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Hamiltonian four-field model.

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The four-field model for nonlinear tokamak fluid dynamics is a generalization of reduced MHD that allows for slow evolution ($\omega < \omega_*$), long mean-free-path electron dynamics, and various effects of plasma compressibilty, in a simple albeit non-rigorous way. Like previous models it reproduces such features of kinetic and FLR physics as the "semi-collisional" conductivity, gyroviscosity-modified nonlinear convection, curvature-modified drift-tearing instability and diffusion in a stochastic magnetic field. Also like its predecessors it omits temperature gradients and kinetic effects of magnetic trapping. Finally, unlike previous work (but like the underlying physics it atttempts to represent), it not only conserves energy but is a Hamiltonian dynamical system.

Note that the Hamiltonian property is much stronger than energy conservation; it implies both conservation of phase space volume as well as the proper invariance of generalized helicities (Casimir invariants).

First recall the following conventional normalized field variables: $\psi = (\varepsilon B_T a)^{-1} A_\zeta$, where ε is the inverse aspect ratio and A_ζ is the toroidal component of the vector potential; $\phi = c\Phi/(\varepsilon v_A B_T a)$, where Φ is the electrostatic potential and v_A is the Alfven speed; $v = V_{\|}/(\varepsilon v_A)$, where $V_{\|}$ is the ion parallel velcocity; and $p = (\beta/\varepsilon)[(n/n_c) -1]$, where n is the plasma density. It is also convenient to introduce a velocity stream function, F, according to $(1 + \tau\beta\delta^2\nabla^2)F = \phi + \delta\tau p$. This contains the expected combination of electric and diamagnetic drifts, with FLR corrections. We finally use the parallel current density $J \equiv \nabla^2{}_\perp\psi$ and parallel vorticity $W = \nabla^2{}_\perp F$ in order to obtain the four-field model shown in Eqs. (1)-(4). Here $h \equiv (R - R_0)/a$ is a normalized "horizontal" distance (R is the major radius and R_0 the major radius of the magnetic axis), and the three constant parameters are $\beta \equiv 8\pi n_c T_e/B_T{}^2$, where n_c measures the central plasma density and B_T measures the toroidal magnetic field; $\delta \equiv c/(2\omega_{pi}a)$, where ω_{pi} is the ion plasma frequency and a is the plasma radius; and $\tau \equiv T_i/T_e$. The "inner bracket" is conventional,

$$[f,g] \equiv \zeta\cdot\nabla_\perp f \times \nabla_\perp g,$$

where ζ is a unit vector in the toroidal direction, and the parallel gradient operator is defined by $\nabla_{\|} f \equiv \partial f/\partial\zeta + [f,\psi]$.

The conserved field energy is given by

$$H \equiv {}^1\!/_2 \langle\, |\nabla_\perp F|^2 + |\nabla_\perp \psi|^2 + (1+\tau)p^2/(2\beta) + v^2 \,\rangle,$$

where the angular brackets denote an integral over the system volume. This energy is easily understood in terms of fluid kinetic, magnetic field and thermal energies.

To demonstrate the Hamiltonian character of this system we introduce field variables ξ^i according to

$$\xi^1 = \nabla_\perp{}^2(F - \delta\tau p/2), \quad \xi^2 = \psi,$$
$$\xi^3 = p + 2\beta h, \quad \xi^4 = v.$$

Thus ξ^1 is a modified vorticity; ξ^2 measures the poloidal magnetic flux; ξ^3 is a curvature-corrected electron pressure; and ξ^4 measures the ion parallel velocity. The system energy then takes

$$(\partial/\partial t)W + [F,W] + \nabla_{\|}J + (1+\tau)(1+\tau\delta^2\beta\nabla_\perp{}^2)[h,p] = \delta\tau\nabla_\perp\cdot[p+2\beta h,\nabla_\perp F] + (1/2)\tau^2\delta^3\beta\nabla_\perp{}^2[p+2\beta h,W] - (1/2)\tau\delta\beta\nabla_\perp{}^2\nabla_{\|}(v+2\delta J), \quad (1)$$

$$(\partial/\partial t)\psi + \nabla_{\|}\phi - \delta\nabla_{\|}p = 0, \quad (2)$$

$$(\partial/\partial t)p + [\phi, p+2\beta h] = \beta\{2\delta[p,h] - \nabla_{\|}(v+2\delta J)\}, \quad (3)$$

$$(\partial/\partial t)v + [\phi,v] + (1/2)\nabla_{\|}[p+\tau(p-\delta\beta W)] = \tau\delta^2\beta[v,\nabla_\perp{}^2(F-\delta\tau p)]+2\delta\tau\beta[v,h]. \quad (4)$$

the form

$$H[\xi] \equiv {}^1/_2\langle\, |\nabla_\perp(\nabla_\perp^{-2}\xi^1) + {}^{\delta\tau}/_2\nabla_\perp(\xi^3 - 2\beta h)\,|^2 + |\nabla_\perp\xi^2|^2 + (1+\tau)(\xi^3 - 2\beta h)^2/(2\beta) + (\xi^4)^2\,\rangle$$

where $\nabla_\perp^{-2}$ is the inverse of the Laplacian. H is to be considered a Hamiltonian functional of the ξ^i, with functional derivatives $H_i \equiv \delta H/\delta\xi^i$The Hamiltonian version of Eqs. (1) - (4) is given by

$$\partial\xi^i/\partial t = \{\xi^i, H\}, \qquad (5)$$

where the noncanonical Poisson bracket is defined for arbitrary functionals F and G by

$$\{F,G\} = \langle C^{ij}{}_k\xi^k[F_i, G_j] + C^{ij}{}_2(F_i\partial G_j/\partial\zeta - G_i\partial F_j/\partial\zeta)\rangle. \qquad (6)$$

Here $F_i \equiv \delta F/\delta\xi^i$, and a sum over paired indices is implicit. The coefficient matrix $C^{ij}{}_k$, symmetric with respect to its upper indices, has the following nonzero components:

$$\begin{aligned} C_k{}^{1j} &= C_k{}^{j1} = \delta_{kj},\\ C_k{}^{23} &= C_k{}^{32} = 2\delta\beta\delta_{k2},\\ C_k{}^{33} &= 2\delta\beta\delta_{k3},\\ C_k{}^{34} &= C_k{}^{43} = -\beta\delta_{k2},\\ C_k{}^{44} &= -\delta\tau(\delta_{k3} - 2\delta\beta\delta_{k1}). \end{aligned} \qquad (7)$$

We remark that Eqs. (6) and (7) define a true Poisson bracket: it is antisymmetric, it satisifies the Jacobi identity,

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\} = 0,$$

and acts as a dervation, in the sense that

$$\{F,GH\} = \{F,G\}H + G\{F,H\}.$$

Note also that $C^{ij}{}_k$ is a simple matrix, at least in the sense of being sparse.

The new FLR and compressibility terms appearing in Eqs. (1) - (4) can be understood as follows. FLR corrections appear multiplied by $\tau\delta^2\beta$ or $\tau\delta\beta$, measuring the squared ion gyrodradius, ρ_i^2. Such terms occur in combination with the expected Laplacian factor, and have a well known interpretation in terms of averages over the Larmor orbit. The FLR terms manifest on the right-hand side of Eq. (1) describe, in particular, nonlinear diamagnetic convection and ion gyroviscosity. In linear theory these terms reproduce the ion drift frequency corrections of gyrokinetic theory.

Another type of FLR correction is most apparent in Eq. (4): the $\delta\beta W$ correction to the ion pressure. It can be identified with a well-known residue from "gyroviscous cancellation;" thus gyroviscosity is known to modify the ion scalar pressure in an FLR plasma according to $p_i \rightarrow p_i[1 - (2\Omega_i)^{-1}\mathbf{b}\cdot\nabla\times\mathbf{V}_i]$, where Ω_i is the ion gyrofrequency, $\mathbf{V}_i$ the ion fluid velocity and $\mathbf{b}$ a unit vector in the direction of the magnetic field. When this correction factor is expressed in terms of normalized variables and reduced for large aspect ratio it becomes $p-\delta\beta W$.

The remaining corrections of interest involve the plasma compressibility, given by the right-hand side of Eq. (3). The term involving h is perpendicular compressibility, resulting from curvature of the magnetic field, while the term involving $\nabla_\parallel$ is parallel compressibility of the electron flow, $V_{\parallel e} \propto v+2\delta J$. A new feature of the Hamiltonian model is the appearance of compressibility terms in Eq. (1), as seen, for example, in its last term. The contribution of compressibility to the shear-Alfven law, although often omitted, is easily understood. First of all, the vorticity of diamagnetic flow, $\zeta\cdot\nabla\times(d/dt)(\zeta\times\nabla p)$, evidently involves the Laplacian of the compressibility, $\nabla^2(dp/dt)$. Secondly, gyroviscosity can be shown to contribute terms of the same form. Equation (1) displays the sum of these two contributions, which, together with the factor of (1/2), also occur in the rigorous version.

The Hamiltonian four-field model, drastically simpler than models obtained from systematic ordering procedures, is not rigorously asymptotic. The selection of FLR effects it contains can be characterized as the *minimal* additions to a cold-ion theory yielding the following physical properties: (i) correct cold-ion limit; (ii) physical treatment of ion diamagnetic effects; and (iii) Hamiltonian structure.♦