THE MAXWELL–VLASOV EQUATIONS AS A CONTINUOUS HAMILTONIAN SYSTEM

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The well-known Maxwell–Vlasov equations that describe a collisionless plasma are cast into hamiltonian form. The dynamical variables are the physical although noncanonical variables $\boldsymbol{E}$, $\boldsymbol{B}$ and f. We present a Poisson bracket which acts on these variables and the energy functional to produce the equations of motion.

Systems of partial differential equations that possess hamiltonian structure are of great importance in physics. Historically, the first step to quantization has been the recognition of classical hamiltonian form. More recently, the underlying Hamiltonian structure of equations such as those of Korteweg and de Vries [1], Benjamin and Ono [2], and Benney [3] have been discovered. For these equations the useful variables are noncanonical. In addition the use of noncanonical variables in discrete systems has been of practical advantage [4]. Recently, the equations of eulerian hydrodynamics and ideal MHD have been shown to be hamiltonian in terms of the physical although noncanonical variables [5]. The great difficulty encountered when attempting to cast the Maxwell–Vlasov system into hamiltonian form has been in the search for a set of canonical variables. This difficulty is compounded by the functional coupling between functions of field variables and functions of phase space variables. We have circumvented this problem by producing a Poisson bracket in terms of the noncanonical variables, $f_\alpha(\boldsymbol{r},\boldsymbol{v},t)$, $\boldsymbol{E}(\boldsymbol{x},t)$ and $\boldsymbol{B}(\boldsymbol{x},t)$. Here f_α is the distribution function of species α, $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field.

In the following we informally define the mathematics underlying what is meant by a continuous hamiltonian system; the mathematics is tailored to the system at hand. The important result is essentially the Poisson bracket defined by eq. (9). This is the bracket for the full Maxwell–Vlasov system. A reduced form of this bracket is then shown to produce the Poisson–Vlasov system. Finally a formulation of this reduced system with the distribution function as the sole variable is presented.

We consider the following equations:

$$\boldsymbol{B}_t(\boldsymbol{x},t) = -\nabla\times\boldsymbol{E}(\boldsymbol{x},t)\,, \tag{1}$$

$$\boldsymbol{E}_t(\boldsymbol{x},t) = \nabla\times\boldsymbol{B}(\boldsymbol{x},t) - \sum_\alpha e_\alpha \int_{\mathrm{R}_1} \boldsymbol{v} f_\alpha(z,t) P(z|\boldsymbol{x})\,\mathrm{d}z\,, \tag{2}$$

$$f_{\alpha t}(z,t) = -\boldsymbol{v}\cdot\frac{\partial f_\alpha(z,t)}{\partial \boldsymbol{r}} - \frac{e_\alpha}{m_\alpha}\int_{\mathrm{R}_2}[\boldsymbol{E}(\boldsymbol{x},t) + \boldsymbol{v}\times\boldsymbol{B}(\boldsymbol{x},t)]\cdot\frac{\partial f_\alpha(z,t)}{\partial\boldsymbol{v}}\,P(z|\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{3}$$

Eq. (1) is Faraday's law; eq. (2) is Ampère's law with the inclusion of the displacement current. (We use rationalized gaussian units with the speed of light set to unity.) The remaining two Maxwell equations will take their usual role as initial conditions. Eq. (3) is the Vlasov equation where e_α and m_α are the signed charge and mass, respectively, of species α. Formally, we treat f_α as a function of the phase space variable $z \equiv (\boldsymbol{r},\boldsymbol{v})$ while $\boldsymbol{E}$ and $\boldsymbol{B}$ are functions of the field variable $\boldsymbol{x}$; hence, we have used the operator $P(z|\boldsymbol{x}) \equiv \delta(\boldsymbol{x}-\boldsymbol{r})$, where $\delta(\boldsymbol{x})$ is the Dirac delta function, in the coupling terms of eqs. (2) and (3). The region of integration $\mathrm{R}_1 \equiv \mathrm{A}\times\mathbf{R}^3$ where $\mathbf{R} = (-\infty,\infty)$ and $\mathrm{A}\subset\mathbf{R}^3$. $\mathrm{R}_2 \equiv \mathrm{A}$.

We seek to represent these equations in the form

$$\partial\chi^i/\partial t = [\chi^i, \hat{H}]\,,\quad i = 0, 1, \ldots, 6\,, \tag{4}$$

where

$$\chi^i \equiv f_\alpha\,, \quad \text{for } i=0\,,$$
$$\equiv \boldsymbol{E}\,, \quad \text{for } i=1,2,3\,,$$
$$\equiv \boldsymbol{B}\,, \quad \text{for } i=4,5,6\,,$$

the hamiltonian functional

$$\hat{H}\{\chi^i\} \equiv \sum_\alpha \int_{\mathrm{R}_1} \tfrac{1}{2} m_\alpha v^2 f_\alpha \,\mathrm{d}z + \int_{\mathrm{R}} \tfrac{1}{2}(E^2+B^2)\,\mathrm{d}\boldsymbol{x}\,,$$

and the operator [,] is the Poisson bracket (which is complicated since the χ^i's are noncanonical).

Before presenting the bracket we now lay some groundwork. Suppose that the solutions of eqs. (1), (2) and (3) are contained in a vector space $\omega = \omega_1 \times \omega_2$ (over $\mathbf{R}$) which is the direct product of subspace ω_1 whose elements are functions of z, and subspace ω_2 whose elements are functions of $\boldsymbol{x}$. Then observe that the operator $P(z|\boldsymbol{x})$ is used in eqs. (2) and (3) to map elements of one subspace to the other. Now suppose that ω is equipped with the following inner product

$$\langle g|h\rangle = \int_{\mathrm{R}_1} g_1 h_1 \,\mathrm{d}z + \int_{\mathrm{R}_2} g_2 h_2 \,\mathrm{d}\boldsymbol{x}\,, \tag{5}$$

where all $g \in \omega$ have the form $g \equiv (g_1, g_2)$ such that $g_1 \in \omega_1$ and $g_2 \in \omega_2$. (Clearly eq. (5) satisfies the necessary algebraic properties for an inner product.) We define our Poisson bracket in terms of this inner product and a skew-symmetric matrix operator on ω as follows:

$$[\hat{F}, \hat{G}] = \sum_{\alpha,i,j} \langle \delta\hat{F}/\delta\chi^i | O^{ij} \delta\hat{G}/\delta\chi^j\rangle\,. \tag{6}$$

The quantities $\hat{F}$ and $\hat{G}$ are elements of Ω, a vector space (over $\mathbf{R}$) of Frechet differentiable functionals of the functions f_α, $\boldsymbol{E}$ and $\boldsymbol{B}$ [differentiable with respect to the L^2 norm defined by eq. (5)]. For example Ω contains elements of the form

$$\hat{F}\{\chi^i\} = \hat{F}_1\{\chi^0\} + \hat{F}_2\{\chi^i\}, \quad i \neq 0\,, \tag{7}$$

where

$$\hat{F}_\beta\{\chi^i\} = \int_{\mathrm{R}_\beta} F_\beta(x_\beta, \chi_k^{i_\beta})\,\mathrm{d}x_\beta\,, \quad \beta = 1,2\,,$$

with $x_1 \equiv z$, $x_2 \equiv \boldsymbol{x}$, $i_1 = 0$ and $i_2 = 1, 2, 3$. The subscript k on $\chi_k^{i_\beta}$ means that F_β is in general a function of all partial derivatives of $\chi_k^{i_\beta}$ of degree k. Clearly the hamiltonian functional $\hat{H}\{\chi^i\}$ is of the form of eq. (7) with $k = 0$. Ω also contains arbitrary C^∞ functions of χ_k^i for all i. In eq. (6) the quantity $\delta F/\delta\chi^i$ is the functional derivative of $\hat{F}$ with respect to χ^i; it is defined by

$$(\mathrm{d}/\mathrm{d}\epsilon)\hat{F}\{\chi^0(z) + \epsilon w(z)\}|_{\epsilon=0} \equiv \langle \delta\hat{F}/\delta\chi^0 | w\rangle\,,$$

and similarly for $i \neq 0$. Observe $w(z)$, $\delta\hat{F}/\delta\chi^0 \in \omega_1$ and for $i \neq 0$ $\delta\hat{F}/\delta\chi^i \in \omega_2$. [Note $\delta\chi^0(z)/\delta\chi^0(z') = \delta(z-z')$ and $\delta\chi^i(\boldsymbol{x})/\delta\chi^i(\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}')$; since for the hamiltonian $k=0$, functional differentiation reduces to ordinary partial differentiation of its integrands: $\delta\hat{H}/\delta f_\alpha = \frac{1}{2} m_\alpha v^2$, $\delta\hat{H}/\delta\boldsymbol{E} = \boldsymbol{E}$ and $\delta\hat{H}/\delta\boldsymbol{B} = \boldsymbol{B}$.] The bracket defined by eq. (6) is a bilinear function which maps $\Omega \times \Omega$ to Ω. In addition we require that the operator O^{ij} endow our bracket with the following properties:

(i) $[\hat{F}, \hat{F}] = 0$ for every $\hat{F} \in \Omega$. Since Ω is defined over $\mathbf{R}$ this is equivalent to $[\hat{F}, \hat{G}] = -[\hat{G}, \hat{F}]$ for $\hat{F}, \hat{G} \in \Omega$.

(ii) The Jacobi identity

$$[\hat{E}, [\hat{F}, \hat{G}]] + [\hat{F}, [\hat{G}, \hat{E}]] + [\hat{G}, [\hat{E}, \hat{F}]] = 0\,,$$

for every $\hat{E}, \hat{F}, \hat{G} \in \Omega$.

Clearly these are the usual properties possessed by a Poisson bracket. A vector space together with a bracket which has these properties defines a Lie algebra [6].

In summary, by a hamiltonian system we mean a system of partial differential equations which possess an integral invariant and a bracket with properties (i) and (ii), such that the system is amenable to the form of eq. (4).

We now introduce the following operator (cosymplectic form):

$$(O^{ij}) = \left[\begin{array}{c|c|c} -\{f_\alpha, \cdot\} & -\boldsymbol{P}_E^{2\to 1} & -\boldsymbol{P}_B^{2\to 1} \\ \hline \boldsymbol{P}_E^{1\to 2} & 0_3 & D \\ \hline \boldsymbol{P}_B^{1\to 2} & -D & 0_3 \end{array}\right]\,. \tag{8}$$

This is a 7×7 matrix operator which maps ω into ω. 0_3 is a 3×3 array of zeros and $D \equiv [\epsilon_{ijk}(\partial/\partial x_j)]$ where ϵ_{ijk} is the Levi-Civita tensor. The braces in the upper left hand entry are used to indicate the usual Poisson bracket $\{g, h\} = \partial g/\partial \boldsymbol{r} \cdot \partial h/\partial \boldsymbol{v} - \partial g/\partial \boldsymbol{v} \cdot \partial h/\partial \boldsymbol{r}$. This entry maps ω_1 into ω_1. The lower right hand block (delineated by the double hash marks) maps ω_2 into ω_2. The off-diagonal elements

$$\boldsymbol{P}_E^{2\to 1} \equiv \frac{e_\alpha}{m_\alpha} \int_{R_2} (\cdot) \frac{\partial f_\alpha}{\partial \boldsymbol{v}} P \, \mathrm{d}\boldsymbol{x} ,$$

and

$$\boldsymbol{P}_B^{2\to 1} \equiv \frac{e_\alpha}{m_\alpha} \int_{R_2} (\cdot) \left(\frac{\partial f_\alpha}{\partial \boldsymbol{v}} \times \boldsymbol{v}\right) P \, \mathrm{d}\boldsymbol{x} ,$$

send elements of ω_2 into ω_1. The off-diagonal elements

$$\boldsymbol{P}_E^{1\to 2} \equiv \frac{e_\alpha}{m_\alpha} \int_{R_1} (\cdot) \frac{\partial f_\alpha}{\partial \boldsymbol{v}} P \, \mathrm{d}z ,$$

and

$$\boldsymbol{P}_B^{1\to 2} \equiv \frac{e_\alpha}{m_\alpha} \int_{R_1} (\cdot) \left(\frac{\partial f_\alpha}{\partial \boldsymbol{v}} \times \boldsymbol{v}\right) P \, \mathrm{d}z ,$$

send elements of ω_1 into ω_2.

Since O^{ij} is skew-symmetric, property (i) is satisfied. We have proved the Jacobi identity for the first and last terms of eq. (9) below. For the cross terms we have done so for a class of restricted functionals of the form of eq. (7) with $k = 0$. (These properties depend upon integrations by parts and subsequent neglect of surface terms. We require $F_1(z, \chi_k^0)$ to vanish as $|v| \to \infty$ for all elements of Ω. The surface terms obtained by spatial integrations by parts can also be assumed to vanish [7].) Writing out eq. (6) we obtain

$$[\hat{F}, \hat{G}] = [\hat{F}, \hat{G}]_1 + [\hat{F}, \hat{G}]_2 , \tag{9}$$

where

$$[\hat{F}, \hat{G}]_1 \equiv \sum_\alpha \left\{ \int_{R_1} f_\alpha \left\{ \frac{\delta \hat{F}}{\delta f_\alpha}, \frac{\delta \hat{G}}{\delta f_\alpha} \right\} \mathrm{d}z - \int_{R_1} \frac{\delta \hat{F}}{\delta f_\alpha} \boldsymbol{P}_E^{2\to 1} \cdot \left(\frac{\delta \hat{G}}{\delta \boldsymbol{E}}\right) \mathrm{d}z - \int_{R_1} \frac{\delta \hat{F}}{\delta f_\alpha} \boldsymbol{P}_B^{2\to 1} \cdot \left(\frac{\delta \hat{G}}{\delta \boldsymbol{B}}\right) \mathrm{d}z \right\} , \tag{10}$$

$$[\hat{F}, \hat{G}]_2 \equiv \sum_\alpha \left\{ \int_{R_2} \frac{\delta \hat{F}}{\delta \boldsymbol{E}} \cdot \boldsymbol{P}_E^{1\to 2} \left(\frac{\delta \hat{G}}{\delta f_\alpha}\right) \mathrm{d}\boldsymbol{x} + \int_{R_2} \frac{\delta \hat{F}}{\delta \boldsymbol{B}} \cdot \boldsymbol{P}_B^{1\to 2} \left(\frac{\delta \hat{G}}{\delta f_\alpha}\right) \mathrm{d}\boldsymbol{x} \right\} + \int_{R_2} \left(\frac{\delta \hat{F}}{\delta \boldsymbol{E}} \cdot \nabla \times \frac{\delta \hat{G}}{\delta \boldsymbol{B}} - \frac{\delta \hat{G}}{\delta \boldsymbol{B}} \cdot \nabla \times \frac{\delta \hat{F}}{\delta \boldsymbol{E}}\right) \mathrm{d}\boldsymbol{x} . \tag{11}$$

Substituting the χ^i and $\hat{H}$ into eq. (9) renders eqs. (1), (2), and (3) in the form of eq. (4). Observe that the first term of eq. (10) yields the Vlasov equation without the coupling term; the last term of eq. (11) yields the Maxwell equations in vacuum. The remaining terms yield the coupling terms.

Neglecting all the terms which contain $\boldsymbol{B}$ in eq. (9) and $\hat{H}$ produces a bracket for the Vlasov–Poisson equations [i.e., eqs. (2) and (3) with $\boldsymbol{B} = 0$; the constraint $\nabla \times \boldsymbol{E} = 0$ is required of functionals of Ω]. This system can be further reduced by eliminating $\boldsymbol{E}$ in the Vlasov equation in terms of f_α [‡1]. The bracket thus obtained is

$$[\hat{F}, \hat{G}] = \sum_\alpha \int_{R_1} f_\alpha(z) \left\{ \frac{\delta \hat{F}}{\delta f_\alpha}, \frac{\delta \hat{G}}{\delta f_\alpha} \right\} \mathrm{d}z . \tag{12}$$

From eq. (12) we obtain the Vlasov equation in the form

$$\frac{\delta f_\alpha}{\delta t} = \left[\frac{\delta f_\alpha}{\delta f_\alpha}, \frac{\delta \hat{H}_E}{\delta f_\alpha}\right] = -\{f_\alpha, H_p^\alpha\} ,$$

where

$$\hat{H}_E\{f_\alpha\} = \sum_\alpha \left(\int_{R_1} H_1^\alpha(z) f_\alpha(z) \, \mathrm{d}z + \frac{1}{2} \int_{R_1} \int_{R_1} f_\alpha(z) f_\alpha(z') H_2^\alpha(z|z') \, \mathrm{d}z \, \mathrm{d}z' \right) ,$$

$H_1^\alpha = \frac{1}{2} m_\alpha v^2$ and $H_2^\alpha = e_\alpha^2/|\boldsymbol{r} - \boldsymbol{r}'|$. H_p^α is the particle hamiltonian which we observe, $H_\mathrm{p}^\alpha = \delta \hat{H}_E/\delta f_\alpha$.

‡1 This reduced form was derived by Allan N. Kaufman in collaboration with the author while both were guests at the Aspen Center for Physics. It has been brought to our attention that J. Gibbons has independently obtained this result [8].

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