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# Hamiltonian Systems, from Topology to Applications through Analysis

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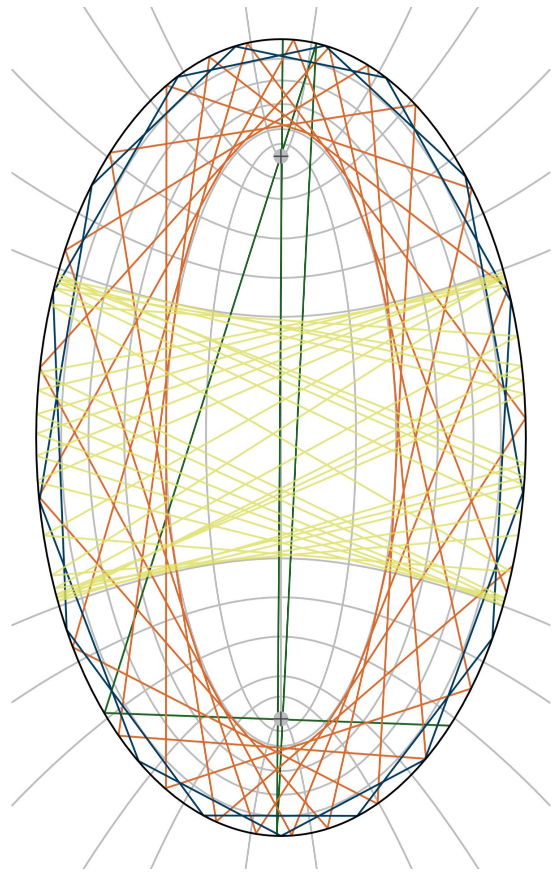
## Overview

The special historic form of dynamical systems credited to Hamilton encompasses a vast array of fundamental and applied research. It is a basic form for physical law that has engendered development in many areas of mathematics, including analysis of ordinary and partial differential equations, topology, and geometry. Our jumbo program at MSRI has brought together a broad spectrum of mathematicians and scientists with research spanning emphasis on the applied to the rigorous.

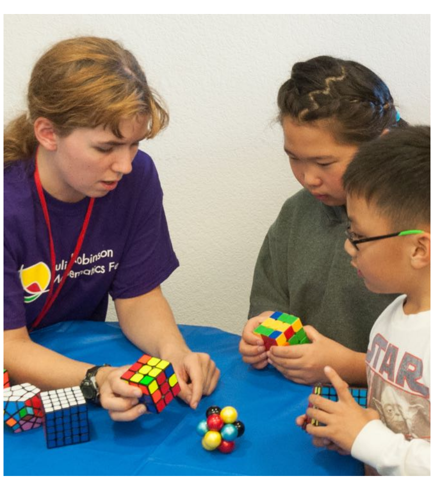
It is hard to choose a starting point in the long history of Hamiltonian dynamics and its concomitant variational principles. One can go as far back as ancient Greece (Euclid, Heron), onward to the inspirational work of Fermat's principle of geometric optics (17th century), and up to the voluminous works of Lagrange (18th century). Fermat stated that the path taken by light going from one point to another in some medium is the path that minimizes (or, more accurately, extremizes) the travel time. This implies the law of optical reflection (the angle of incidence equals the angle of reflection) and Snell's law of refraction.

William Rowan Hamilton (19th century) studied the propagation of the phase in optical systems guided by Fermat's principle and realized that one could generalize it and adapt it to particle mechanics. Here is a very brief description of Hamiltonian mechanics.

*(continued on page 4)*



Multiple trajectories for a billiard inside an ellipse: One of the popular Hamiltonian systems is the "billiard problem."



David Eisenbud

## Come and Celebrate . . .

← Our recent **Celebration of Mind** (page 8).

The **Summer Research for Women** program created and first hosted in 2017 (page 9). →

↓ Mafia, hats, and more in the **Puzzles**, as Elwyn and Joe welcome a new contributor (page 15)!

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# Hamiltonian Systems, from Topology to Applications through Analysis

*(continued from page 1)*

A mechanical system is described by its configuration space, mathematically a smooth manifold $Q$ whose points, $q \in Q$, are understood as positions. The system is described by a Lagrangian function $L(q,\dot{q}) : TQ \to \mathbb{R}$, depending on the position $q$ and the velocity $\dot{q}$. One considers the action functional

$$S[q(t)] = \int_{t_0}^{t_1} L(q,\dot{q})\, dt, \quad \delta q(t_0) = \delta q(t_1) = 0.$$

The time evolution of the mechanical system is described as a variational principle that the action $S$ be extremal. This implies the Euler–Lagrange equations,

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right).$$

Going from the tangent bundle $TQ$ to the cotangent bundle $T^*Q$, one introduces momenta $p = \partial L/\partial \dot{q}$ and the Hamiltonian function $T^*Q \to \mathbb{R}$, given by the Legendre transform $H(q,p) = p\dot{q} - L(q,\dot{q})$. In the phase space $T^*Q$, the motion is described by Hamilton's first order differential equations,

$$\dot{p} = -\frac{\partial H}{\partial q} \quad \text{and} \quad \dot{q} = \frac{\partial H}{\partial p}.$$

The Hamiltonian form is not limited to finite-dimensional systems — indeed, action functionals are fundamental to the development of 20th century field theories in physics. A case in point is $\phi^4$ field theory that has the action functional

$$S[\phi] = \int_{t_0}^{t_1} dt \int d^3x\, \mathcal{L}(\phi, \partial\phi),$$

with the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}m^2\phi^2 - \frac{g}{4!}\phi^4.$$

The Legendre transform of this system gives the associated Hamiltonian form of the partial differential equations,

$$\partial_t \pi = -\frac{\delta H}{\delta \phi} \quad \text{and} \quad \partial_t \phi = \frac{\delta H}{\delta \pi},$$

where $\delta H/\delta\phi$ denotes the functional derivative and the Hamiltonian functional is

$$H = \int d^3x \left(\frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4\right),$$

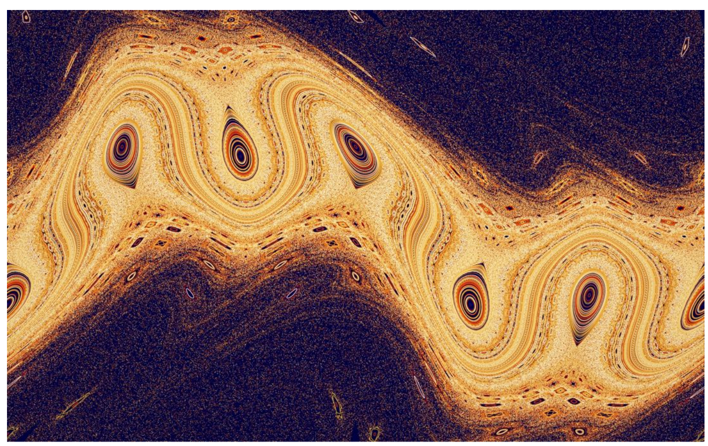
where $m$ and $g$ are parameters of the theory. By direct calculation one obtains a nonlinear wave equation as a Hamiltonian system.

There are many ramifications and generalizations, practical and aesthetic, of both the finite and infinite forms of Hamiltonian systems. We describe some of those addressed by our MSRI program below.

## Dynamics on Symplectic Manifolds

The theoretical underpinning of Hamiltonian mechanics is symplectic geometry. (The term symplectic was adopted by H. Weyl to avoid any connotation of complex numbers that his previously-used term, "complex group," had suffered. It is simply the Greek adjective corresponding to the word "complex.") The cotangent bundle $T^*Q$ of a smooth manifold carries a canonical symplectic structure $\omega = dp \wedge dq$, a closed non-degenerate differential 2-form. Other manifolds, not necessarily cotangent bundles, may also cary a symplectic structure. Hamilton's equations of motion describe the Hamiltonian vector field (or the symplectic gradient) $X_H$ defined by the formula $\omega(X_H, \cdot) = -dH$.

An important example of a Hamiltonian system, and one of the research topics in this program, is the motion of a charged particle in a magnetic field. Mathematically, a magnetic field is represented by a closed differential 2-form $\beta$ on a Riemannian manifold $Q$. One considers the twisted symplectic structure $\omega + \pi^*(\beta)$ on the cotangent bundle $T^*Q$, where $\pi : T^*Q \to Q$ is the projection. The motion of a charge is described as the Hamiltonian vector field of the energy function $||p||^2/2$ (the norm is taken using the metric on $Q$) with respect to the twisted symplectic structure. One of the problems of contemporary interest is the existence of periodic motions of the charge.



Iteration of the Standard Nontwist map (del-Castillo-Negrete et al., *Physica D* **91**, 1 (1996)) that shows invariant tori (continuous curves) embedded in a chaotic sea of orbits. (Figure courtesy of George Miloshevich.)

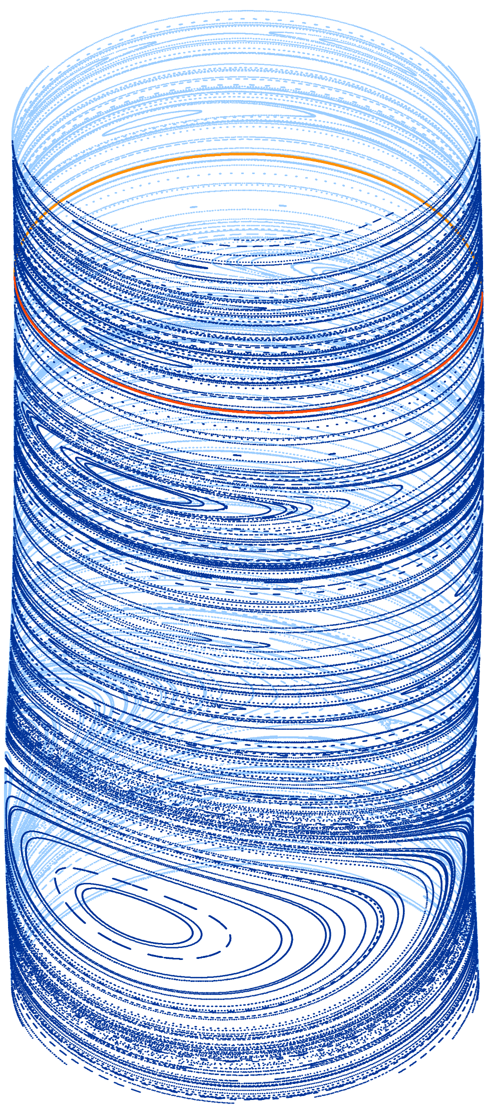
Other theoretical backgrounds of Hamiltonian dynamics include the calculus of variations, Morse theory, and Floer theory. Floer theory, in particular, is designed for the study of symplectic dynamics of Arnold's conjecture concerning fixed points of Hamiltonian diffeomorphisms (flows of time-dependent Hamiltonian vector fields). In the simplest case, this conjecture states that an area and center of mass preserving diffeomorphism of a torus has at least three, and generically at least four, distinct fixed points (proved in the 1980s by Conley and Zehnder).

## Integrability and Chaos

Hamiltonian systems exhibit a wide variety of dynamical behavior, from very regular (completely integrable) to chaotic. In a completely integrable system (for example, Kepler's problem in celestial mechanics), the motion is typically confined to tori that have half the dimension of the phase space, and the motion on these invariant tori is described by constant vector fields.

While completely integrable systems are interesting, they are extremely rare. The Kolmogorov–Arnold–Moser (KAM) theory studies small perturbations of integrable systems; its fundamental result is that some invariant tori persist under small perturbations.

If the dimension of the phase space is greater than two, the invariant tori do not separate the space, and a phase trajectory may escape to infinity. This, and related phenomena, are known as Arnold diffusion. Arnold diffusion is one of the focal points of this topical semester, and many experts in this field are participants of the program. KAM theory and the theory of Arnold diffusion have many applications, in particular, in celestial mechanics (for example, "Is the solar system stable?")



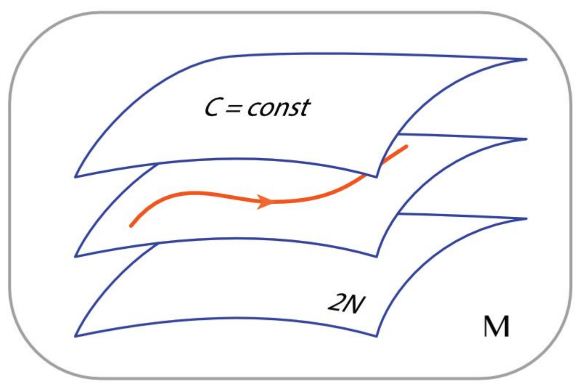
Normally hyperbolic invariant cylinder in a Froeschlé map. This type of object is relevant to the mechanism of Arnold diffusion; for example, moving around a 2-torus in a four dimensional manifold. (Figure courtesy of Alex Haro.)

## Flows on Poisson Manifolds

Poisson manifolds are a generalization of symplectic manifolds, where a manifold $M$ has instead of a symplectic 2-form a Poisson bracket $\{,\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ for arbitrary functions $f, g, h \in C^\infty(M)$ satisfying

1. Bilinearity: $\{f+\lambda g, h\} = \{f, h\} + \lambda\{g, h\} \quad \lambda \in \mathbb{R}$
2. Skew symmetry: $\{f, g\} = -\{g, f\}$
3. Jacobi identity: $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$
4. Leibniz Rule: $\{fg, h\} = g\{f, h\} + f\{g, h\}$

Here, the Poisson bracket $\{f, g\} = J(df \wedge dg)$, written in terms of the Poisson bivector $J$, generates the vector field $X_f = \{f, \cdot\}$ whose integral curves are the trajectories of the Hamiltonian dynamics of interest. Because the bivector does not have the usual form, systems of this type are sometimes called noncanonical Hamiltonian systems. Unlike conventional Poisson brackets, the noncanonical Poisson brackets of Poisson manifolds are degenerate with special invariants know as Casimir invariants $C \in C^\infty(M)$ that satisfy $\{C, f\} = 0$ for all $f \in C^\infty(M)$. Because of this degeneracy flows are constrained to submanifolds that are in fact generically symplectic manifolds.




Cartoon of Poisson manifold with its foliation by symplectic leaves.

The study of Poisson manifolds is important both because of intrinsic mathematical interest and because infinite-dimensional versions of such flows generated by noncanonical Poisson brackets describe many physical systems. These flows are systems of partial differential equations that have a Hamiltonian form given by

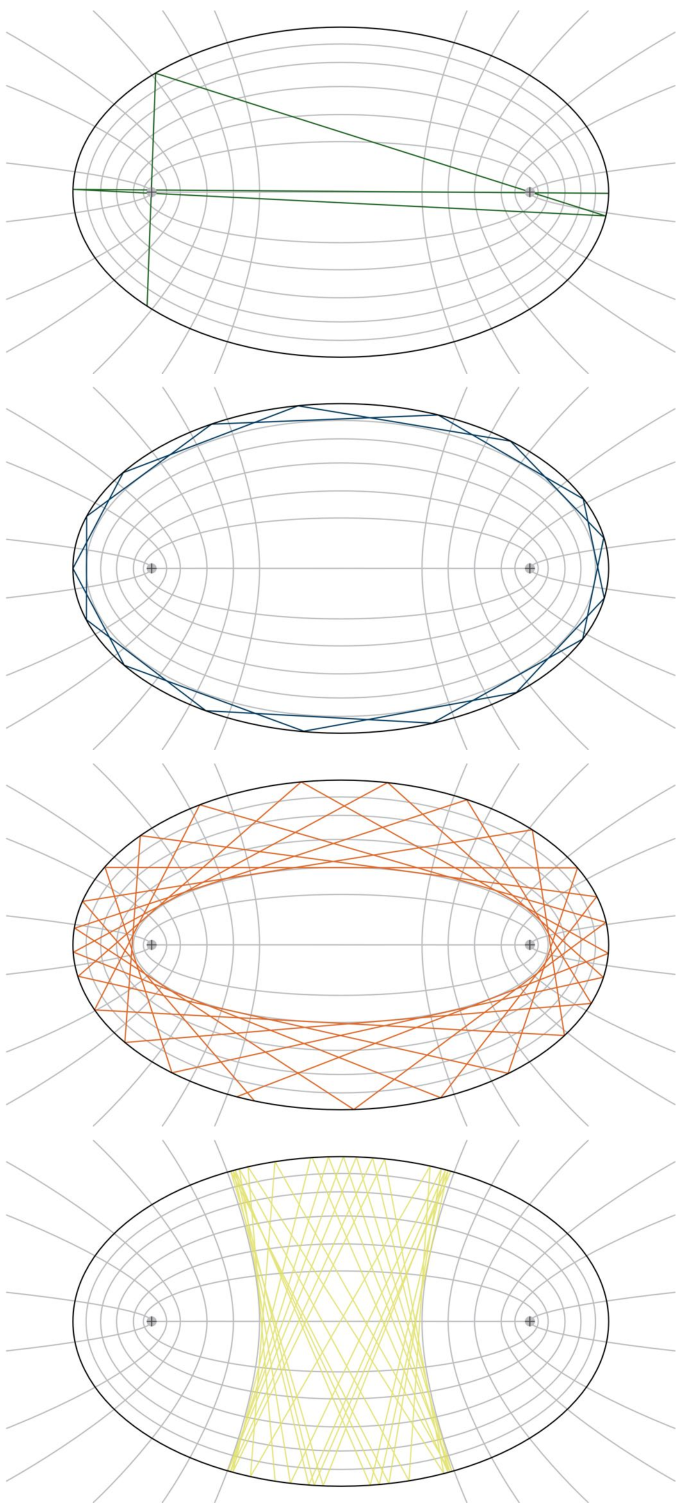
$$\partial_t \chi = \mathcal{J}(\chi) \frac{\delta H}{\delta \chi},$$

where $\chi$ denotes the set of dependent field variables and $\mathcal{J}$ is a Poisson operator that is a generalization of the Poisson tensor $J$ of the Hamiltonian bivector for finite systems.

## Mathematical Billiards

One of the popular Hamiltonian systems is a mathematical billiard that describes the motion of a mass-point in a domain, subject to specular reflections off the boundary. Many mechanical systems with elastic collisions — that is, collisions in which the energy and momentum are preserved — are described as billiard systems.

On a basic level, billiards can be viewed as a study of non-smooth Hamiltonian vector fields, one motivated by a physically relevant setup. The study of billiards was put forward by Birkhoff, who observed (in 1927) that "... in this problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting qualitative questions need to be considered."



Trajectories for a billiard inside an ellipse, broken out from the cover image. (Figure courtesy of Vadim Kaloshin.)

The billiard system can be considered as a continuous-time Hamiltonian system with discontinuities corresponding to the reflections. It can be also considered as a discrete-time system, that is, a transformation acting on the oriented lines, thought of as segments of a billiard trajectory. The space of oriented lines (rays of light) carries a symplectic structure, and the billiard ball map is a symplectic transformation.

Just like general Hamiltonian systems, planar billiards exhibit a full spectrum of dynamical behaviors, from completely integrable to chaotic. A completely integrable example is the billiard inside an ellipse: the interior of the billiard table is foliated by caustics which are confocal conics.

A notoriously hard problem in this area is known as Birkhoff's conjecture: *If a neighborhood of the boundary of a billiard table is foliated by caustics, then it is an ellipse*. There was a very substantial progress made toward the proof of this conjecture and its variation (for example, algebraic integrability, where the conserved quantity is a polynomial in momentum). A number of the main players in this field participate in the program.

## Applications

Given the history of its development, it comes as no surprise that Hamiltonian dynamics plays an important role in modern theoretical physics and an array of applications, a few of which we note.

The Hamiltonian dynamics of bodies under the influence of gravity, celestial mechanics, is of course of basic importance for understanding the dynamics of the solar system and beyond (planets and stars). However, it also is needed for practical satellite navigation, for communication and space exploration, and the tracking of possibly deleterious astroids.

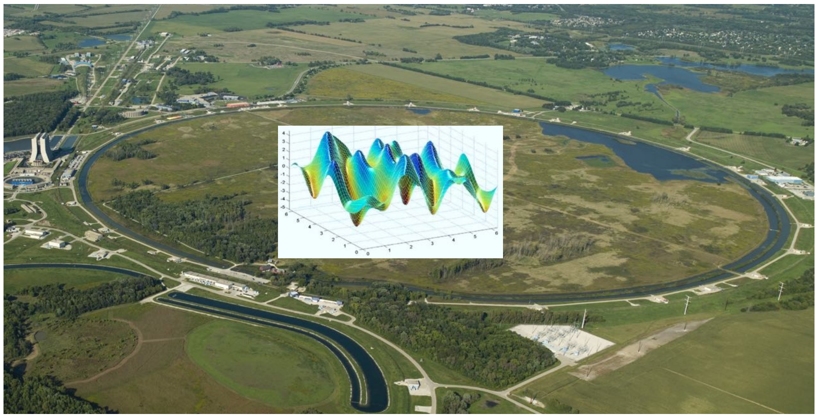
The Hamiltonian dynamics of gravitating bodies is a special case of so-called natural Hamiltonian systems where $H(q,p) = T(q,p) + V(q)$ with $T(q,p)$ the kinetic energy (mathematically, a Riemannian metric on the fiber of $T^*Q$) and $V(q)$ the potential energy. Examples of natural Hamiltonian systems are spring systems, pendula, particles in potential wells, and the N-body problem with $q_i \in \mathbb{R}^3$, $i = 1, \dots, N$, where

$$H(q,p) = \sum_{i=1}^{N} \frac{\|p_i\|^2}{2m_i} + \sum_{i,j=1}^{N} \frac{c_{ij}}{|q_i - q_j|}$$

and $c_{ij}$ represents the interaction. In the context of celestial mechanics, the interaction represents gravitational attraction, but the N-body setup can also describe the electrostatic interaction of repelling electrons, electrons and ions (protons) that attract each other, and the collection of both that occurs in plasmas.

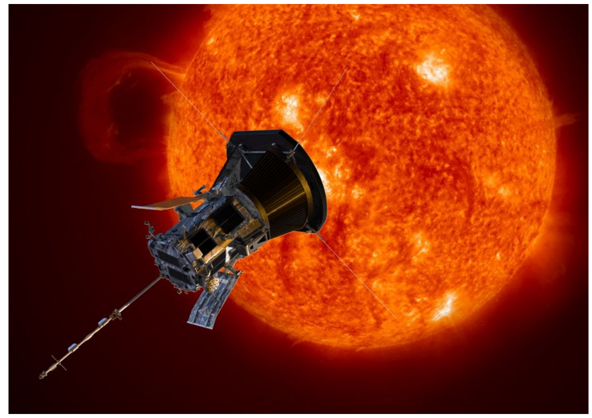
If charged particles experience the full electromagnetic interaction then the Hamiltonian is given by

$$H(q,p) = \frac{\|p - eA(q,t)\|^2}{2m} + e\phi(q,t),$$



Fermilab

The Tevatron — a large, 1 km-radius particle accelerator at Fermilab near Chicago — accelerated particles to near the speed of light and stored them for many hours. Understanding the particle orbits in such a system is a complicated Hamiltonian dynamics problem of estimating minute deviations from integrability. (Inset courtesy of Martin Berz.)



NASA/Johns Hopkins APL/Steve Gribben

The Parker Solar Probe, already in orbit around the sun, will characterize plasma near the sun (within 10 solar radii) and investigate magnetic fields, energetic particles, and the creation of the solar wind that impacts the Earth's environment and affects space weather.

where the magnetic field $B = \nabla \times A$ and the electric field $E = -\nabla\phi - \partial_t A$. The dynamics under (relativistic versions of) such Hamiltonians is of great importance for the design of particle accelerators such as Fermilab where one must corral and accelerate particles with sufficient luminosity to probe the nature of elementary particles. In addition accelerator technology is essential for the medical physics of radiotherapy, radiology, nuclear medicine, and oncology.

The Hamiltonian dynamics of charged particles is of fundamental importance for understanding naturally occurring plasmas such as that near the sun that will be explored with the Parker Solar probe, which will investigate the origin of the solar wind that impacts the earth's magnetosphere. In addition the Hamiltonian dynamics of charged particles is essential for understanding the laboratory plasmas created in large machines such as the ITER Tokamak that is being built in France. Particle dynamics in magnetic fields is particularly important for the design of such devices that use strong magnetic field to confine plasma inside a solid torus with the goal of producing controlled thermonuclear fusion to generate power.

In addition to the finite-dimensional systems described above, there are infinite-dimensional systems — field theories — that describe collective motion. An important example is the Vlasov–Maxwell system, a kinetic theory that takes into account the fact that charges in addition to producing electric fields have motions, currents, that produce magnetic fields and electromagnetic wavelike motions. A reduced form of this system describes average features of large scale stellar dynamics.

Other field theories include Euler's equations of fluid mechanics; shallow water theory and the quasi-geostrophic equations, important equations of geophysical fluid dynamics that describe aspects of atmospheric and ocean dynamics; magnetohydrodynamics and two-fluid theory, fluid theories of plasma physics that include the magnetic and electric fields as dynamical variables; and sundry additional equations that describe various aspects of media treated as a continuum.

All of these infinite-dimensional systems are Hamiltonian systems, and they all possess the form of flows on infinite-dimensional Poisson manifolds as described above. Indeed, the discovery of their Hamiltonian form provided a major impetus for the study of Poisson manifolds begun in the 1980s.

## Structure-Preserving Computation

Given that Hamiltonian systems are defined in terms of a differential 2-form, it comes as no surprise that various structures are preserved by the dynamics. In fact, the 2-form itself is preserved in the sense $\pounds_{X_H}\omega = 0$, where $\pounds_{X_H}$ is the Lie derivative with the Hamiltonian vector field $X_H$. One understands that Hamiltonian dynamics is a one-parameter temporal map of the phase space manifold to itself by a canonical transformation (symplectomorphism) that preserves symplectic area defined by the 2-form.

Numerical algorithms that preserve this structure are known as symplectic integrators. In addition to the exact conservation of a symplectic area, which can be wedged together to make a notion of volume preservation, symplectic integrators prevent the Hamiltonian (energy) from deviating significantly from its theoretically constant value. Thus they provide improved performance for long-time computation.

Recently, the variational form of Hamiltonian dynamics has been exploited to obtain variational integrators based on discretizing the variational principle, giving rise to desirable preservation of geometric structure. Poisson integrators are numerical algorithms that have exact conservation of the symplectic leaves of a Poisson manifold, as well as being symplectic on symplectic leaves.

Currently, one major challenge is to extract from Hamiltonian partial differential equations finite-dimensional (semi-discrete) systems of ordinary differential equations that inherit their parent Hamiltonian form and then to implement the resulting system with a symplectic or Poisson integrator. This is particularly difficult for infinite-dimensional systems that have noncanonical Poisson brackets.

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