{\mathrm{c}}),\end{aligned} \tag{4.33}$$

we finally obtain

$$\hat{h}(\omega)=\frac{\nu kU''(x_{\mathrm{c}})}{p(\omega)|U'(x_{\mathrm{c}})|}[|F(x_{\mathrm{c}},\omega+\mathrm{i}0)|^2+|G(x_{\mathrm{c}},\omega+\mathrm{i}0)|^2], \tag{4.34}$$

where $x_{\mathrm{c}}=U^{-1}(\omega/k)$ should be read as a function of ω. According to the definition (3.6) of ν, this expression indicates that $\hat{h}(\omega)$ is negative for all $\omega\in\sigma_{\mathrm{c}}$ and we conclude that Q is negative semi-definite on the continuous spectrum. ■

If $Q=\langle\bar{\xi},\mathcal{H}\xi\rangle>0$ for some $\xi\in X+\mathrm{i}X$, then $\sigma\backslash\sigma_{\mathrm{c}}$ must not be null and there exists at least one pair of complex eigenvalues, say ω_j and $\bar{\omega}_j$ with $\mathrm{Im}\,\omega_j>0$, which correspond to growing and damping modes, respectively. Since $\mathcal{H}$ is actually a real self-adjoint operator, the condition $Q=\langle\xi,\mathcal{H}\xi\rangle>0$ for some $\xi\in X$ comes to the same conclusion. Thus, we have proved the instability if $Q>0$ for some $\xi\in X$.

Conversely, if the flow is unstable due to the presence of several pairs of complex eigenvalues $\sigma\backslash\sigma_{\mathrm{c}}=\{\omega_j,\bar{\omega}_j\in\mathbb{C}|\mathrm{Im}\,\omega_j>0,j=1,2,\ldots\}$, the constant of motion Q must be indefinite in the corresponding eigenspaces, as shown in [6,7,50] for a Hamiltonian function. Indeed, the solution ξ is subject to the following modal decomposition:

$$\xi=\sum_j(a_j\hat{\xi}_j\,\mathrm{e}^{-\mathrm{i}\omega_jt}+b_j\bar{\hat{\xi}}_j\,\mathrm{e}^{-\mathrm{i}\bar{\omega}_jt})+\cdots, \tag{4.35}$$

where $\hat{\xi}_j$ is the eigenfunction for ω_j and $a_j,b_j\in\mathbb{C}$ are the mode amplitudes which depend on ξ_0. As usual, the eigenfunctions $\{\hat{\xi}_j,\bar{\hat{\xi}}_j|j=1,2,\ldots\}$ constitute a non-orthogonal basis and its dual basis is provided by the eigenfunctions $\{\hat{w}_j,\bar{\hat{w}}_j|j=1,2,\ldots\}$ of $\mathcal{L}$, where $\hat{w}_j=-U''\hat{\xi}_j$ holds from proposition 4.2. This leads to the following orthogonality relations (see Sec. III of [16] for details):

$$\langle \hat{\xi}_l, U'' \hat{\xi}_j \rangle = \langle \bar{\hat{\xi}}_l, U'' \hat{\xi}_j \rangle = \langle \bar{\hat{\xi}}_j, U'' \hat{\xi}_j \rangle = 0 \ \ (l \neq j), \quad \langle \hat{\xi}_j, U'' \hat{\xi}_j \rangle \neq 0. \tag{4.36}$$

By substituting this modal decomposition into Q and using the above orthogonality, we obtain

$$Q|_{\sigma \backslash \sigma_c} = \nu \sum_j [a_j \bar{b}_j p(\omega_j) \langle \hat{\xi}_j, U'' \hat{\xi}_j \rangle + \text{c.c.}], \tag{4.37}$$

whose sign is clearly indefinite. For example, by setting either $(a_j, b_j) = (1, 1)$ or $(a_j, b_j) = (1, -1)$, we can make $Q|_{\sigma \backslash \sigma_c} > 0$. Thus, we have proved that the equation (2.1) is spectrally stable if and only if $Q = \langle \bar{\xi}, \mathcal{H}\xi \rangle \leq 0$ for all $\xi \in X + \mathrm{i}X$, namely, if and only if $Q = \langle \xi, \mathcal{H}\xi \rangle \leq 0$ for all $\xi \in X$.

(d) Extension of search space

Our remaining task is to extend the search space from X to $\mathbf{L}^2$. Maximization of $Q = \langle \xi, \mathcal{H}\xi \rangle$ on $\mathbf{L}^2$ is, in practice, more tractable than that on X, since the variational problem $\lambda_{\max} = \max Q/\|\xi\|^2_{\mathbf{L}^2}$ simply searches the maximum eigenvalue $\lambda_{\max}$ of the self-adjoint operator $\mathcal{H}$. Let us consider the eigenvalue problem $(\lambda - \mathcal{H})\hat{\xi} = 0$ and rewrite $\mathcal{H}$ in the form of

$$\mathcal{H} = \nu U'' \prod_{n=1}^{N} (U - U_{\mathrm{I}n}) + \mathcal{R}, \tag{4.38}$$

where $\mathcal{R}$ represents the sum of all operators that involve at least one multiplication of $\mathcal{G}$,

$$\mathcal{R} = \nu \sum_{l=1}^{N} \left(\prod_{j=1}^{N-l} (U - U_{\mathrm{I}j}) \right) U'' \mathcal{G} \left(\prod_{n=N-l+2}^{N} [(U - U_{\mathrm{I}n}) + U''\mathcal{G}] \right) U'', \tag{4.39}$$

and hence $\mathcal{R} : \mathbf{L}^2 \to X$. It follows from the condition (3.6) that $\mathcal{H}$ has a continuous spectrum for the negative side, $\min[\nu U'' \prod_{n=1}^{N} (U - U_{\mathrm{I}n})] \leq \lambda \leq 0$. On the other hand, for $\lambda > 0$, the eigenvalue problem is non-singular and can be rewritten as follows:

$$\hat{\xi} = \frac{1}{\lambda - \nu U'' \prod_{n=1}^{N} (U - U_{\mathrm{I}n})} \mathcal{R}\hat{\xi}. \tag{4.40}$$

Since $\mathcal{R}\hat{\xi} \in X$, this eigenfunction $\hat{\xi}$ inevitably belongs to X. If $\mathcal{H}$ has such a positive discrete eigenvalue, the corresponding eigenfunction $\hat{\xi} \in X$ directly proves the instability $Q = \langle \hat{\xi}, \mathcal{H}\hat{\xi} \rangle > 0$. Conversely, if $Q \leq 0$ for all $\xi \in \mathbf{L}^2$, then obviously $Q \leq 0$ for all $\xi \in X \subset \mathbf{L}^2$. Therefore, we may replace the search space X by $\mathbf{L}^2$; thus, the proof of theorem 3.1 is completed.

We can further extend this idea as follows:

Corollary 4.8. *The stability condition* (3.7) *in theorem* 3.1 *can be replaced by*

$$Q = \langle w, \mathcal{H}_{\mathrm{v}} w \rangle \leq 0 \quad \textit{for all } w \in \mathbf{L}^2, \tag{4.41}$$

where $w = -U''\xi$ *and, hence,*

$$\mathcal{H}_{\mathrm{v}} = \frac{\nu}{U''} \prod_{n=1}^{N} [U - U_{\mathrm{I}n} + U''\mathcal{G}]. \tag{4.42}$$

Proof. Since $(\nu/U'') \prod_{n=1}^{N} (U - U_{\mathrm{I}n}) < 0$ follows from the assumptions, the operator $\mathcal{H}_{\mathrm{v}}$ is found to be bounded; $\exists C > 0$ such that $\langle w, \mathcal{H}_{\mathrm{v}} w \rangle < C\|w\|^2_{\mathbf{L}^2}$ for all $w \in \mathbf{L}^2$. Suppose that we find a function $\hat{w} \in \mathbf{L}^2$ that makes Q positive

$$0 < \frac{\langle \hat{w}, \mathcal{H}_{\mathrm{v}} \hat{w} \rangle}{\|\hat{w}\|^2_{\mathbf{L}^2}} < C. \tag{4.43}$$

Then, consider a sequence $\xi_m \in \mathbf{L}^2$, $m = 1, 2, \ldots, \infty$, that satisfies $\|\hat{w} + U''\xi_m\|_{\mathbf{L}^2} \to 0$ as $m \to \infty$. Since $\langle \xi_m, \mathcal{H}\xi_m \rangle \to \langle \hat{w}, \mathcal{H}_{\mathrm{v}} \hat{w} \rangle$ as $m \to \infty$, $\langle \xi_m, \mathcal{H}\xi_m \rangle$ also becomes positive when m is sufficiently large.

On the other hand, if $\langle w, \mathcal{H}_{\mathrm{v}} w\rangle \le 0$ for all $w \in \mathbf{L}^2$, then obviously $\langle \xi, \mathcal{H}\xi\rangle = \langle U''\xi, \mathcal{H}_{\mathrm{v}} U''\xi\rangle \le 0$ for all $\xi \in \mathbf{L}^2$. ■

We will actually adopt corollary 4.8 in the subsequent sections, because this variational problem for $w \in \mathbf{L}^2$ is more beneficial than that for $\xi \in \mathbf{L}^2$, both analytically and numerically. This fact is evident from the corresponding eigenvalue problem

$$(\lambda - \mathcal{H}_{\mathrm{v}})\hat{w} = 0. \tag{4.44}$$

The operator $\mathcal{H}_{\mathrm{v}}$, which is again written as

$$\mathcal{H}_{\mathrm{v}} = \frac{\nu}{U''}\prod_{n=1}^{N}(U - U_{\mathrm{I}n}) + \frac{1}{U''}\mathcal{R}\frac{1}{U''} \tag{4.45}$$

has a continuous spectrum, but it is remarkable that the upper edge of this continuous spectrum, $\lambda_{\mathrm{u}} = \max[(\nu/U'')\Pi_{n=1}^{N}(U - U_{\mathrm{I}n})]$, is separated from the origin ($\lambda_{\mathrm{u}} < 0$). Owing to this property, the variational problem for $w \in \mathbf{L}^2$ is useful for investigating the stability boundary at $\lambda = 0$ without suffering from any singularity.

5. Comparison with existing results

In this section, we explore several alternative representations of our variational stability criterion by assuming that we have somehow *solved* Rayleigh's equation under specific conditions. As a consequence of this exploration, we reproduce existing stability theorems and gain a clear-cut understanding of the onset of instability.

(a) Single inflection point

First consider the case of a single inflection point with the condition $(U - U_{\mathrm{I}})/U'' < 0$ for all $x \in [-L, L]$, since the opposite case $(U - U_{\mathrm{I}})/U'' > 0$ is always stable. According to corollary 4.8, we maximize Q with respect to $w \in \mathbf{L}^2$, where the corresponding eigenvalue problem (4.44) is simply

$$\lambda\hat{w} = \frac{U - U_{\mathrm{I}}}{U''}\hat{w} + \mathcal{G}\hat{w}. \tag{5.1}$$

We are interested in whether a positive eigenvalue $\lambda > 0$ exists or not, for its existence is the necessary and sufficient condition for instability. By focusing on $\lambda > \lambda_{\mathrm{u}}$, where $\lambda_{\mathrm{u}} = -\min[(U_{\mathrm{I}} - U)/U''] < 0$, the eigenvalue problem is transformed into

$$\hat{\phi}'' - k^2\hat{\phi} + \frac{1}{\lambda + (U_{\mathrm{I}} - U)/U''}\hat{\phi} = 0 \tag{5.2}$$

and

$$\hat{\phi}(-L) = \hat{\phi}(L) = 0, \tag{5.3}$$

using $\hat{\phi} = \mathcal{G}\hat{w}$. If (5.2) is viewed as $\hat{\phi}'' + f(x, \lambda, k)\hat{\phi} = 0$, we can apply Sturm's oscillation theorem [51] to this equation with respect to both λ and k. As the two parameters $\lambda > \lambda_{\mathrm{u}}$ and $k > 0$ increase, $f(x, \lambda, k)$ decreases everywhere on $[-L, L]$ and hence the general solution $\hat{\phi}$ becomes less oscillatory (i.e. the interval between any two zeros of $\hat{\phi}$ expands). When $k^2 \ge 1/(\lambda - \lambda_{\mathrm{u}})$, it becomes non-oscillatory, $f(x, \lambda, k) \le 0$, and unable to satisfy (5.3). It follows that the eigenvalue λ is bounded by

$$\lambda < \frac{1}{k^2} + \lambda_{\mathrm{u}}. \tag{5.4}$$

If $k^2 > -\lambda_{\mathrm{u}}^{-1}$, no positive eigenvalue $\lambda > 0$ exists and, hence, the flow U is stable for such large k.

Since marginal stability occurs at $\lambda = 0$ in (5.2), we analyse the equation

$$\mathcal{E}_{\mathrm{I}}\hat{\phi} := -\hat{\phi}'' + k^2\hat{\phi} - \frac{U''}{U_{\mathrm{I}} - U}\hat{\phi} = 0. \tag{5.5}$$

If this solution is somehow available, we obtain the following stability criterion.

Corollary 5.1. *If $U(x)$ satisfies (A 1) and (A 2) and has a single inflection point x_I, and $\phi_\mathrm{c}(x)$ denotes the solution of*

$$\mathcal{E}_\mathrm{I}\phi_\mathrm{c} = 0, \quad \phi_\mathrm{c}(-L) = 0, \quad \phi_\mathrm{c}'(-L) = 1, \tag{5.6}$$

then (2.1) is spectrally stable if and only if $\phi_\mathrm{c}(L) \geq 0$.

Proof. According to lemma 4.3, ϕ_c does not have zero on $[-L, x_\mathrm{I}]$ other than $x = -L$ and has at most one zero on $[x_\mathrm{I}, L]$. Note that, by increasing λ from 0, the general solution $\hat{\phi}$ of (5.2) becomes less oscillatory than ϕ_c. If $\phi_\mathrm{c}(L) < 0$, $\phi_\mathrm{c}(x)$ has one zero on $[x_\mathrm{I}, L]$ and hence there must be one eigenvalue $\lambda \in [0, 1/k^2 + \lambda_\mathrm{u}]$ for which $\hat{\phi}$ satisfies (5.2) and (5.3).

Conversely, if $\phi_\mathrm{c}(L) \geq 0$, then $\phi_\mathrm{c}(x)$ does not have zero on $-L < x < L$ and the solution $\hat{\phi}$ of (5.2) cannot satisfy the boundary condition (5.3) when $\lambda > 0$, i.e. there is no positive eigenvalue $\lambda > 0$. ■

In particular, we can obtain an analytical solution ϕ_c for the case of $k \to 0$ as

$$\phi_\mathrm{c}(x) = [U(-L) - U_\mathrm{I}][U(x) - U_\mathrm{I}] \int_{-L}^{x} \frac{1}{[U(s) - U_\mathrm{I}]^2}\,\mathrm{d}s. \tag{5.7}$$

Then, the necessary and sufficient stability condition $\phi_\mathrm{c}(L) \geq 0$ becomes

$$\left.\frac{1}{U'(s)[U(s) - U_\mathrm{I}]}\right|_{-L}^{L} + \int_{-L}^{L} \frac{U''(s)}{U'^2(s)[U(s) - U_\mathrm{I}]}\,\mathrm{d}s \geq 0, \tag{5.8}$$

which agrees with the result of Rosenbluth & Simon [44]. (Note, the typographical error in the final eqn (4) of this paper, in which w'^3 should be replaced by w'^2.)

Another equivalent approach is to solve the equation $\mathcal{E}_\mathrm{I}\phi_\mathrm{c} = 0$ with boundary conditions $\phi_\mathrm{c}(-L) = \phi_\mathrm{c}(L) = 0$ and with a *derivative jump* at $x = x_\mathrm{I}$

$$\phi_\mathrm{c}(x_\mathrm{I} + 0) = \phi_\mathrm{c}(x_\mathrm{I} - 0) \quad \text{and} \quad \alpha := \phi_\mathrm{c}'(x_\mathrm{I} + 0) - \phi_\mathrm{c}'(x_\mathrm{I} - 0). \tag{5.9}$$

In other words, we solve $\mathcal{E}_\mathrm{I}\phi_\mathrm{c} = -\alpha\delta(x - x_\mathrm{I})$ or

$$-\phi_\mathrm{c} + \mathcal{G}\left(\frac{U''}{U_\mathrm{I} - U}\phi_\mathrm{c}\right) = \alpha g(x, x_\mathrm{I}), \tag{5.10}$$

where g is defined in (2.3). By introducing a normalization $\int_{-L}^{L}(-\phi_\mathrm{c}'' + k^2\phi_\mathrm{c})\,\mathrm{d}x = 1$ for ϕ_c, we can determine α as

$$\alpha = -1 + \int_{-L}^{L} \frac{U''}{U_\mathrm{I} - U}\phi_\mathrm{c}\,\mathrm{d}x, \tag{5.11}$$

and arrive at the integral equation (5.12). This approach reproduces the stability criterion obtained by Balmforth & Morrison [45].

Corollary 5.2. *If $U(x)$ satisfies (A 1) and (A 2) and has a single inflection point x_I, and $\phi_\mathrm{c}(x)$ denotes the solution of*

$$-\phi_\mathrm{c}(x) + g(x, x_\mathrm{I}) + \int_{-L}^{L} [g(x, s) - g(x, x_\mathrm{I})]\frac{U''(s)}{U_\mathrm{I} - U(s)}\phi_\mathrm{c}(s)\,\mathrm{d}s = 0, \tag{5.12}$$

then (2.1) is spectrally stable if and only if

$$-1 + \int_{-L}^{L} \frac{U''}{U_\mathrm{I} - U}\phi_\mathrm{c}\,\mathrm{d}x < 0, \tag{5.13}$$

Proof. According to lemma 4.3, $\phi_\mathrm{c}(x)$ does not have zero on $-L < x < L$ and its sign should be always positive $\phi_\mathrm{c}(x) > 0$ due to the normalization. If $\alpha > 0$, we can eliminate this derivative jump by increasing λ from 0, since the general solution $\hat{\phi}$ of (5.2) becomes less oscillatory than ϕ_c. Therefore, there must be an eigenvalue $\lambda \in [0, 1/k^2 + \lambda_\mathrm{u}]$ for which $\hat{\phi}$ satisfies (5.2) and (5.3) without the derivative jump.

Conversely, if $\alpha \leq 0$, this derivative jump gets large as λ increases from 0 and, hence, there is no positive eigenvalue $\lambda > 0$. ■

(b) Multiple inflection points

Here, we address the problem of multiple inflection points. Recall from proposition 4.4 that neutrally stable eigenmodes may exist only at the frequencies $\omega = kU_{\mathrm{I}n}$, $n = 1, 2, \ldots, N$. In the same manner as for the case of a single inflection point, we consider the equations for the neutrally stable eigenmodes

$$\mathcal{E}_{\mathrm{I}n}\hat{\phi}_{\mathrm{c}} := -\hat{\phi}_{\mathrm{c}}'' + k^2\hat{\phi}_{\mathrm{c}} - \frac{U''}{U_{\mathrm{I}n} - U}\hat{\phi}_{\mathrm{c}} = 0 \tag{5.14}$$

and

$$\hat{\phi}_{\mathrm{c}}(-L) = \hat{\phi}_{\mathrm{c}}(L) = 0, \tag{5.15}$$

for every inflection point $x_{\mathrm{I}n}$, $n = 1, 2, \ldots, N$. Since these equations do not have non-trivial solutions for general k, we seek them for some characteristic values of k, in the same spirit as Tollmien's approach [39–43].

Proposition 5.3. *Let $U(x)$ satisfy (A 1) and (A 2). For each inflection point $x_{\mathrm{I}n}$, there is at most one critical wavenumber $k_n > 0$ at which the equation*

$$\mathcal{E}_{\mathrm{I}n}|_{k_n}\hat{\phi}_{\mathrm{c}} = 0, \quad \hat{\phi}_{\mathrm{c}}(-L) = \hat{\phi}_{\mathrm{c}}(L) = 0, \tag{5.16}$$

has a non-trivial solution $\hat{\phi}_{\mathrm{c}}$, where $\mathcal{E}_{\mathrm{I}n}|_{k_n}$ denotes the operator $\mathcal{E}_{\mathrm{I}n}$ at $k = k_n$.

Proof. According to lemma 4.3, the solution $\phi_{\mathrm{c}}(x)$ of $\mathcal{E}_{\mathrm{I}n}\phi_{\mathrm{c}} = 0$ satisfying $\phi_{\mathrm{c}}(-L) = 0$ and $\phi_{\mathrm{c}}'(-L) = 1$ has at most one zero on $-L < x \le L$. This $\phi_{\mathrm{c}}(x)$ becomes less oscillatory as k increases from 0 to ∞ and eventually has no zero for $k^2 > \max[U''/(U_{\mathrm{I}n} - U)]$. Therefore, there exists at most one value k_n of k for which $\phi_{\mathrm{c}}(L) = 0$ holds. ■

Without loss of generality, let us focus on an inflection point $x_{\mathrm{I}1}$ and assume that there is a critical wavenumber $k_1 > 0$ for it. Namely, we have a solution $\hat{w}_{\mathrm{c}} = -\hat{\phi}_{\mathrm{c}}'' + k_1^2\hat{\phi}_{\mathrm{c}} \in \mathbf{L}^2$ that satisfies

$$-\hat{w}_{\mathrm{c}} + \frac{U''}{U_{\mathrm{I}1} - U}\mathcal{G}|_{k_1}\hat{w}_{\mathrm{c}} = 0, \quad \text{or} \quad (\mathcal{L}|_{k_1} - U_{\mathrm{I}1})\hat{w}_{\mathrm{c}} = 0. \tag{5.17}$$

Now, we again invoke corollary 4.8 and consider the *self-adjoint* eigenvalue problem (4.44). The above neutrally stable eigenfunction $\hat{w}_{\mathrm{c}}$ clearly corresponds to the marginally stable eigenfunction ($\lambda = 0$) of (4.44) at $k = k_1$, namely, $\mathcal{H}_{\mathrm{v}}|_{k_1}\hat{w}_{\mathrm{c}} = 0$.

Let us continuously change the parameter k in the neighbourhood of k_1 and investigate how an eigenvalue λ and an eigenfunction $\hat{w}$ deviate from $\lambda = 0$ and $\hat{w} = \hat{w}_{\mathrm{c}}$, respectively. By differentiating the identity,

$$0 = \int_{-L}^{L} \hat{w}(\lambda - \mathcal{H}_{\mathrm{v}})\hat{w}\,\mathrm{d}x, \tag{5.18}$$

with respect to k and setting $k = k_1$, we obtain

$$\begin{aligned} 0 &= \int_{-L}^{L} \hat{w}_{\mathrm{c}} \left(\left.\frac{\partial\lambda}{\partial k}\right|_{k_1} - \left.\frac{\partial\mathcal{H}_{\mathrm{v}}}{\partial k}\right|_{k_1} \right) \hat{w}_{\mathrm{c}}\,\mathrm{d}x \\ &= \int_{-L}^{L} \hat{w}_{\mathrm{c}} \left[\left.\frac{\partial\lambda}{\partial k}\right|_{k_1} - \frac{\nu}{U''} \left.\frac{\partial\mathcal{L}}{\partial k}\right|_{k_1} (U_{\mathrm{I}1} - U_{\mathrm{I}2})(U_{\mathrm{I}1} - U_{\mathrm{I}3})\cdots(U_{\mathrm{I}1} - U_{\mathrm{I}N}) \right] \hat{w}_{\mathrm{c}}\,\mathrm{d}x, \end{aligned} \tag{5.19}$$

where (5.17) has been used. Since

$$\frac{\partial\mathcal{L}}{\partial k} = U''\frac{\partial\mathcal{G}}{\partial k} = -2kU''\mathcal{G}\mathcal{G}, \tag{5.20}$$

we obtain

$$\left.\frac{\partial\lambda}{\partial k}\right|_{k_1} \|\hat{w}_{\mathrm{c}}\|_{\mathbf{L}^2}^2 = -2k_1\nu(U_{\mathrm{I}1} - U_{\mathrm{I}2})(U_{\mathrm{I}1} - U_{\mathrm{I}3})\cdots(U_{\mathrm{I}1} - U_{\mathrm{I}N})\|\hat{\phi}_{\mathrm{c}}\|_{\mathbf{L}^2}^2. \tag{5.21}$$

Similar relations are available for the other critical wavenumbers $k_2, k_3, \dots, k_N$ if they exist. In the view of condition (3.6), one can distinguish the sign of $\partial\lambda/\partial k|_{k_n}$ from (5.21) as follows;

$$\operatorname{sgn} \left.\frac{\partial\lambda}{\partial k}\right|_{k_n} = \operatorname{sgn}[U'''(x_{\mathrm{In}})U'(x_{\mathrm{In}})] = \operatorname{sgn}[(U'^2)''(x_{\mathrm{In}})], \tag{5.22}$$

which agrees with Tollmien and Lin's result [39–41]. In other words, if the absolute value of the background vorticity $|U'(x)|$ has a local maximum (or minimum) at $x = x_{\mathrm{In}}$, then a positive eigenvalue $\lambda > 0$ emerges at $k = k_n$ as k decreases (or increases).

We note that there is no positive eigenvalue $\lambda > 0$ of (4.44) in the limit of $k \to \infty$. As k continuously changes from ∞ to 0, the number of positive eigenvalues increases (or decreases) by one when k passes through k_n that is associated with the inflection point x_{In} satisfying $(U'^2)''(x_{\mathrm{In}}) < 0$ (or > 0). We can summarize these facts into the following stability criterion.

Corollary 5.4. *Let $U(x)$ satisfy (A 1) and (A 2). Suppose that, for every inflection points x_{In}, $n = 1, 2, \dots, N$, the critical wavenumbers $k_n > 0$, $n = 1, 2, \dots, N$, are either solved or proven to be non-existent according to proposition 5.3. Then, equation (2.1) is spectrally unstable if and only if $N^+ - N^- > 0$, where*

N^+: number of the critical wavenumbers k_n that satisfy $k < k_n$ and $(U'^2)''(x_{\mathrm{In}}) < 0$,
N^-: number of the critical wavenumbers k_n that satisfy $k < k_n$ and $(U'^2)''(x_{\mathrm{In}}) > 0$.

When $N^+ - N^-$ is positive, it corresponds to the number of positive eigenvalues λ of (4.44). This number cannot be greater than the number of the inflection points x_{In} satisfying $(U'^2)''(x_{\mathrm{In}}) < 0$, i.e. the number of local maxima of $|U'(x)|$.

A similar result to corollary 5.4 is shown by Lin [42,43] as a rigorous justification of Tollmien's method. While he treats a larger class of flows than ours, his criterion is sufficient but not necessary for instability in the presence of multiple inflection points [43]. Balmforth & Morrison [45] have also discussed the case of multiple inflection points in the same manner as corollary 5.2, where the derivative jump α_n is evaluated for each inflection point x_{In} and then $\alpha_n < 0$ (or $\alpha_n > 0$) corresponds to $k < k_n$ (or $k > k_n$). However, in this work the importance of $\operatorname{sgn}(U'^2)''(x_{\mathrm{In}})$ was not observed.

6. Numerical tests

Finally, we exhibit numerical results to illustrate the practicability of our method. For three velocity profiles $U(x)$, we compare the results of two different numerical codes: one code solves the Rayleigh equation (2.4) directly for complex eigenvalues $c = \omega/k \in \mathbb{C}$, while the other code solves for the eigenvalues $\lambda_1, \lambda_2, \dots,$ of the self-adjoint operator $\mathcal{H}_{\mathrm{v}}$ in descending order. In fact, the maximum eigenvalue λ_1 can be more easily determined by the variational problem, $\lambda_1 = \max Q/\|w\|^2_{\mathbf{L}^2}$, and the flow $U(x)$ is spectrally unstable if and only if $\lambda_1 > 0$ (see corollary 4.8).

The first example is

$$U(x) = \tanh(x), \quad x \in [-\infty, \infty], \tag{6.1}$$

which is well known to be unstable for $0 < k < 1$. The result is shown in figure 2, where we also plot $\tilde{\lambda}_1 = \max Q/\|\xi\|^2_{\mathbf{L}^2}$ for comparison (the damping eigenvalue $\operatorname{Im} c < 0$ is not plotted since its presence is trivial). As expected from the results of §4*d*, $\lambda = 0$ is the upper edge of the continuous spectrum of $\mathcal{H}$. Since the eigenfunction $\hat{\xi}_1$ becomes singular, i.e. $\|\hat{\xi}_1\|_{\mathbf{L}^2} \to \infty$, as $\tilde{\lambda}_1 \to +0$, the curve of $\tilde{\lambda}_1$ is tangent to the marginal line $\lambda = 0$ and the critical wavenumber $k = 1$ is not so evident. On the other hand, the upper edge of the continuous spectrum of $\mathcal{H}_{\mathrm{v}}$ is less than zero, $\lambda_{\mathrm{u}} = \max[\tanh(x)/\tanh''(x)] = -0.5 < 0$, and hence the maximum eigenvalue λ_1 of $\mathcal{H}_{\mathrm{v}}$ smoothly intersect with $\lambda = 0$ at $k = 1$ in figure 2. Thus, for the purpose of drawing the stability boundary, the variational principle with respect to the norm $\|w\|_{\mathbf{L}^2}$ is seen to be numerically efficient and accurate.

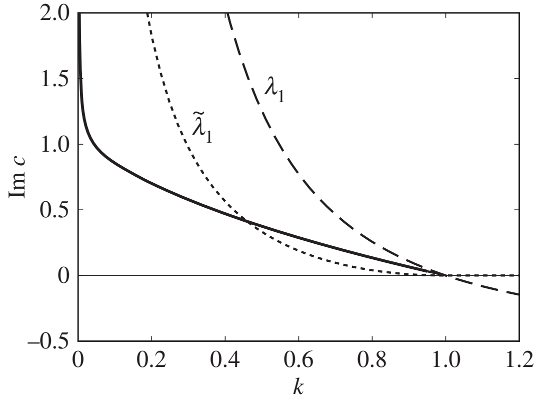

Figure 2. Growth rate Im c (where Re $c \equiv 0$), $\lambda_1 = \max Q/\|w\|^2_{L^2}$ and $\tilde{\lambda}_1 = \max Q/\|\xi\|^2_{L^2}$ versus wavenumber k for the shear flow $U(x) = \tanh(x)$.

The second example is

$$U(x) = x + 5x^3 + 1.62\tanh[4(x - 0.5)], \quad x \in [-1, 1], \tag{6.2}$$

which was previously addressed by Balmforth & Morrison [45]. This flow has three inflection points

$$\left.\begin{aligned} x_{I1} &= -0.069, \quad U_{I1} = -1.65,\\ x_{I2} &= 0.622, \quad U_{I2} = 2.55\\ \text{and} \quad x_{I3} &= 0.665, \quad U_{I3} = 3.07, \end{aligned}\right\} \tag{6.3}$$

at which $(U'^2)''$ is positive, negative and positive, respectively. Only for x_{I2} and x_{I3}, do the critical wavenumbers $k_2 \simeq 1.2$ and $k_3 \simeq 0.4$ exist. As predicted in corollary 5.4, the instability occurs only for finite wavenumbers $k_3 < k < k_2$. In figure 3, the positive signature of the maximum eigenvalue λ_1 certainly agrees with this unstable regime. In practice, our variational approach can directly prove the instability at a fixed k without knowing the existence of nor the values k_1, k_2 and k_3.

The third example is

$$U(x) = x - 0.02 + \frac{\sin[8(x - 0.02)]}{16}, \quad x \in [-1, 1], \tag{6.4}$$

which has five inflection points

$$\left.\begin{aligned} x_{I1} &= -0.765, \quad U_{I1} = -0.785,\\ x_{I2} &= -0.373, \quad U_{I2} = -0.393,\\ x_{I3} &= 0.020, \quad U_{I3} = 0.0,\\ x_{I4} &= 0.413, \quad U_{I4} = 0.393\\ \text{and} \quad x_{I5} &= 0.805, \quad U_{I5} = 0.785. \end{aligned}\right\} \tag{6.5}$$

For this example, there exist three critical wavenumbers k_1, k_3 and k_5 for the inflection points x_{I1}, x_{I3} and x_{I5}, all of which have $(U'^2)''$ negative. Therefore, three unstable eigenvalues emerge at k_1, k_3 and k_5 with different phase speeds U_{I1}, U_{I3} and U_{I5}, respectively. Thus, three eigenvalues λ_1, λ_2 and λ_3 of our variational problem completely predict the onsets of instabilities, as shown in figure 4.

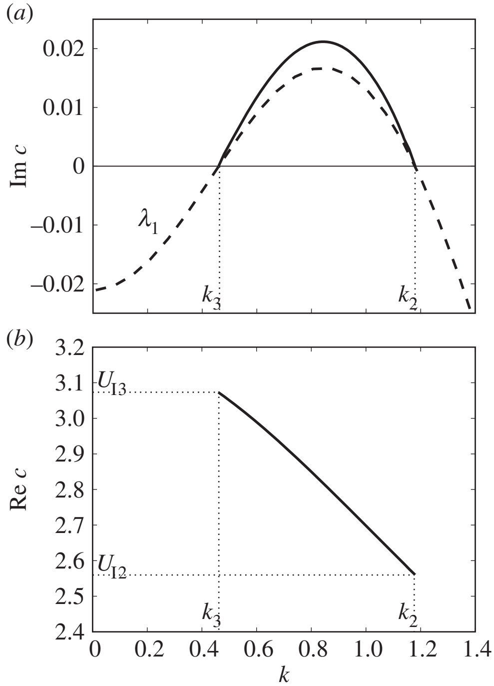

Figure 3. (*a*,*b*) Growth rate (Im *c*) and phase speed (Re *c*) versus wavenumber *k* for the shear flow $U(x) = x + 5x^3 + 1.62\tanh[4(x - 0.5)]$. The dashed line is $\lambda_1 = \max Q/\|w\|^2_{\mathbf{L}^2}$.

7. Summary

We have investigated the linear stability of inviscid plane parallel shear flow (Rayleigh's equation) as a typical example of an infinite-dimensional and non-self-adjoint eigenvalue problem that originates upon linearizing a Hamiltonian system. By assuming monotonicity and analyticity of the shear profile, a necessary and sufficient condition for spectral stability was obtained in the form of a variational criterion (theorem 3.1). Our theory is based on (i) the existence of the infinite number of constants of motion Q_f (proposition 3.2), whose definition includes an arbitrary real polynomial $f(c)$ and (ii) the rigorous derivation of the Kreĭn signature (i.e. the signature of $\delta^2 H$) for the continuous spectrum. Since the energy, $\delta^2 H = Q_f$ with $f(c) = c$, is generally indefinite due to the presence of both positive and negative energy modes, we have chosen a special $f(c)$ such that $Q = Q_f$ becomes negative semi-definite $Q|_{\sigma_c} \le 0$ (proposition 4.7) for the neutrally stable spectrum $\sigma_c \subset \mathbb{R}$, which is mostly the continuous spectrum in the present case. Then, a positive signature of the quadratic form $Q = \langle \xi, \mathcal{H}\xi \rangle$ implies existence of an unstable eigenmode. Since $\mathcal{H}$ is self-adjoint, we were able to prove instability must occur if some test function ξ (virtual displacement) exists that makes Q positive, which is analytically and numerically easier to do than solving Rayleigh's equation. Moreover, the singularity at the stability boundary (due to the continuous spectrum) was shown to be removed technically by maximizing Q with respect to the vorticity disturbance $w \in \mathbf{L}^2$, instead of the displacement $\xi \in \mathbf{L}^2$. However, we remark that, unlike the Rayleigh–Ritz method, neither $\max Q/\|\xi\|^2_{\mathbf{L}^2}$ nor $\max Q/\|w\|^2_{\mathbf{L}^2}$ are quantitatively related to the maximum growth rate of instability.

Our variational criterion is an improvement of previous sufficient stability criteria [11,33]. Given that Rayleigh's equation has been solved under a specific condition, we have also reproduced the earlier results of the Nyquist method [44,45] and Tollmien's analysis of the neutral modes [39–41].

In this paper, we have imposed the assumptions (A 1) and (A 2) on velocity profile $U(x)$ to simplify the discussion. The relaxation of these assumptions is possible to some extent, but it

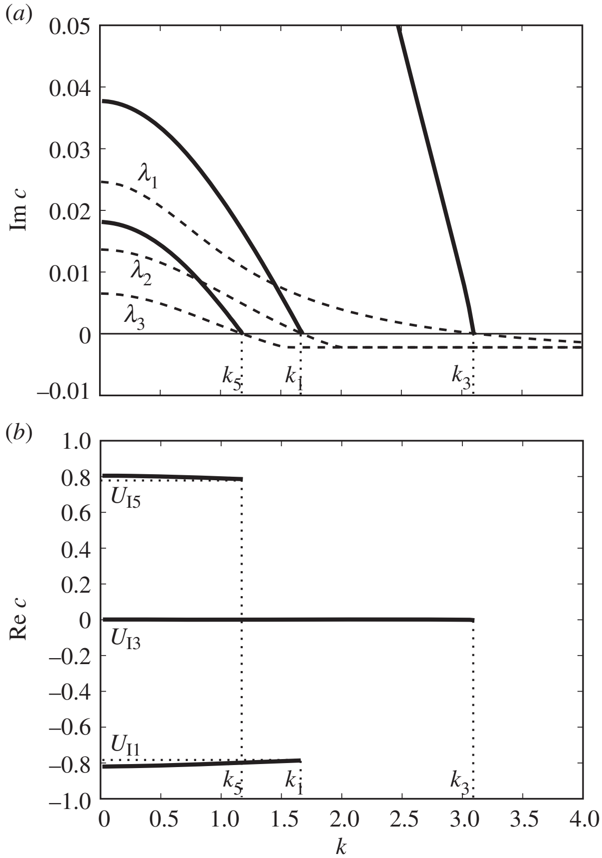

Figure 4. (*a*) Growth rate (Im c) and (*b*) phase speed (Re c) versus wavenumber k for the shear flow $U(x) = x - 0.02 + \sin[8(x - 0.02)]/16$. The dashed lines are eigenvalues λ_1, λ_2, λ_3 of $\mathcal{H}_v$.

would be difficult to overcome the following difficulties: (i) if analyticity is not assumed and U'' is only continuous, special care is needed for piecewise-linear regions of U. In such a region, say $[x_1, x_2]$, all points are regarded as inflection points and we expect that the variational problem would become a minimax problem like $\min_{x_I \in [x_1,x_2]} \max_{\xi \in \mathbf{L}^2} Q > 0$, which is not so analytically tractable. (ii) If monotonicity is not assumed, a serious difficulty arises when the sign of U'' is not identical at the locations of multiple critical layers for a phase speed $c = \omega/k \in \mathbb{R}$. Since at this frequency $\omega = kc$ belongs to degenerate multiple continuous spectra whose signature is indefinite, our technique for constructing Q breaks down.

In conclusion, we note that our variational approach will be applicable to rather simple equilibrium profiles which are free from the above difficulties. However, there is a large class of fluid and plasma systems with existing sufficient stability criteria (e.g. MHD [52,53] with flow) that have Kreĭn-like signature (or action–angle variables) for a continuous spectrum. This is the key ingredient needed for constructing the quadratic form. Thus, our techniques are available for a large class of applications governed by other dynamical systems. We will report our additional results in future publications.

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Appendix A. Brief review of the isovortical variation

Consider the vorticity equation $\partial_t \boldsymbol{w} = \mathrm{curl}(\boldsymbol{u} \times \boldsymbol{w})$ where $\boldsymbol{w} = \mathrm{curl}\,\boldsymbol{u}$. For a given displacement vector field $\boldsymbol{\xi}$, let us generate a variation $\boldsymbol{w}_0 \to \boldsymbol{w}_\epsilon$ around a steady-state $\boldsymbol{w}_0 = \mathrm{curl}\,\boldsymbol{u}_0$ by solving

$$\partial_\epsilon \boldsymbol{w}_\epsilon = \mathrm{curl}(\boldsymbol{\xi} \times \boldsymbol{w}_\epsilon), \quad \boldsymbol{w}_\epsilon|_{\epsilon=0} = \boldsymbol{w}_0, \tag{A 1}$$

in terms of a small parameter $\epsilon \in \mathbb{R}$. This so-called isovortical variation automatically preserves the Kelvin's circulation law (or the topology of the vorticity $\boldsymbol{w}_0$), where we usually expand $\boldsymbol{w}_\epsilon$ as

$$\boldsymbol{w}_\epsilon = \boldsymbol{w}_0 + \epsilon\,\mathrm{curl}(\boldsymbol{\xi} \times \boldsymbol{w}_0) + \frac{\epsilon^2}{2}\mathrm{curl}[\boldsymbol{\xi} \times \mathrm{curl}(\boldsymbol{\xi} \times \boldsymbol{w}_0)] + O(\epsilon^3) \tag{A 2}$$

$$=: \boldsymbol{w}_0 + \epsilon\delta\boldsymbol{w}_0 + \frac{\epsilon^2}{2}\delta^2\boldsymbol{w}_0 + O(\epsilon^3). \tag{A 3}$$

For the steady-state $\boldsymbol{w}_0 = (0, 0, U'(x))$ and two-dimensional motion $\boldsymbol{\xi} = (\xi_x, \xi_y, 0)$ discussed in this paper, the first variation $\delta\boldsymbol{w}_0 = \mathrm{curl}(\boldsymbol{\xi} \times \boldsymbol{w}_0)$ is reduced to $\delta\boldsymbol{w}_0 = (0, 0, -U''\xi_x)$. Then, the second variation of the Hamiltonian $H = \int |\boldsymbol{u}|^2/2d^3x$ around $\boldsymbol{u}_0$ results in (3.3).

If the dynamics of $\boldsymbol{\xi}$ is taken into account, it must be related to $\delta\boldsymbol{u}_0 = \mathcal{P}(\boldsymbol{\xi} \times \boldsymbol{w}_0)$ (where $\mathcal{P}$ is the projection operator to the solenoidal vector field) by

$$\partial_t\boldsymbol{\xi} + (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{\xi} - (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u}_0 = \delta\boldsymbol{u}_0 + O(\epsilon) \tag{A 4}$$

(see Sec. 6 of [54] or [52,53]). When $O(\epsilon)$ is neglected in the linear analysis, this is indeed the adjoint equation of the linearized vorticity equation. In this paper, its x component $\partial_t\xi_x + U\partial_y\xi_x = \delta u_{0x}$ corresponds to (3.1).

Appendix B. Proof of $\sigma_c \backslash \sigma_I$ being the continuous spectrum

Let $\omega = kc \in \sigma_c\backslash\sigma_I$, namely, $\omega = kU(x_c)$ and $W(\omega \pm \mathrm{i}0) \neq 0$. Using lemma 4.3, one finds $\Phi_<(x_c, \omega) \neq 0$ and $\Phi_>(x_c, \omega) \neq 0$, which implies that there exists $\xi_0 \in X + \mathrm{i}X$ such that $\Phi(x_c, \omega \pm \mathrm{i}0) \neq 0$. Then, $\Xi(x, \Omega) = (\Omega - k\mathcal{L}^*)^{-1}\xi_0 = -k\Phi(x, \Omega)/(\Omega - kU)$ becomes singular $\Xi(x, \omega \pm \mathrm{i}0) \notin X + \mathrm{i}X$ as $\Omega \to \omega \pm \mathrm{i}0$, namely, the resolvent operator $(\omega - k\mathcal{L}^*)^{-1}$ is unbounded.

Moreover, the range of $\omega - k\mathcal{L}^*$ is dense in $X + \mathrm{i}X$ as shown below. Therefore, ω is not in the residual spectrum but the continuous spectrum.

Lemma B.1. *Let $\omega = kc \in \sigma_c\backslash\sigma_I$. For any given $\eta_0 \in X + \mathrm{i}X$ and $\epsilon > 0$,*

$$\exists\eta \in X + \mathrm{i}X \quad \text{s.t.} \quad \|(\omega - k\mathcal{L}^*)\eta - \eta_0\|_{X+\mathrm{i}X} < \epsilon. \tag{B 1}$$

Proof. Define a neighbourhood of x_c as $B_{\epsilon_1} := [x_c - \epsilon_1, x_c + \epsilon_1]$ with $0 < \epsilon_1 \in \mathbb{R}$. Let us consider $\xi_0 = \mathcal{G}\theta \in X + \mathrm{i}X$ where $\theta \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ is given by

$$\theta := \begin{cases} -\eta_0'' + k^2\eta_0 & \text{on } [-L, L]\backslash B_{\epsilon_1}, \\ 0 & \text{on } B_{\epsilon_1}. \end{cases} \tag{B 2}$$

As in (4.11), we generate $\Phi(x, \Omega)$ with this ξ_0 and define $\phi_*(x, \omega)$ as

$$\phi_*(x, \omega) = \frac{\Phi(x_c, \omega - \mathrm{i}0)\Phi(x, \omega + \mathrm{i}0) - \Phi(x_c, \omega + \mathrm{i}0)\Phi(x, \omega - \mathrm{i}0)}{\Phi(x_c, \omega - \mathrm{i}0) - \Phi(x_c, \omega + \mathrm{i}0)} \in \mathbf{H}_0^1 + \mathrm{i}\mathbf{H}_0^1, \tag{B 3}$$

which indeed exists due to proposition 4.5. Then, ϕ_* satisfies $\phi_*(x_c, \omega) = 0$ and, from (4.6),

$$-\phi_*'' + k^2\phi_* - \frac{U''}{c - U}\phi_* = -\frac{1}{k}\theta. \tag{B 4}$$

In the neighbourhood B_{ϵ_1}, this ϕ_* must be the *regular* Frobenius series solution $\phi_*(x, \omega) = \mathrm{const.} \times \Phi_1(x, \omega)$ since the right-hand side of (B 4) is zero and $\phi_*(x_c, \omega) = 0$ (recall (4.18), where $\Phi_1(x_c, \omega) = 0$

and $\Phi_2(x_c, \omega) = 1$). If we set

$$\eta = -\frac{\phi_*}{c - U}, \tag{B 5}$$

then this η is still regular on B_{ϵ_1} and hence $\eta \in \mathbf{H}_0^1 + \mathrm{i}\mathbf{H}_0^1$. Moreover, $\eta \in X + \mathrm{i}X$ because $\phi_*'' \in \mathbf{L}^2 + \mathrm{i}\mathbf{L}^2$ follows from (B 4). The relation (B 4) is transformed into $(\omega - k\mathcal{L}^*)\eta = \mathcal{G}\theta$, which implies $(\omega - k\mathcal{L}^*)\eta \in X + \mathrm{i}X$. Therefore, we obtain $(\omega - k\mathcal{L}^*)\eta - \eta_0 = \mathcal{G}(\theta + \eta_0'' - k^2\eta_0)$, where

$$\|\theta + \eta_0'' - k^2\eta_0\|_{\mathbf{L}^2+\mathrm{i}\mathbf{L}^2} \leq 2\epsilon_1 \sup_{x\in[-L,L]} |\eta_0'' - k^2\eta_0|. \tag{B 6}$$

By adopting the definition $\|\cdot\|_{X+\mathrm{i}X} := \|\mathcal{G}^{-1}\cdot\|_{\mathbf{L}^2+\mathrm{i}\mathbf{L}^2}$ for simplicity, the required result is obtained by making ϵ_1 small such that $2\epsilon_1 \sup_{x\in[-L,L]} |\eta_0'' - k^2\eta_0| < \epsilon$. ■

References

1. Drazin PG, Howard LN. 1966 Hydrodynamic stability of parallel flow of inviscid fluid. *Adv. Appl. Mech.* **9**, 1–89. (doi:10.1016/S0065-2156(08)70006-1)
2. Bernstein IB, Frieman EA, Kruskal MD, Kulsrud RM. 1958 An energy principle for hydromagnetic stability problems. *Proc. R. Soc. Lond. A* **244**, 17–40. (doi:10.1098/rspa.1958.0023)
3. Morrison PJ. 1998 Hamiltonian description of the ideal fluid. *Rev. Mod. Phys.* **70**, 467–521. (doi:10.1103/RevModPhys.70.467)
4. Rayleigh JWS. 1945 (reissue of 1877 version) *The theory of sound*, 2nd edn. New York, NY: Dover.
5. Kreǐn MG. 1950 A generalization of some investigations on linear differential equations with periodic coefficients. *Dokl. Akad. Nauk SSSR A* **73**, 445–448.
6. Moser JK. 1958 New aspects in the theory of stability of Hamiltonian systems. *Comm. Pure Appl. Math.* **11**, 81–114. (doi:10.1002/cpa.3160110105)
7. MacKay R. 1986 Stability of equilibria of Hamiltonian systems. In *Nonlinear Phenomena and Chaos* (ed. S Sarkar), pp. 254–270. Bristol, UK: Adam Hilger.
8. Morrison PJ, Kotschenreuther M. 1990 The free energy principle, negative energy modes and stability. In*Nonlinear World: IV International Workshop on Nonlinear and Turbulent Processes in Physics* (eds VG Bar'yakhtar, VM Chernousenko, NS Erokhin, AB Sitenko, VE Zakharov), pp. 910–932. Singapore: World Scientific.
9. Oberman C, Kruskal M. 1965 Some constants of the linearized motion of Vlasov plasmas. *J. Math. Phys.* **6**, 327–335. (doi:10.1063/1.1704284)
10. Case KM. 1965 Constants of the linearized motion of Vlasov plasmas. *Phys. Fluids* **8**, 96–101. (doi:10.1063/1.1761106)
11. Barston EM. 1991 On the linear stability of inviscid incompressible plane parallel flow. *J. Fluid Mech.* **233**, 157–163. (doi:10.1017/S0022112091000435)
12. Morrison PJ, Pfirsch D. 1992 Dielectric energy versus plasma energy, and Hamiltonian action–angle variables for the Vlasov equation. *Phys. Fluids B* **4**, 3038–3057. (doi:10.1063/1.860415)
13. Morrison PJ. 2000 Hamiltonian description of Vlasov dynamics: action–angle variables for the continuous spectrum. *Trans. Theory Stat. Phys.* **29**, 397–414. (doi:10.1080/00411450008205881)
14. Balmforth NJ, Morrison PJ. 2002 Hamiltonian description of shear flow. In *Large-scale atmosphere–ocean dynamics II* (eds J Norbury, I Roulstone), pp. 117–142. Cambridge, UK: Cambridge University Press.
15. Morrison PJ. 2003 Hamiltonian description of fluid and plasma systems with continuous spectra. In *Nonlinear processes in geophysical fluid dynamics* (eds OU Velasco Fuentes, J Sheinbaum, J Ochoa), pp. 53–69. Dordrecht, The Netherlands: Kluwer.
16. Hirota M, Fukumoto Y. 2008 Energy of hydrodynamic and magnetohydrodynamic waves with point and continuous spectra. *J. Math. Phys.* **49**, 083101. (doi:10.1063/1.2969275)
17. Hirota M. 2010 Action–angle representation of waves and instabilities in flowing plasmas. *J. Plasmas Fusion Res. SERIES* **9**, 463–470.
18. Hagstrom GI, Morrison PJ. 2010 On Krein-like theorems for noncanonical Hamiltonian systems with continuous spectra: application to Vlasov–Poisson. *Trans. Theory Stat. Phys.* **39**, 466–501. (doi:10.1080/00411450.2011.566484)

19. Morrison PJ, Hagstrom GI. 2014 Continuum Hamiltonian Hopf bifurcation I. In *Nonlinear physical systems—spectral analysis, stability and bifurcations* (eds O Kirillov, D Pelinovsky), pp. 247–282. London, UK: ISTE; Hoboken, NJ: John Wiley.
20. Hagstrom GI, Morrison PJ. 2014 Continuum Hamiltonian Hopf bifurcation II. In *Nonlinear physical systems—spectral analysis, stability and bifurcations* (eds O Kirillov, D Pelinovsky), pp. 283–310. London, UK: ISTE; Hoboken, NJ: John Wiley.
21. Grillakis M. 1990 Analysis of the linearization around a critical point of an infinite dimensional Hamiltonian system. *Comm. Pure Appl. Math.* **43**, 299–333. (doi:10.1002/cpa.3160430302)
22. Kapitula T, Kevrekidis PG, Sandstede B. 2004 Counting eigenvalues via the Krein signature in infinite-dimensional Hamiltonian systems. *Physica D* **195**, 263–282. (doi:10.1016/j.physd.2004.03.018)
23. Kirillov O, Pelinovsky D. 2014 *Nonlinear physical systems—spectral analysis, stability and bifurcations*. London, UK: ISTE; Hoboken, NJ: John Wiley.
24. Rayleigh JWS. 1880 On the stability, or instability, of certain fluid motions. *Proc. Lond. Math. Soc.* **11**, 57–72.
25. Fjørtoft R. 1950 Application of integral theorems in deriving criteria of stability for laminar flows and for the baroclinic circular vortex. *Geofys. Publ.* **17**, 1–52.
26. Arnold VI. 1966 An a priori estimate in the theory of hydrodynamic stability. *Izv. Vyssh. Uchebn. Zaved. Mat.* [*Sov. Math. J.*] **5**, 3–5.
27. Arnold VI. 1966 Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* **16**, 319–361. (doi:10.5802/aif.233)
28. Arnold VI. 1969 The Hamiltonian nature of the Euler equations in the dynamics of a rigid body and of an ideal fluid. *Usp. Mat. Nauk.* [*Sov. Math. Usp.*] **24**, 225–226. (doi:10.5802/air.233)
29. Kruskal MD, Oberman C. 1958 On the stability of plasma in static equilibrium. *Phys. Fluids* **1**, 275–280. (doi:10.1063/1.1705885)
30. Hazeltine RD, Holm DD, Marsden JE, Morrison PJ. 1984 Generalized Poisson brackets and nonlinear Liapunov stability-application to reduced MHD. In *International Conference on Plasma Physics Proceedings 1* (eds MQ Tran, ML Sawley), p. 203. Lausanne: Ecole Polytechnique Federale de Lausanne.
31. Holm DD, Marsden JE, Ratiu T, Weinstein A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**, 1–116. (doi:10.1016/0370-1573(85)90028-6)
32. Morrison PJ, Eliezer S. 1986 Spontaneous symmetry breaking and neutral stability in the noncanonical Hamiltonian formalism. *Phys. Rev. A* **33**, 4205–4214. (doi:10.1103/PhysRevA.33.4205)
33. Arnold VI. 1978 *Mathematical methods of classical mechanics*. Berlin, Germany: Springer.
34. Morrison PJ, Pfirsch D. 1989 Free-energy expressions for Vlasov equilibria. *Phys. Rev. A* **40**, 3898–3910. (doi:10.1103/PhysRevA.40.3898)
35. Morrison PJ, Pfirsch D. 1990 The free energy of Maxwell–Vlasov equilibria. *Phys. Fluids B* **2**, 1105–1113. (doi:10.1063/1.859246)
36. Morrison PJ. 1982 Poisson brackets for fluids and plasmas. In *Mathematical methods in hydrodynamics and integrability in related dynamical systems* (eds M Tabor, Y Treve), pp. 13–46. AIP Conference Proceedings no. 88. New York, NY: AIP.
37. Maddocks J, Sachs R. 1993 On the stability of KdV multi-solitons. *Comm. Pure Appl. Math.* **46**, 867–901. (doi:10.1002/cpa.3160460604)
38. Bottman N, Nivala M, Deconinck B. 2011 Elliptic solutions of the defocusing NLS equation are stable. *J. Phys. A* **44**, 285201. (doi:10.1088/1751-8113/44/28/285201)
39. Tollmien W. 1935 Ein allgemeines Kriterium der Instabilitat laminarer Gescgwindigkeits-verteilungen. *Nachr. Wiss Fachgruppe, Göttingen, Math. Phys.* **1**, 79–114.
40. Lin CC. 1945 On the stability of two-dimensional parallel flows Part II—stability in an inviscid fluid. *Quart. Appl. Math.* **3**, 218–234.
41. Lin CC. 1955 *The theory of hydrodynamic stability*. Cambridge, UK: Cambridge University Press.
42. Lin Z. 2003 Instability of some ideal plane flows. *SIAM J. Math. Anal.* **35**, 318–356. (doi:10.1137/S0036141002406266)
43. Lin Z. 2005 Some recent results on instability of ideal plane flows. *Contemp. Math.* **371**, 217–229. (doi:10.1090/conm/371/06857)
44. Rosenbluth MN, Simon A. 1964 Necessary and sufficient conditions for the stability of plane parameter inviscid flow. *Phys. Fluids* **7**, 557–558. (doi:10.1063/1.1711237)

45. Balmforth NJ, Morrison PJ. 1999 A necessary and sufficient instability condition for inviscid shear flow. *Stud. Appl. Math.* **102**, 309–344. (doi:10.1111/1467-9590.00113)
46. Chen XL, Morrison PJ. 1991 A sufficient condition for the ideal instability of shear flow with parallel magnetic field. *Phys. Fluids B* **3**, 863–865. (doi:10.1063/1.859841)
47. Rosencrans SI, Sattinger DH. 1966 On the spectrum of an operator occurring in the theory of hydrodynamic stability. *J. Math. Phys.* **45**, 289–300.
48. Case KM. 1960 Stability of inviscid plane Couette flow. *Phys. Fluids* **3**, 143–148. (doi:10.1063/1.1706010)
49. Drazin PG, Reid WH. 1981 *Hydrodynamic stability*. Cambridge, UK: Cambridge University Press.
50. Whittaker ET. 1947 *A treatise on the analytical dynamics of particles and rigid bodies*, 4th edn. Cambridge, UK: Cambridge University Press.
51. Ince EL. 1926 *Ordinary differential equations*. New York, NY: Dover.
52. Frieman E, Rotenberg M. 1960 On hydromagnetic stability of stationary equilibria. *Rev. Mod. Phys.* **32**, 898–902. (doi:10.1103/RevModPhys.32.898)
53. Andreussi T, Morrison PJ, Pegoraro F. 2013 Hamiltonian magnetohydrodynamics: Lagrangian, Eulerian, and dynamically accessible stability-theory. *Phys. Plasmas* **20**, 092104. (doi:10.1063/1.4819779)
54. Fukumoto Y, Hirota M. 2008 Elliptical instability of a vortex tube and drift current induced by it. *Phys. Scr.* **T132**, 014041. (doi:10.1088/0031-8949/2008/T132/014041)