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SOME OBSERVATIONS REGARDING BRACKETS AND DISSIPATION

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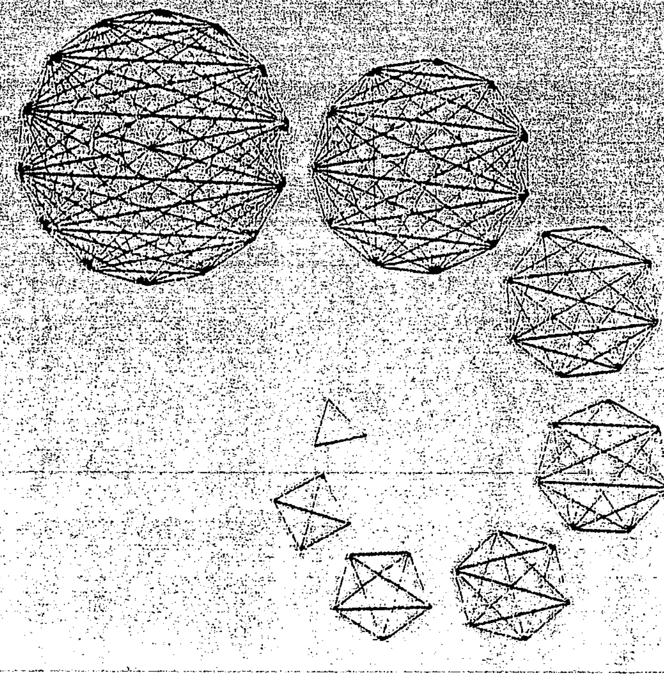

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SOME OBSERVATIONS REGARDING BRACKETS AND DISSIPATION

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Abstract

Some ideas relating to a bracket formulation for dissipative systems are considered. The formulation involves a bracket that is analogous to a generalized Poisson bracket, but possesses a symmetric component. Such a bracket is presented for the Navier-Stokes equations.

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Many of the fundamental nondissipative equations describing fluids and plasmas have been shown to be Hamiltonian field theories in terms of generalized Poisson brackets (GPB). For review see [1-4]. Here we discuss a formalism for entropy producing conservative systems. As an example, the Navier-Stokes equations are considered. (This report is a companion to [5] where plasma kinetic equations are treated. A nonconservative system was discussed in [6]. Other formalisms were presented in [7-10].)

Recall that a GPB is a bilinear, antisymmetric operator that is a derivation on functionals and satisfies the Jacobi identity. The GPB need not be the usual Poisson bracket; hence fields that do not possess standard or canonical form can sometimes still be expressed as follows:

$$\frac{\partial\psi^i}{\partial t} = \{\psi^i, \mathcal{H}\} \qquad i = 1,2,\ldots N \quad , \tag{1}$$

where the Hamiltonian functional $\mathcal{H}$ is the "generator of time translation" and the quantities ψ^i are the field components. For two functionals F and G GPB's typically have the form

$$\{F,G\} = \int \frac{\delta F}{\delta\psi^i} O^{ij} \frac{\delta G}{\delta\psi^j} \, d\tau \quad , \tag{2}$$

where $\delta F/\delta\psi^i$, the functional derivative, is defined by $\left.\frac{d}{d\varepsilon} F[\psi^i+\varepsilon\delta\psi]\right|_{\varepsilon=0} = \int \frac{\delta F}{\delta\psi^i} \delta\psi \, d\tau$. $d\tau$ is a volume element; and O^{ij} is an operator that in light of antisymmetry must be anti-self-adjoint.

Systems that are dissipative would not a priori be expected to fit into the form of Eq. (1). Indeed it is not clear what functional should be the "generator of time translation", and which algebraic properties of a binary bracket operator will lead to a rich structure.

We address the first point above by recalling that in classical thermodynamics the equilibrium state can be obtained by either the energy or entropy extremum principles. In this sense we view the energy, a function of the extensive variables, as the "generator of equilibria", or alternatively the entropy can generate equilibria. Moreover, additional extremum principles exist in terms of the thermodynamic potentials. For dynamical systems an extension of this is to choose from among these quantities the "generator of time translation".

In particular an appealing choice is a quantity we call the "generalized free energy". In the energy formulation of thermodynamics the equilibrium state is obtained by extremizing the energy at constant entropy. This can be achieved by varying the following:

$$F_\lambda = E + \lambda S \quad , \tag{3}$$

where E is the energy, S is the entropy and λ is a Lagrange multiplier. A natural generalization of this for dynamical systems is to add to the Hamiltonian quantities known as Casimirs or "generalized entropy" functionals. These are functionals that, due to degeneracy in a GPB, are conserved for all Hamiltonians; i.e. they commute with all functionals. Such quantities, independent of the GPB formalism, have previously been used to obtain variational principles for plasma equilibria [11-14]; such principles are useful for obtaining linear stability criteria. Recently, using the GPB formalism, nonlinear stability results have been obtained by using Casimirs [15, 16]. Thus generalizing Eq. (3) we obtain

$$\mathcal{G} = \mathcal{H} + \mathcal{S} \tag{4}$$

where $\mathcal{S}$ is a Casimir. (Observe that we have dropped the Lagrange multipliers since typically Casimirs involve free functions; see Eq. (22) below.) The reason that the quantity $\mathcal{G}$ of Eq. (4) is an appealing "generator of time translation" is that by analogy critical points of $\mathcal{G}$ correspond to both thermodynamic and dynamic equilibria. $\mathcal{G}$ so defined is what we have termed the "generalized free energy."

It remains to describe the binary bracket operator that together with $\mathcal{G}$ produces the equations of motion; i.e., in the form

$$\frac{\partial\psi^i}{\partial t} = \{\{\psi^i, \mathcal{G}\}\} \quad , \tag{5}$$

where the double braces are used for the dissipative generalization of Eq. (2). Just as any operator can be split into self-adjoint and anti-self-adjoint parts, we split the bracket of Eq. (5) into the sum of an antisymmetric GPB and a symmetric component. For two functionals F and G we have

$$\{\{F,G\}\} = \{F,G\} + (F,G) \tag{6}$$

where $\{F,G\}$ has the form of Eq. (2) with an anti-self-adjoint operator

O^{ij} and (F,G) is given by

$$(F,G) = \int \frac{\delta F}{\delta\psi^i} M^{ij} \frac{\delta G}{\delta\psi^j} d\tau \quad . \tag{7}$$

Here, M^{ij} is to be self-adjoint and hence (F,G) is symmetric under the interchange of F and G.

Equation (5) thus becomes

$$\frac{\partial\psi^i}{\partial t} = \{\psi^i,\mathcal{G}\} + (\psi^i,\mathcal{G}) = (O^{ij} + M^{ij}) \frac{\delta\mathcal{G}}{\delta\psi^j} \quad . \tag{6}$$

From Eq. (6) it is clear that critical points of $\mathcal{G}$; i.e. points where $\delta\mathcal{G}/\delta\psi^i = 0$, correspond to dynamical equilibrium, since clearly $\partial\psi^i/\partial t = 0$. Also Eq. (6) can be rewritten as

$$\frac{\partial\psi^i}{\partial t} = \{\psi^i,\mathcal{H}\} + (\psi^i,\mathcal{G}) \quad , \tag{7}$$

since the difference between $\mathcal{H}$ and $\mathcal{G}$ is a Casimir. From Eq. (7) we see that the dynamics is split into Hamiltonian and non-Hamiltonian parts. Moreover, if the symmetric bracket has the degeneracy property $(\mathcal{H},G) = 0$ for all functionals G, Eq. (7) becomes

$$\frac{\partial\psi^i}{\partial t} = \{\psi^i,\mathcal{H}\} + (\psi^i,\mathcal{S}) \quad . \tag{8}$$

Thus the time rate of change of the generalized entropy is given by

$$\frac{d\mathcal{S}}{dt} = (\mathcal{S},\mathcal{S}) \quad . \tag{9}$$

From Eq. (9) it is clear that definiteness of the symmetric bracket is equivalent to an H-theorem. The ideas of degeneracy and definiteness first appeared in [7] and were subsequently employed in [5, 8-10].

We now consider the Navier-Stokes equations

$$\frac{\partial v_i}{\partial t} = - v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k} \tag{10}$$

$$\frac{\partial s}{\partial t} = - v_k \frac{\partial s}{\partial x_k} + \frac{\sigma_{ik}}{\rho T}\frac{\partial v_i}{\partial x_k} - \frac{1}{\rho T}\frac{\partial q_k}{\partial x_k} \tag{11}$$

$$\frac{\partial \rho}{\partial t} = - \frac{\partial}{\partial x_k}(\rho v_k) \quad , \tag{12}$$

Equation (10) is the equation of motion, where v_i is the i^{th} $(i = 1,2,3)$ component of the velocity field, which is assumed to be a function of the spatial coordinate x_k as well as time t. Repeated sum notation is assumed. As usual, p is the pressure, ρ is the mass density and T is the temperature. The heat equation, Eq. (11) is written in terms of the entropy per unit mass s, in order to explicitly show those terms that instigate entropy production. The quantities σ_{ik} and q_k are the viscosity stress tensor and the conductive heat flux density respectively. They are given by the following constitutive relations:

$$\sigma_{ik} = \eta \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial v_t}{\partial x_t}\right) + \zeta\, \delta_{ik}\frac{\partial v_t}{\partial x_t} \tag{13}$$

$$q_k = - \kappa \frac{\partial T}{\partial x_k} \quad , \tag{14}$$

where η and ζ are the viscosity coefficients, which are in general positive functions of p and T. The thermal conductivity is κ, which may in addition be a function of $|\nabla T|$. The system of equations given by (10)-(12) is closed by the thermodynamic relations

$$p = \rho^2 \frac{\partial U}{\partial \rho} \tag{15}$$

and

$$T = \frac{\partial U}{\partial s} \quad , \tag{16}$$

where $U(\rho,s)$ is the internal energy per unit mass; $U(\rho,s)$ is assumed to be a known function of ρ and s.

The Navier-Stokes equations, as given, are known to conserve the energy

$$\mathcal{H} = \int \left(\frac{1}{2}\rho v^2 + \rho U(\rho,s)\right) d^3\mathbf{x} \quad , \tag{17}$$

but produce entropy as a result of the terms of Eq. (11) involving σ_{ik} and q_k ; i.e. by viscous dissipation and heat flux. Before presenting the symmetric bracket that produces these terms we review the Hamiltonian structure for the Euler equations (i.e. $\sigma_{ik}, q_k \to 0$) as given in [17] (see also [2]).

The Hamiltonian in this case is the total energy functional of Eq. (17). The equations of motion, continuity and entropy are given by

$$\frac{\partial v_i}{\partial t} = \{v_i, \mathcal{H}\} \tag{18}$$

$$\frac{\partial \rho}{\partial t} = \{\rho, \mathcal{H}\} \tag{19}$$

$$\frac{\partial s}{\partial t} = \{s, \mathcal{H}\} \tag{20}$$

where the GPB, $\{,\}$, is given by

$$\{F,G\} = - \int \Bigg(\frac{\delta F}{\delta \rho} \vec{\nabla} \cdot \frac{\delta G}{\delta \vec{v}} + \frac{\delta F}{\delta \vec{v}} \cdot \vec{\nabla} \frac{\delta G}{\delta \rho} +$$

$$\frac{\delta F}{\delta \vec{v}} \cdot \left[\frac{(\vec{\nabla}\times\vec{v})}{\rho} \times \frac{\delta G}{\delta \vec{v}} \right] + \frac{\vec{\nabla} s}{\rho} \cdot \left[\frac{\delta F}{\delta s} \frac{\delta G}{\delta \vec{v}} - \frac{\delta F}{\delta \vec{v}} \frac{\delta G}{\delta s} \right] \Bigg) d^3x \; . \tag{21}$$

Upon inserting the quantities shown on the right hand side of Eqs. (18)-(20), into Eq. (21) and performing the indicated operations one obtains, as noted, the invicid adiabatic limit of Eqs. (10)-(12).

The Casimirs for the bracket given by Eq. (21) are the total mass $M = \int \rho \, d^3x$ and a generalized entropy functional $\mathcal{S}_f = \int \rho f(s) \, d^3x$, where f is an arbitrary function of s. The latter quantity is added to the energy [Eq. (17)] to produce the generalized free energy of Eq. (4): $\mathcal{G} = \mathcal{H} + \mathcal{S}_f$.

In order to obtain the dissipative terms, we introduce the following symmetric bracket:

$$(F,G) = \frac{1}{\lambda} \int \Bigg\{ \frac{1}{\rho} \frac{\delta F}{\delta v_i} \frac{\partial}{\partial x_k} \left[\frac{\sigma_{ik}}{\rho} \frac{\delta G}{\delta s} \right] + \frac{1}{\rho} \frac{\delta G}{\delta v_i} \frac{\partial}{\partial x_k} \left[\frac{\sigma_{ik}}{\rho} \frac{\delta F}{\delta s} \right]$$

$$+ \frac{\sigma_{ik}}{T} \frac{\partial v_i}{\partial x_k} \left[\frac{1}{\rho^2} \frac{\delta F}{\delta s} \frac{\delta G}{\delta s} \right] + T^2 \kappa \frac{\partial}{\partial x_k} \left[\frac{1}{\rho T} \frac{\delta F}{\delta s} \right] \frac{\partial}{\partial x_k} \left[\frac{1}{\rho T} \frac{\delta G}{\delta s} \right]$$

$$+ T \, \Lambda_{ikmn} \frac{\partial}{\partial x_m} \left[\frac{1}{\rho} \frac{\delta F}{\delta v_n} \right] \frac{\partial}{\partial x_k} \left[\frac{1}{\rho} \frac{\delta G}{\delta v_i} \right] \Bigg\} d^3x \quad , \tag{23}$$

where

$$\Lambda_{ikmn} = \eta\,(\delta_{ni}\delta_{mk} + \delta_{nk}\delta_{mi} - \frac{2}{3}\,\delta_{ik}\delta_{mn}) + \zeta\,\delta_{ik}\,\delta_{mn}, \qquad (24)$$

from which we note that $\sigma_{ik} = \Lambda_{ikmn}\partial v_n/\partial x_m$, and λ is an arbitrary constant. In addition to symmetry this bracket possesses the following properties:

(a) There are degeneracies associated with the momentum functional $\vec{P} = \int \rho\vec{v}\, d^3x$ and energy functional H; i.e. $(\vec{P}, G) = (\mathcal{H}, G) = 0$ for all functionals G.

(b) For all functionals the bracket is definite with sign depending upon λ. This is clear for the term that depends upon κ (recall $\kappa > 0$), but it is not immediately apparent for the remaining terms, so we rewrite the bracket as follows:

$$(F,G) = \frac{1}{\lambda}\int\left\{T\Lambda_{ikmn}\left[\frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\frac{\delta F}{\delta v_k}\right) - \frac{1}{\rho T}\frac{\partial v_i}{\partial x_k}\frac{\delta F}{\delta s}\right]\right.$$
$$\left.\times\left[\frac{\partial}{\partial x_m}\left(\frac{1}{\rho}\frac{\delta G}{\delta v_n}\right) - \frac{1}{\rho T}\frac{\partial v_m}{\partial x_n}\frac{\delta G}{\delta s}\right] + \kappa T^2\,\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta F}{\delta s}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta G}{\delta s}\right]\right\} d^3x \;.$$

Definiteness arises from the fact that $\Lambda_{ikmn}\, a_{ik}a_{mn} > 0$ for any (a_{ik}). An important ramification of definiteness occurs for the functional $\mathcal{S}_f$. Definiteness in this case corresponds to an H-theorem, which is valid even though the function f remains arbitrary.

(c) If we let $f = \lambda s$ upon inserting $\mathcal{Q}$ into Eq. (23) with $\vec{v}$, ρ, and s we obtain

$$(v_j,\mathcal{Q}) = \frac{1}{\rho}\frac{\partial}{\partial x_k}\sigma_{jk} \tag{25}$$

$$(\rho,\mathcal{Q}) = 0 \tag{26}$$

$$(s,\mathcal{Q}) = \frac{\sigma_{ik}}{\rho T}\frac{\partial v_i}{\partial x_k} + \frac{1}{\rho T}\frac{\partial}{\partial x_k}\left(\kappa\frac{\partial T}{\partial x_k}\right). \tag{27}$$

Equations (25)-(27) yield the dissipative terms of the Navier-Stokes equations. Since $\mathcal{S}$ is a Casimir, the Navier-Stokes equations are given by

$$\frac{\partial v_j}{\partial t} = \{\{v_j,\mathcal{Q}\}\}$$

$$\frac{\partial \rho}{\partial t} = \{\{\rho,\mathcal{Q}\}\}$$

$$\frac{\partial s}{\partial t} = \{\{s,\mathcal{Q}\}\} \quad .$$

Observe that had we chosen a nonlinear f Eqs. (25) and (27) would obtain additional dependence upon s.

In closing, we point out that for general systems, symmetry in transport coefficients is related to bracket symmetry. For the purpose of illustration we demonstrate this by replacing the scalar conductivity κ by a tensor κ_{ij}. Usually anisotropy arises because of the presence of a magnetic field B, as in the case of a crystal or conducting fluid. Here we ignore the dependence of κ_{ij} on B, but evidently the formalism presented here for the Navier-Stokes equations can be extended to magnetohydrodynamics with constitutive relations arising from small Larmor radius corrections [18].

If we replace the penultimate term of Eq. (23) by

$$\int \left(T^2 \kappa_{ij} \frac{\partial}{\partial x_i} \left[\frac{1}{\rho T} \frac{\delta F}{\delta s} \right] \frac{\partial}{\partial x_j} \left[\frac{1}{\rho T} \frac{\delta G}{\delta s} \right] \right) d^3x \quad , \tag{28}$$

then in order to maintain symmetry in the bracket it is necessary for $\kappa_{ij} = \kappa_{ji}$. This corresponds to Onsager symmetry since here $\kappa_{ij}(B) = \kappa_{ij}(-B)$. The contribution to the heat equation that is produced by Eq. (28) is

$$\frac{1}{\rho T} \frac{\partial}{\partial x_i} \kappa_{ij} \frac{\partial T}{\partial x_j} \quad .$$

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References

1. E.C.G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective (Wiley 1974). Chapt. 9 deals with GPBs for ordinary differential equations.

2. P.J. Morrison, "Poisson Brackets for Fluids and Plasmas", in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, eds. M. Tabor and Y. Treve (AIP Conf. Proc. 88, 1982), 13-46.

3. J. Marsden, A. Weinstein, T. Ratiu, R. Schmid and R. Spencer, "Hamiltonian Systems with Symmetry, Coadjoint Orbits and Plasma Physics" in Modern Developments in Analytical Mechanics, (Proc. IUTAM-ISIMM, 1982).

4. J.E. Marsden and P.J. Morrison, "Noncanonical Hamiltonian Field Theory and Reduced MHD", Contemporary Math. 28 (1984). Am. Math. Soc.

5. P.J. Morrison, "Bracket Formulation for Irreversible Classical Fields", Physics Letters 100A, 423 (1984).

6. P.J. Morrison and R.D. Hazeltine, "Hamiltonian Formulation of Reduced Magnetohydrodynamics" Physics of Fluids 27, 886 (1984).

7. M. Grmela, talk presented at AMS workshop, Boulder, CO, July 1983, "Particle and Bracket Formulations of Kinetic Equations", Contemporary Math.28 (1984). Am. Math. Soc.

8. A.N. Kaufman, "Dissipative Hamiltonian Systems: A Unifying Principle", Physics Letters 100A, 419 (1984).

9. P. Similon, private communication.

10. M. Grmela, "Bracket Formulation of Dissipative Fluid Mechanics Equations", preprint (1984).

11. W. Newcomb, appendix of I. Bernstein, Physical Review 109, (1958).

12. M.D. Kruskal and C.R. Oberman, "On the Stability of Plasma in Static Equilibrium", Physics of Fluids 1, 275 (1958).

13. T.K. Fowler, "Lyapunov's Stability Criteria for Plasmas", J. Math. Phys. 4, 559 (1963).

14. M. Rosenbluth, "Topics in Microinstabilities", in Advanced Plasma Theory, (Academic 1964) 136.

15. D.D. Holm, J.E. Marsden, T. Ratiu and A. Weinstein, "Nonlinear Stability Conditions and A Priori Estimates for Barotropic Hydrodynamics", Physics Letters 98A, 15 (1983). (Also in preparation "A Priori Estimates for Nonlinear Stability of Fluids and Plasmas".)

16. R.D. Hazeltine, D.D. Holm,, J.E. Marsden and P.J. Morrison, "Generalized Poisson Brackets and Nonlinear Liapunov Stability - Appliction to Reduced MHD, "Center for Pure and Applied Math. Report PAM-214/NSRMHD, Berkeley, CA (1984). Submitted to ICPP proceedings Lausanne (1984).

17. P.J. Morrison and J.M. Greene, "Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal Magnetohydrohynamics", Phys. Rev. Lett. 45, 790 (1980).

18. W.B. Thompson, An Introduction to Plasma Physics, (Pergamon 1962) 227.

Erratum

Page 3, Line 2: Essential ideas (energy conservation, definiteness, etc.) in the context of brackets for dissipation actually appeared earlier in

> A. N. Kaufman and P. J. Morrison, "Algebraic Structure of the Plasma Quasilinear Equations," Phys. Lett **88A**, 405 (1982),

which evaded the memories of both of these authors* during the independent preparations of their back-to-back papers of Refs. [5] and [8].

* A. N. Kaufman private communication October (2007).

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