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On energy conservation in extended magnetohydrodynamics

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A systematic study of energy conservation for extended magnetohydrodynamic models that include Hall terms and electron inertia is performed. It is observed that commonly used models do not conserve energy in the ideal limit, i.e., when viscosity and resistivity are neglected. In particular, a term in the momentum equation that is often neglected is seen to be needed for conservation of energy. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4890955]

I. INTRODUCTION

Ideal magnetohydrodynamics (MHD) has a long history of wide application to various types of plasmas, including those of relevance in astrophysics, geophysics, and nuclear fusion science. However, it is well-known to plasma physicists that ideal MHD is deficient in many respects and that some of these deficiencies are accounted for by extending Ohm's law as in the early work of Refs. 1 and 2 (see also Refs. 3 and 4). And, it is also well-known that the inclusion of additional terms in Ohm's law breaks the frozen flux condition of ideal MHD and gives rise to specific regions where magnetic reconnection takes place and that this phenomenon is important for energy transfer. Reconnection is most well-studied when it is induced by resistivity (e.g., Ref. 5), but there is significant work on the Hamiltonian reconnection afforded by other effects such as electron inertia (e.g., Refs. 6–8). Because a variety of effects can be important, many researchers have investigated various extended MHD models both analytically and numerically, and many reduced models based on geometric reduction have been used as well (e.g., Refs. 9–11).

Different versions of extended MHD models have been implemented, e.g., Refs. 2, 5, and 12–14, and some of the relationships between these models and their limitations seem to be unknown. All of these models differ only in the choice of the generalization of Ohm's law, with the momentum equation being the same as that for usual MHD. We will see that in some instances, energy conservation requires modification of the momentum equation.

The goal of this paper is to sort out which extended MHD models conserve energy and which do not. Upon returning to the original two-fluid derivation of extended MHD of Lüst,[1] we find that energy conservation requires a term that is often neglected in the momentum equation and that retention of this term is consistent with the appropriate ordering. Various reductions of the full extended MHD model are investigated, including Hall MHD (HMHD) and Inertial MHD (IMHD).

This paper is organized as follows. In Sec. II, we describe briefly extended MHD and generalize the thermodynamics of the fluid to allow for more general equations of state with the inclusion of electron pressure and anisotropic pressure of the form of Chew, Goldberger, and Low.[15] Next, in Sec. III, we begin our discussion of energy conservation. We first consider HMHD, which is a well-known consistent model in its own right and verify its energy conservation including the generalized thermodynamics. Then, we introduce IMHD by employing an ordering of the full extended MHD in which the terms of Ohm's law of HMHD are dominated. Thus, we are able to treat the energy conservation of IMHD independently, but our results apply to the full extended MHD model without the ordering. Since various IMHD-like models in the literature do not conserve energy, these are of main concern. For this reason, in Sec. IV, we systematically determine which MHD models with electron inertia conserve energy and which do not—the results are summarized in Table I. Finally, we conclude in Sec. V, where we discuss some limitations and possibilities.

II. EXTENDED MHD AND THERMODYNAMICS

In this section, we first state the extended MHD model. As is well-known, such a one-fluid model can be derived from kinetic theory (see, e.g., Refs. 2, 13, and 16), but we begin with the results of Lüst,[1] who appears to be the first to derive the generalized Ohm's law for a one-fluid model by adding and subtracting individual electron and ion fluid equations, enforcing quasineutrality and expanding in the smallness of the electron mass. His derivation yields a term in the one-fluid momentum equation that is often neglected and is necessary for energy conservation. Next, we extend Lüst's model by completing the thermodynamics and, in addition, we show how one can incorporate anisotropic pressure into the thermodynamics.

A. Extended MHD

The assumptions of quasineutrality and smallness of the electron mass compared to the ion mass leads to a model that we will refer to as *extended MHD*. It is given by the following:

the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho \boldsymbol{V}), \tag{1}$$

the momentum equation

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TABLE I. Classification of energy conserving IMHD models. The values of the epsilons in the generalized Ohm's law are listed in the first four columns, and the generalized Ohm's law is described by the fifth column. The epsilon values in the momentum equation are listed in the sixth column. When the total energy is conserved "Yes" is written in the last column, otherwise the deficit terms are written in the last column. Note that for incompressible plasma, there is no $\epsilon_{\rm cp}$—term in the generalized Ohm's law; consequently, we write "–" in the third column for this case.

$\epsilon_{\rm ti}$	$\epsilon_{\rm ad}$	$\epsilon_{\rm cp}$	$\epsilon_{\rm ohm}$	Ohm's law $\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}=$	$\epsilon_{\rm mom}$	Conserved?
Compressible plasma						
1	1	1	1	$\frac{m_e}{e^2n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{VJ}+\boldsymbol{JV})\right)$	1	Yes
1	1	1		$\frac{m_e}{e^2n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{VJ})\right)$		Yes
1				$\frac{m_e}{e^2n}\frac{\partial \boldsymbol{J}}{\partial t}$		$\frac{m_e}{e^2n}\frac{\lvert\boldsymbol{J}\rvert^2}{2}\frac{\nabla\cdot(n\boldsymbol{V})}{n}$
1	1			$\frac{m_e}{e^2n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{J}\right)$		$\frac{m_e}{e^2n}\lvert\boldsymbol{J}\rvert^2(\nabla\cdot\boldsymbol{V})$
1	1	1	1	$\frac{m_e}{e^2n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{VJ}+\boldsymbol{JV})\right)$		$\frac{m_e}{e}\boldsymbol{V}\cdot\left((\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}\right)$
Incompressible plasma						
1		–		$\frac{m_e}{e^2n_0}\frac{\partial \boldsymbol{J}}{\partial t}$	1	Yes
1	1	–		$\frac{m_e}{e^2n_0}\left(\frac{\partial \boldsymbol{J}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{J}\right)$		Yes
1	1	–	1	$\frac{m_e}{e^2n_0}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{VJ}+\boldsymbol{JV})\right)$		Yes
1	1	–	1	$\frac{m_e}{e^2n_0}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{VJ}+\boldsymbol{JV})\right)$		$\frac{m_e}{e^2n_0}\boldsymbol{V}\cdot((\boldsymbol{J}\cdot\nabla)\boldsymbol{J})$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right)=-\nabla p+\boldsymbol{J}\times\boldsymbol{B}-\frac{m_e}{e}(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}\,, \tag{2}$$

and the generalized Ohm's law

$$\begin{aligned}\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}-\frac{\boldsymbol{J}}{\sigma}&=\frac{1}{en}(\boldsymbol{J}\times\boldsymbol{B}-\nabla p_e)\\&\quad+\frac{m_e}{e^2n}\left[\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{VJ}+\boldsymbol{JV})\right]\\&\quad-\frac{m_e}{e^2n}(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}\,,\end{aligned} \tag{3}$$

where ρ is the mass density of plasma, $\boldsymbol{V}$ is the bulk velocity, p is the pressure, $\boldsymbol{B}$ is the magnetic field, $\boldsymbol{J}=\nabla\times\boldsymbol{B}/\mu_0$ is the current density, m_e is the electron mass, e is the elementary charge, n is the number density of each species of charged particles, $\boldsymbol{E}$ is the electric field, σ is the conductivity, and p_e is the electron pressure.

Although this generalized Ohm's law arises upon subtracting the individual electron and ion fluid momentum equations, with velocity fields $\mathbf{v}_e$ and $\mathbf{v}_i$, respectively, and contains electron momentum dynamics, we will refer to this system of Eqs. (1)–(3), supplemented by the thermodynamics of Sec. II B, as a single fluid model.

B. Extended thermodynamics

In the original article of Lüst[1] and to our knowledge elsewhere, the thermodynamics of the two-fluid to one-fluid extended MHD reduction has not been treated in full generality. Lüst assumes polytropic laws for p_i and p_e and constructs the extended MHD pressure as $p=p_i+p_e$, in accordance with Dalton's law on the addition of partial pressures. Because of quasineutrality, $\rho=\rho_i+\rho_e=m_in+m_en=mn$, where $m=m_i+m_e$.

Here, we generalize this by using entropy as the second thermodynamic variable for each species. With this choice, the thermodynamics of species $\alpha\in\{e,i\}$ is determined from an internal energy function $\mathcal{U}_\alpha(\rho_\alpha,s_\alpha)$, the internal energy per unit mass m_α, where s_α is the entropy per unit mass m_α. Since $1/\rho_\alpha$ is the specific volume, this thermodynamic representation is the one in terms extensive variables, with the intensive quantities determined by

$$T_\alpha=\frac{\partial \mathcal{U}_\alpha}{\partial s_\alpha}\quad\text{and}\quad p_\alpha=\rho_\alpha^2\frac{\partial \mathcal{U}_\alpha}{\partial \rho_\alpha}\,. \tag{4}$$

For isothermal processes $\mathcal{U}_\alpha=\kappa(s_\alpha)\ln(\rho_\alpha)$, while for a polytropic equation of state, $\mathcal{U}_\alpha=\kappa(s_\alpha)\rho_\alpha^{\gamma-1}/(\gamma-1)$, whence $p_\alpha=\kappa(s_\alpha)\rho_\alpha^\gamma$. For these choices, one can substitute the variable p_α in lieu of s_α, as is more common in plasma physics.

For the ideal, energy conserving, fluid, the Fourier and other heat flux terms are dropped and the two entropies obey the advection equations

$$\frac{\partial s_\alpha}{\partial t}+\mathbf{v}_\alpha\cdot\nabla s_\alpha=0\,. \tag{5}$$

Since entropy is extensive, it is natural to introduce the total entropy for one-fluid extended MHD as follows:

$$s=(m_is_i+m_es_e)/m\,. \tag{6}$$

In addition, if Hall MHD is to have a complete set of thermodynamic variables, one must retain the electron entropy, s_e. Using $\boldsymbol{V}=(m_i\boldsymbol{v}_i+m_e\boldsymbol{v}_e)/m$ and $\boldsymbol{J}=en(\boldsymbol{v}_i-\boldsymbol{v}_e)$, a simple calculation gives

$$\frac{\partial s}{\partial t}=-\boldsymbol{V}\cdot\nabla s \tag{7}$$

and

$$\frac{\partial s_e}{\partial t}=-\boldsymbol{V}\cdot\nabla s_e+\frac{1}{en}\boldsymbol{J}\cdot\nabla s_e\,, \tag{8}$$

where we have dropped terms of order m_e/m.

Again, appealing to the extensive property of energy, the total internal energy per unit volume of the two species is given by

$$\rho\mathcal{U}=\rho_i\mathcal{U}_i(\rho_i,s_i)+\rho_e\mathcal{U}_e(\rho_e,s_e)\,. \tag{9}$$

Because of quasineutrality, n is the only density variable; for our purposes, it is sufficient to rewrite this, again correct to order m_e/m, as

$$n\mathfrak{U}(n,s,s_e)=n\mathfrak{U}_i(n,s)+n\mathfrak{U}_e(n,s_e)\,, \tag{10}$$

where $m\mathcal{U}=:\mathfrak{U}$ and $m_\alpha\mathcal{U}_\alpha=:\mathfrak{U}_\alpha$. Evidently, the pressures satisfy $p=\rho^2\partial\mathcal{U}/\partial\rho=n^2\partial\mathfrak{U}/\partial n=p_i+p_e$. Consistent with the derivation of extended MHD, we can expand (10) in the smallness of m_e/m_i. Because the only thermodynamic deviation from single fluid MHD occurs in the Hall term ∇p_e, it is sufficient to retain only the leading order in this expansion in order to ensure energy conservation. This will be shown in Sec. III A where we examine the total energy conservation for Hall MHD.

Since it is not widely known, we review here the thermodynamics for anisotropic pressure in a magnetofluid model, which to our knowledge was first given in Ref. 17 (see also Ref. 18). The generalization follows upon adding $B = |\boldsymbol{B}|$ as an additional thermodynamic variable, i.e., now $\mathcal{U}(\rho, s, B)$, and the parallel and perpendicular pressures are given by

$$p_{||} = \rho^2 \frac{\partial \mathcal{U}}{\partial \rho} \quad \text{and} \quad \Delta p = -\rho B \frac{\partial \mathcal{U}}{\partial B}\,, \tag{11}$$

where $\Delta p = p_{||} - p_{\perp}$.

Expressions (11) can be seen to be consistent with the natural extensive thermodynamic variables (ρ^{-1}, s, B), all being specific quantities. The intensive thermodynamic dual variables associate with this set are ($p_{||}$, T, M) with the magnetization M being given by

$$M := \frac{\Delta p}{\rho B} = -\frac{\partial \mathcal{U}}{\partial B}\,. \tag{12}$$

In (12), $\Delta p = p_{||} - p_{\perp}$ is related to the work done when magnetic flux is held fixed. Note the magnetic moment $\mu = mv_{\perp}^2/(2B)$; thus, the magnetization per unit volume is $\mathcal{M} = nmv_{\perp}^2/(2B)$ or one can argue macroscopically $\mathcal{M} \sim p/B$. Therefore, the specific magnetization would be $M \sim p/(\rho B)$; thus, (12) makes sense as a relative magnetization. Alternatively, one can use H as the variable thermodynamically conjugate to B by introducing the "total" energy, $\mathcal{U}_{tot} = \rho\mathcal{U} + B^2/2\mu_0$. Then, the conjugate to B is given by $\partial\mathcal{U}_{tot}/\partial B = H = -\mathcal{M} + B/\mu_0$, as expected. In Ref. 17, it was shown that this way of introducing anisotropy is energy conserving. For simplicity, in the following, we will restrict to isotropic pressure, but the generalization is straightforward.

III. ENERGY CONSERVATION

Because the examination of general energy conservation of extended MHD is complicated, we divide the calculation into two parts. We first consider HMHD and then its complement IMHD. Results for total energy follow upon superposing the calculations. In addition, we give an ordering for IMHD, an energy conserving model in its own right.

A. Hall MHD

Setting $m_e = 0$ in (2) and (3) gives resistive HMHD. Since electron inertia is absent, the energy is expected to be composed of the sum of kinetic, internal, and magnetic, i.e.,

$$H_H := \int_D d^3x \left(\rho \frac{|\boldsymbol{V}|^2}{2} + \rho\,\mathcal{U} + \frac{|\boldsymbol{B}|^2}{2\mu_0} \right), \tag{13}$$

where it remains to determine the function $\mathcal{U}$ that will ensure conservation of H_H when the resistivity σ^{-1} is set to zero. Determination of $\mathcal{U}$ will be tantamount to the determination of the entropy dynamics.

Upon calculating dH_H/dt, it is readily seen that the MHD terms cancel as usual and that the Hall term $\boldsymbol{J}\times\boldsymbol{B}$ produces the energy flux $\boldsymbol{B}\times(\boldsymbol{B}\times\boldsymbol{J})/(\mu_0 en)$. Consequently, only the thermodynamic terms are of concern for the energy of (13) to satisfy a conservation law. Assuming $\mathcal{U}(\rho, s, s_e)$ and $p = \rho^2\partial\mathcal{U}/\partial\rho$, one obtains the usual internal energy flux of MHD, $(p + \rho\mathcal{U})\boldsymbol{V}$. Thus, we are left with the following upon neglect of surface terms:

$$\frac{dH_H}{dt} = \int_D d^3x \left(\frac{1}{en}\boldsymbol{J}\cdot\nabla p_e + \frac{\rho}{en}\frac{\partial\mathcal{U}}{\partial s_e}\boldsymbol{J}\cdot\nabla s_e \right), \tag{14}$$

where use has been made of (7) and (8).

For barotropic electron pressure, $\mathcal{U}$ has no dependence on the electron entropy s_e and, consequently, only the first term of (14) is present and $\nabla p_e/(en) = \nabla(\rho\mathcal{U})'/e$, where prime denotes d/dn. Then, upon integration by parts we obtain

$$\frac{dH_H}{dt} = \int_D d^3x \left(\boldsymbol{J}\cdot\nabla(\rho\mathcal{U})'/e\right) = \int_D d^3x\,\nabla\cdot\left(\boldsymbol{J}(\rho\mathcal{U})'/e\right), \tag{15}$$

using $\nabla\cdot\boldsymbol{J} = 0$. Thus, for this case, the energy flux is $-\boldsymbol{J}(\rho\mathcal{U})'/e$.

The barotropic model is incomplete, since electron pressure can change at fixed electron density. To account for this, we have included the electron entropy in the dynamics via $\mathcal{U}$. Using (10) with $p_e = n^2\partial\mathfrak{U}_e/\partial n$ and $\rho\partial\mathcal{U}/\partial s_e = n\partial\mathfrak{U}_e/\partial s_e$, (14) becomes

$$\begin{aligned}
\frac{dH_H}{dt} &= \frac{1}{e}\int_D d^3x \left[\frac{1}{n}\boldsymbol{J}\cdot\nabla\left(n^2\frac{\partial\mathfrak{U}_e}{\partial n}\right) + \frac{\partial\mathfrak{U}_e}{\partial s_e}\boldsymbol{J}\cdot\nabla s_e \right],\\
&= \frac{1}{e}\int_D d^3x \left[\nabla\cdot\left(\boldsymbol{J}n\frac{\partial\mathfrak{U}_e}{\partial n}\right) + \frac{\partial\mathfrak{U}_e}{\partial n}\boldsymbol{J}\cdot\nabla n \quad + \frac{\partial\mathfrak{U}_e}{\partial s_e}\boldsymbol{J}\cdot\nabla s_e \right]\\
&= \frac{1}{e}\int_D d^3x \nabla\cdot\left[\boldsymbol{J}\left(n\frac{\partial\mathfrak{U}_e}{\partial n} + \mathfrak{U}_e\right)\right],
\end{aligned} \tag{16}$$

yielding $-\boldsymbol{J}(n\partial\mathfrak{U}_e/\partial n + \mathfrak{U}_e)/e$ as another contribution to the energy flux.

Thus, we conclude that HMHD has the integrand of (13) as an energy density, say $\mathcal{E}_H$, and this quantity satisfies a conservation law of the form $\partial\mathcal{E}_H/\partial t + \nabla\cdot\boldsymbol{J}_H = 0$ for an energy flux $\boldsymbol{J}_H$ given by

$$\boldsymbol{J}_H = \boldsymbol{J}_{MHD} + \frac{B^2}{\mu_0 en}\boldsymbol{J}_{\perp} - \frac{1}{e}\left(n\frac{\partial\mathfrak{U}_e}{\partial n} + \mathfrak{U}_e\right)\boldsymbol{J}\,, \tag{17}$$

where $\boldsymbol{J}_{MHD}$ is the usual MHD energy flux.

B. Inertial MHD

Now let us consider the remaining terms of extended MHD. The last term on the right-hand-side of the momentum Eq. (2) exists due to electron inertia, and we will see that its retention is crucial when electron inertia terms are included in Ohm's law. We will call the second term on the left-hand-side of the generalized Ohm's law (3) the "nonlinear term," the third term on the left-hand-side the "collision term," the first line on the right-hand-side the "Hall term," and the remaining terms on the right-hand-side the "electron inertia terms." To compare the sizes of these terms, we use the following dimensionless numbers:

$$R_M := \frac{\text{Nonlinear term}}{\text{Collision term}} = \sigma\mu_0 UL,$$
$$C_H := \frac{\text{Hall term}}{\text{Collision term}} = \frac{\sigma B}{en},$$
$$C_I := \frac{\text{Electron inertia term}}{\text{Collision term}} = \frac{\sigma m_e}{e^2 n\tau},$$

where U, L, B, and τ are the characteristic velocity scale, length scale, magnitude of magnetic field, and time scale of current change, respectively. Here, R_M is the usual magnetic Reynolds number or Lundquist number as it is sometimes called. The two Hall terms are comparable in size if $B^2 \sim p_e$, i.e., $\beta_e \sim 1$.

For IMHD, we focus on the situation where the electron inertia term is larger than the collision and Hall terms; however, the nonlinear term is still considered to be comparable with the electron inertia term, that is,

$$R_M \gg 1, \quad C_I \gg 1, \quad \frac{C_I}{C_H} \gg 1.$$

Since the last inequality is equivalent to

$$\frac{C_I}{C_H} = \frac{m_e}{eB}\frac{1}{\tau} = \frac{1}{\Omega_e \tau} \gg 1,$$

where Ω_e is the electron gyro-frequency, this relation can be interpreted as saying that the characteristic time scale of the current change is much shorter than the gyro-period of the electron.

With the above ordering, extended MHD reduces to the IMHD model given by the following set of equations:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{V}), \tag{18}$$

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} - \epsilon\frac{m_e}{e}(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}, \tag{19}$$

$$\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} = \epsilon\frac{m_e}{e^2 n}\left[\frac{\partial \boldsymbol{J}}{\partial t} + \nabla\cdot(\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V})\right] - \delta\frac{m_e}{e^2 n}(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}, \tag{20}$$

$$\frac{\partial s}{\partial t} = -\boldsymbol{V}\cdot\nabla s, \tag{21}$$

where s is the entropy per unit mass of the plasma and the last equation means the plasma is adiabatic. Note, we have artificially inserted book keeping parameters ϵ and δ in order to identify terms—in reality, both of these parameters have value unity. The above equations are to be solved with the pre-Maxwell's equations

$$\nabla\times\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \quad \text{and} \quad \nabla\times\boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{22}$$

with the initial condition $\nabla\cdot\boldsymbol{B} = 0$. Note, consistent with quasineutrality and the neglect of the Maxwell displacement current, the current density is solenoidal, $\nabla\cdot\boldsymbol{J} = 0$.

We stress that the energy conservation results we obtain do not depend on the IMHD ordering, but exist with the inclusion of the Hall terms, provided one extends the thermodynamics as in Sec. III A. For IMHD, one only needs to consider $\mathcal{U}(\rho, s)$, but it could also be generalized to include dependence on $|\mathrm{B}|$.

Upon considering a candidate energy for this IMHD model by taking the scalar product of $\boldsymbol{V}$ and the momentum equation, the scalar product of $\boldsymbol{J}$ and the generalized Ohm's law, and using the pre-Maxwell equations, we obtain the following energy relation:

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + \rho\mathcal{U} + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right) + \nabla\cdot\left[\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + p + \rho\mathcal{U} + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2}\right)\boldsymbol{V} + \epsilon\frac{m_e}{e^2 n}(\boldsymbol{V}\cdot\boldsymbol{J})\boldsymbol{J} - \delta\frac{m_e}{2e^3 n^2}|\boldsymbol{J}|^2\boldsymbol{J} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right] = 0. \tag{23}$$

Observe the new term of the energy density of (23), $m_e|\boldsymbol{J}|^2/(2e^2 n)$, which arises from electron inertia and represents the electron kinetic energy density. Note that, from the generalized Ohm's law, because of the dependence of $\boldsymbol{E}\times\boldsymbol{B}$ in the energy flux of (23), the flux includes the time derivative term $\epsilon m_e \partial\boldsymbol{J}/\partial t/(e^2 n)$, so the above formulation is not in the usual conservation form. However, upon integrating the above energy relation over the whole domain D with appropriate boundary conditions, it is revealed that the total energy H, which is defined as

$$H := \int_D d^3x\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + \rho\mathcal{U} + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right),$$

is conserved.

Note, if we were to consider the governing Eqs. (18)–(21) with the full Maxwell's equations, then the following energy relation applies:

$$0 = \frac{\partial}{\partial t}\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + \rho\mathcal{U} + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0} + \frac{\epsilon_0}{2}|\boldsymbol{E}|^2\right) + \nabla\cdot\left[\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + p + \rho U + \epsilon\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2}\right)\boldsymbol{V} + \epsilon\frac{m_e}{e^2 n}(\boldsymbol{V}\cdot\boldsymbol{J})\boldsymbol{J} - \delta\frac{m_e}{2e^3 n^2}|\boldsymbol{J}|^2\boldsymbol{J} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right],$$

and this relation is of the usual conservation form since $\boldsymbol{E}$ is now a dynamical variable.

IV. CLASSIFICATION BY ENERGY CONSERVATION OF IMHD

In this section, we sort IMHD models into energy conserving and non-energy conserving classes.

We first consider the compressible IMHD model composed of the pre-Maxwell equations and the following:

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \boldsymbol{V}) = 0,\\
&\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} - \epsilon_{\mathrm{mom}}\frac{m_e}{e}(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en},\\
&\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} = \epsilon_{\mathrm{ti}}\frac{m_e}{e^2 n}\frac{\partial \boldsymbol{J}}{\partial t} + \epsilon_{\mathrm{ad}}\frac{m_e}{e^2 n}(\boldsymbol{V}\cdot\nabla)\boldsymbol{J} + \epsilon_{\mathrm{cp}}\frac{m_e}{e^2 n}\boldsymbol{J}(\nabla\cdot\boldsymbol{V}) + \epsilon_{\mathrm{ohm}}\frac{m_e}{e^2 n}(\boldsymbol{J}\cdot\nabla)\boldsymbol{V} - \delta\frac{m_e}{e^2 n}(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en},\\
&\frac{\partial s}{\partial t} + (\boldsymbol{V}\cdot\nabla)s = 0\,.
\end{aligned}
$$

Recall, the parameters δ and ϵ were artificially inserted into (18)–(21) as book keeping parameters, but now ϵ has been replaced by several parameters that track the various effects: ϵ_{ti} (current time derivative), ϵ_{ad} (current advection), ϵ_{cp} (compressibility), ϵ_{mom} (term in momentum equation that was ϵ), and ϵ_{ohm} (term in Ohm's law partnered with that in the momentum equation). These parameters are useful for determining how the various terms in the calculation of the energy may cancel.

Proceeding, we see the various terms involved in energy conservation combine as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + \rho\,\mathcal{U} + \epsilon_{\mathrm{ti}}\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\\
&\quad + \nabla\cdot\left[\left(\frac{\rho}{2}|\boldsymbol{V}|^2 + p + \rho\,\mathcal{U} + \epsilon_{\mathrm{ad}}\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2}\right)\boldsymbol{V} + \epsilon_{\mathrm{ohm}}\frac{m_e}{e^2 n}(\boldsymbol{V}\cdot\boldsymbol{J})\boldsymbol{J} - \delta\frac{m_e}{e^3 n^2}\frac{|\boldsymbol{J}|^2}{2}\boldsymbol{J} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]\\
&= (\epsilon_{\mathrm{ti}} - \epsilon_{\mathrm{ad}})\frac{m_e}{e^2 n}\frac{|\boldsymbol{J}|^2}{2}\frac{\nabla\cdot(n\boldsymbol{V})}{n} + (\epsilon_{\mathrm{ad}} - \epsilon_{\mathrm{cp}})\frac{m_e}{e^2 n}|\boldsymbol{J}|^2(\nabla\cdot\boldsymbol{V})\\
&\quad + (\epsilon_{\mathrm{ohm}} - \epsilon_{\mathrm{mom}})\frac{m_e}{e}\boldsymbol{V}\cdot(\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}).
\end{aligned}
\tag{24}
$$

Since our main interest here is with electron inertia models, we do not consider the case where ϵ_{ti} vanishes. Thus, from Eq. (24), we find that the total energy is conserved only when all the epsilon terms are non-vanishing or ϵ_{ti}, ϵ_{ad}, and ϵ_{cp} are non-vanishing. Note, we have conservation for any value of δ. Therefore, we conclude that the epsilon term in the momentum equation is essential for energy conservation in IMHD models.

Second, we consider the incompressible IMHD model which is governed by the pre-Maxwell equations and by the following equations:

$$
\begin{aligned}
&\rho_0\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V}\cdot\nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} - \epsilon_{\mathrm{mom}}\frac{m_e}{e^2 n_0}(\boldsymbol{J}\cdot\nabla)\boldsymbol{J},\\
&\boldsymbol{E} + \boldsymbol{V}\times\boldsymbol{B} = \epsilon_{\mathrm{ti}}\frac{m_e}{e^2 n_0}\frac{\partial \boldsymbol{J}}{\partial t} + \epsilon_{\mathrm{ad}}\frac{m_e}{e^2 n_0}(\boldsymbol{V}\cdot\nabla)\boldsymbol{J} + \epsilon_{\mathrm{ohm}}\frac{m_e}{e^2 n_0}(\boldsymbol{J}\cdot\nabla)\boldsymbol{V} - \delta\frac{m_e}{e^3 n_0^2}(\boldsymbol{J}\cdot\nabla)\boldsymbol{J}\,,
\end{aligned}
$$

together with $\rho = \rho_0 =$ constant or equivalently $n = n_0 =$ constant. Note that ϵ_{cp} does not occur in the above equations because of incompressibility. For this system, energy conservation law is as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}|\boldsymbol{V}|^2 + \epsilon_{\mathrm{ti}}\frac{m_e}{e^2 n_0}\frac{|\boldsymbol{J}|^2}{2} + \frac{|\boldsymbol{B}|^2}{2\mu_0}\right)\\
&\quad + \nabla\cdot\left[\left(\frac{\rho_0}{2}|\boldsymbol{V}|^2 + p + \epsilon_{\mathrm{ad}}\frac{m_e}{e^2 n_0}\frac{|\boldsymbol{J}|^2}{2}\right)\boldsymbol{V} + \epsilon_{\mathrm{ohm}}\frac{m_e}{e^2 n_0}(\boldsymbol{V}\cdot\boldsymbol{J})\boldsymbol{J} - \delta\frac{m_e}{e^3 n_0^2}\frac{|\boldsymbol{J}|^2}{2}\boldsymbol{J} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0}\right]\\
&= (\epsilon_{\mathrm{ohm}} - \epsilon_{\mathrm{mom}})\frac{m_e}{e^2 n_0}\boldsymbol{V}\cdot((\boldsymbol{J}\cdot\nabla)\boldsymbol{J}).
\end{aligned}
$$

Thus, it is revealed that total energy is conserved if $\epsilon_{\mathrm{ohm}} = \epsilon_{\mathrm{mom}} = 0$ or 1, and there are no other conditions on ϵ_{ti} and ϵ_{ad}.

All of our results on energy conservation for IMHD models are summarized in Table I.

V. CONCLUSION

One might argue that the term $-m_e(\boldsymbol{J}\cdot\nabla)\boldsymbol{J}/(e^2 n)$ of (2) may be neglected without consequence, since it is small. However, doing so would amount to the introduction of nonphysical dissipation. Since the whole point of reconnection studies is that a small physical dissipation can have important consequences, one should view any reconnection calculation with this nonphysical dissipation with caution. Note, however, in some geometries, this term may vanish.

Equations (3.7.1)–(3.7.4) of Ref. 19 do not conserve energy, whether or not Maxwell's displacement current is retained. In this reference and elsewhere, it is described how the neglected term in the momentum equation can be recognized by reverting to a two-species kinetic theory, where the pressure tensors for each species are given by

$$
\underline{\underline{P}}^{\alpha} = m_\alpha \int d^3v f_\alpha\,(\mathbf{v} - \mathbf{v}_\alpha)\otimes(\mathbf{v} - \mathbf{v}_\alpha), \tag{25}
$$

with $\alpha \in \{e, i\}$, f_α being the phase space density of species α, and the fluid velocities given as usual by

$$
\mathbf{v}_\alpha = \frac{\int d^3v f_\alpha \mathbf{v}}{\int d^3v f_\alpha}\,. \tag{26}
$$

If we follow Lüst's example for scalar pressure, then the total pressure is the sum of the partial pressures, i.e.,

$$
\mathbb{P} = \mathbb{P}^i + \mathbb{P}^e, \tag{27}
$$

in accordance with Dalton's law. However, it is sometimes suggested that one use pressures defined in terms of the center of mass velocity according to

$$
\mathbb{P}^{\alpha}_{\mathrm{cm}} = m_\alpha \int d^3v f_\alpha\,(\mathbf{v} - \mathbf{V})\otimes(\mathbf{v} - \mathbf{V}), \tag{28}
$$

and define the total pressure by $\mathbb{P}_{\mathrm{cm}} = \mathbb{P}^i_{\mathrm{cm}} + \mathbb{P}^e_{\mathrm{cm}}$. Upon inserting

$$\mathbf{v}_i = \mathbf{V} + \frac{m_e}{m}\frac{\mathbf{J}}{en} \quad \text{and} \quad \mathbf{v}_e = \mathbf{V} - \frac{m_i}{m}\frac{\mathbf{J}}{en}, \tag{29}$$

in (27), an easy calculation gives

$$\mathbb{P} = \mathbb{P}_{\rm cm} - \frac{m_e m_i}{me^2 n}\mathbf{J}\otimes\mathbf{J} \approx \mathbb{P}_{\rm cm} - \frac{m_e}{e^2 n}\mathbf{J}\otimes\mathbf{J}. \tag{30}$$

Thus, one could replace the first and last terms on the right hand side of (2) by $-\nabla\cdot\mathbb{P}_{\rm cm}$ and obtain a tidy equation. However, if one further makes conventional thermodynamic closure assumptions on $\mathbb{P}_{\rm cm}$, e.g., that it is isotropic and either barotropic or adiabatic, then one would be in essence saying that the current is dependent on density, which is unphysical. This unphysical nature is manifest in the resulting violation of energy conservation when this procedure is employed.

In this paper, we have used physical reasoning and direct calculation to obtain conserved energy densities. However, energy should emerge from time translation symmetry by means of Noether's theorem. That this is indeed the case will be reported in future work[20] by deriving the action for extended MHD and then using the Galilean group to construct the usual conservation laws. With this formalism, one also obtains the noncanonical Poisson brackets for this model akin to that of Ref. 21 (see Ref. 22 for review). This leads to the Casimir invariants and opens up the possibility of applying Hamiltonian techniques for stability such as in Refs. 23–26.

Finally, we point out that our starting point was two-fluid theory and gyroviscous effects due to strong magnetic fields have not been incorporated.[23,27,28] This will also be the subject of a future publication.

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Erratum: "On energy conservation in extended magnetohydrodynamics" [Phys. Plasmas 21, 082101 (2014)]

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Because of a typesetter's error, the Incompressible plasma part of Table I has a misplaced 1 in the $\epsilon_{\rm mom}$ column. The correct table (as submitted to Physics of Plasmas) is below.

$\epsilon_{\rm ti}$	$\epsilon_{\rm ad}$	$\epsilon_{\rm cp}$	$\epsilon_{\rm ohm}$	Ohm's law $\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}=$	$\epsilon_{\rm mom}$	Conserved?
Compressible plasma						
1	1	1	1	$\frac{m_e}{e^2 n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{J}+\boldsymbol{J}\boldsymbol{V})\right)$	1	Yes
1	1	1		$\frac{m_e}{e^2 n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{J})\right)$		Yes
1				$\frac{m_e}{e^2 n}\frac{\partial \boldsymbol{J}}{\partial t}$		$\frac{m_e}{e^2 n}\frac{\lvert\boldsymbol{J}\rvert^2}{2}\frac{\nabla\cdot(n\boldsymbol{V})}{n}$
1	1			$\frac{m_e}{e^2 n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{J}\right)$		$\frac{m_e}{e^2 n}\lvert\boldsymbol{J}\rvert^2(\nabla\cdot\boldsymbol{V})$
1	1	1	1	$\frac{m_e}{e^2 n}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{J}+\boldsymbol{J}\boldsymbol{V})\right)$		$\frac{m_e}{e}\boldsymbol{V}\cdot\left((\boldsymbol{J}\cdot\nabla)\frac{\boldsymbol{J}}{en}\right)$
Incompressible plasma						
1		$-$		$\frac{m_e}{e^2 n_0}\frac{\partial \boldsymbol{J}}{\partial t}$		Yes
1	1	$-$		$\frac{m_e}{e^2 n_0}\left(\frac{\partial \boldsymbol{J}}{\partial t}+(\boldsymbol{V}\cdot\nabla)\boldsymbol{J}\right)$		Yes
1	1	$-$	1	$\frac{m_e}{e^2 n_0}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{J}+\boldsymbol{J}\boldsymbol{V})\right)$	1	Yes
1	1	$-$	1	$\frac{m_e}{e^2 n_0}\left(\frac{\partial \boldsymbol{J}}{\partial t}+\nabla\cdot(\boldsymbol{V}\boldsymbol{J}+\boldsymbol{J}\boldsymbol{V})\right)$		$\frac{m_e}{e^2 n_0}\boldsymbol{V}\cdot\left((\boldsymbol{J}\cdot\nabla)\boldsymbol{J}\right)$

TABLE I. Classification of energy conserving IMHD models. The values of the epsilons in the generalized Ohm's law are listed in the first four columns, and the generalized Ohm's law is described by the fifth column. The epsilon values in the momentum equation are listed in the sixth column. When the total energy is conserved "Yes" is written in the last column, otherwise the deficit terms are written in the last column. Note that for incompressible plasma, there is no $\epsilon_{\rm cp}$-term in the generalized Ohm's law; consequently, we write " $-$ " in the third column for this case.