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Extremal Bounds on Drift Wave Growth Rates and Transport

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Abstract

A variational technique is used to obtain bounds on the growth constant γ versus wave number k for plasma drift waves. We find, for $T_i = T_e$,

$$\gamma < \sqrt{2}\omega_* \left(1 + \frac{3}{\sqrt{2}}\eta\right)$$

in usual notation. This agrees closely with dispersion—relation results that have had good success in explaining global confinement times in tokamaks based on transport coefficients of the form $\frac{\gamma}{k^2}$. The present method is easier to calculate and results are of such general nature as to give greater assurance that nothing has been missed. The method is based on the time behavior of a free energy function that is chosen to be a constant of motion for an idealized Maxwellian plasma without currents, and almost constant for small departures from this ideal state. The underlying premise associating the variational technique with drift waves remains conjectural.

Extremal Bounds on Drift Wave Growth Rates and Transport

1. Introduction

It is well known that plasma instability growth constants, γ, can be attributed to free energy sources due to departures from an idealized current free Maxwellian plasma[1,2]. Using this approach, it was shown previously that an upper bound on γ can be obtained from the time behavior of a free energy function of the form

$$H = \sum \int d\vec{x} d\vec{v} \, \frac{f_1^2 T}{2 f_0} + \Phi. \tag{1}$$

Manipulation of Eq.(1) yielded[1,3]

$$\gamma < \frac{1}{2} \text{ Max } \left(H^{-1} \frac{dH}{dt} \right) < \sum \frac{\langle u \rangle}{\lambda_D} \frac{1}{\epsilon^{1/2}} \tag{2}$$

where the sum is over ions and electrons, Φ is the field energy, λ_D is the Debye length, $\langle u \rangle$ is a certain velocity average of

$$\vec{u} = \frac{1}{m} \frac{T}{f_0} \frac{\partial f_0}{\partial \vec{v}} + \vec{v}, \tag{3}$$

and f_1 is a perturbation about an equilibrium distribution f_0. Note that, for the idealized Maxwellian plasma ($f_0 = exp(-\frac{mv^2}{2T})$ for both ions and electrons), $\vec{u} = 0$ and the bound predicts stability, as it should.

The quantity ϵ depends on f_1 and the field perturbations contained in Φ. It is given by

$$\epsilon = \frac{(K + \Phi)^2}{K \Phi}, \tag{4}$$

where K is the first term of H. In the earlier work[1,3], it was noted that $\epsilon \geq 4$ for all cases, thereby setting a rigorous upper bound on γ. This was interpreted as a bound on instabilities of high frequency, since they have the largest growth rates. It was speculated that, interpreting ϵ as a reactive dielectric constant, the bound also had relevance to low frequencies for which ϵ could only be guessed.

The purpose of the present paper is to obtain an improved, variational estimate of ϵ for the low frequency drift waves that are of current interest as a possible explanation of turbulent transport in tokamaks. The resulting bound on γ can then be used to estimate transport coefficients of the form $\chi = \frac{\gamma}{k^2}$.[4]

The paper is arranged as follows. After reviewing the derivation of Eq. (2) in Section 2, the variational expression is developed in Section 3. Results are compared with dispersion relations for electrostatic modes in Section 4 and then applied to tokamaks in Section 5. The validity of the method is discussed in Section 6. Magnetostatic modes and changes in reference frame are discussed in the Appendix.

2. Derivation of the Bound

We first review the derivation of Eq. (2) for the case of electrostatic perturbations. As mentioned, magnetic perturbations are discussed in the Appendix. For electrostatic perturbations,

$$\Phi = \frac{1}{8\pi}\int d\vec{x}(-\nabla\phi)^2 = \frac{1}{2}\sum\int d\vec{x}d\vec{v}qf_1\phi_1 \tag{5}$$

where the sum is over ions and electrons of charge q. Taking the time derivative of H gives

$$\frac{dH}{dt} = -\sum\int d\vec{x}d\vec{v}qf_1\vec{u}\cdot(-\nabla\phi_1). \tag{6}$$

Here d/dt denotes the derivative of f_1 and ϕ_1 as given by the Vlasov equation linearized about the equilibrium solution f_0. Boundary conditions are chosen to discard various surface terms arising from integration by parts, corresponding to no flow of free energy into or out of the volume of integration (f_1 vanishing at $\vec{x},\vec{v}\to\pm\infty$, or periodic in $\vec{x}$).

In Eq. (6), all equilibrium information is contained in $\vec{u}$ (which is closely related to the current). The form of the bound of Eq. (2) is already apparent, it being the maximum of $|\vec{u}|$ whatever the instability mode, while all information that selects one mode from another is contained in $\dot{H}/H$ in the symmetric form $ab/(a^2+b^2)$ where $f_1\sim a$, and$\nabla\phi\sim b$. These properties are made rigorous by the successive application of Schwarz's inequalities in $\vec{v}$ and $\vec{x}$ in Eq. (6). For each component u_i we obtain:

$$\begin{aligned}
&\sum\int d\vec{x}d\vec{v}qf_1u_i(\nabla\phi)_i \\
&\leq \sum\int d\vec{x}\left[\int d\vec{v}\frac{16\pi f_0u_i^2}{T}\right]^{1/2}\left[\int d\vec{v}\frac{f_1^2T}{2f_0}\right]^{1/2}\left[\frac{(\nabla\phi)_i^2}{8\pi}\right]^{1/2} \\
&\leq 2\sum\frac{\langle u_i\rangle}{\lambda_D}\int d\vec{x}\left[\sum\int d\vec{v}\frac{f_1^2T}{2f_0}\right]^{1/2}\left[\frac{(\nabla\phi)^2}{8\pi}\right]^{1/2} \\
&\leq 2\sum\frac{\langle u_i\rangle}{\lambda_D}K^{1/2}\Phi^{1/2}.
\end{aligned} \tag{7}$$

In the first line, we multiplied and divided by f_0 before applying the Schwarz inequality in $\vec{v}$, in order to reproduce the form of K in the final result. In the second line, we used $f_1^2 < \sum f_1^2, (\nabla\phi)_i^2 < (\nabla\phi)^2$ and we removed the u_i^2 average from the $\vec{x}$ integration by taking its maximum value at any $\vec{x}$ within the volume of integration, denoted by "max", with the further notation

$$\left[\int d\vec{v}\frac{16\pi f_0}{T}u_i^2\right]_{max}^{1/2} \equiv \frac{2\langle u_i\rangle}{\lambda_D} \quad . \tag{8}$$

The application of Schwarz's inequality on $\vec{x}$ gives the final line of Eq. (7), with the previous notation for K and Φ. Introducing Eqs. (6) and (7) into Eq. (2) gives the bound, with

$$\langle u \rangle = \sum_i \langle u_i \rangle. \tag{9}$$

The derivation proceeds in much the same way for the full electromagnetic field.[1,3]

Finally we note that the bound on γ can be applied to a restricted set of perturbations (e.g. small $k_\perp$). Let S be any such set, mapped into itself by the linearized Vlasov equation, for which there exists a maximum logarithmic derivative of H:

$$\left(\frac{1}{2}\frac{1}{H}\frac{dH}{dt}\right)_{\text{max on } S} < \alpha_s. \tag{10}$$

Field perturbations in the set S cannot have a growth constant greater than $\mu \cong \alpha_s$, since if μ exceeds α_s even by an infinitesimal amount,

$$\int_0^\infty dt\Phi(t)e^{-2\mu t} < \int_0^\infty dtH(t)e^{-2\mu t} < \int_0^\infty dtH(0)e^{2(\alpha_s-\mu)t} < \infty. \tag{11}$$

We shall now explore restricted classes of perturbations by applying variational methods to the bound.

3. Variational Properties

In this section, we examine the extremal properties of ϵ as the perturbation f_1 is varied. Let $R \equiv K/\Phi$. Extrema occur if

$$\delta\epsilon = \delta\frac{(R+1)^2}{R} = \left(1 - \frac{1}{R^2}\right)\delta R = 0. \qquad (12)$$

Setting $R = 1$ gives a minimum at $\epsilon = 4$ as already noted. Setting $\delta R = 0$ gives an eigenvalue problem, as follows.

Expanding we get

$$\delta R = \frac{1}{\Phi}\sum\int d\vec{x}d\vec{v}\delta f_1\left(f_1\frac{T}{f_0} - Rq\phi_1\right) = 0 \qquad (13)$$

from which extrema occur if

$$f_1 = Rf_0\frac{q\phi_1}{T} \qquad (14)$$

and, by Poisson's equation,

$$-\nabla^2\phi_1 = R\sum q^2\int d\vec{v}\frac{4\pi f_0}{T}\phi_1 = R\sum\frac{1}{\lambda_D^2}\phi_1. \qquad (15)$$

This is an eigenvalue problem in R that selects out different classes of perturbations (different wave numbers, k). In local approximation $\nabla^2 \to -k^2$ (although exact expressions are obtainable). For equal ion and electron temperatures the result is

$$R = \frac{1}{2}k^2\lambda_D^2 \qquad (16)$$

and

$$\epsilon = \frac{(k^2\lambda_D^2/2+1)^2}{k^2\lambda_D^2/2} \xrightarrow[k\lambda_D\ll 1]{} \frac{2}{k^2\lambda_D^2}. \qquad (17)$$

Again the minimum value, at $k\lambda_D = \sqrt{2}$, is 4.

Whereas $R = 1$ gave a minimum, the family of extrema given by Eq. (16) are maxima. To see this, we take $\delta^2\epsilon$ at $\delta\epsilon = \delta R = 0$:

$$\delta^2\epsilon = \frac{2\delta R^2}{R^3} + \left(1 - \frac{1}{R^2}\right)\delta^2 R \xrightarrow[\delta R=0]{} \left(1 - \frac{1}{R^2}\right)\delta^2 R \qquad (18)$$

Again taking the local approximation for simplicity (but not necessity), we simplify $\delta^2 R$ by dropping the $\vec{x}$ integral in Eq. (13) and setting $\phi_1 = (4\pi/k^2)\sum\int q f_1 d\vec{v}$. At $\delta R = 0$,

$$\delta^2 R = \frac{1}{\Phi}\left\{\sum\int d\vec{v}\delta f_1^2 \frac{T}{f_0} - R\frac{4\pi}{k^2}\left(\sum\int d\vec{v} q\delta f_1\right)^2\right\} \tag{19}$$

where we have neglected terms in $\delta^2 f_1$. We rewrite the second term by multiplying and dividing by f_0/T and applying Schwarz's inequality on the double—sum $\sum\int d\vec{v}$:

$$R\frac{4\pi}{k^2}\left(\sum\int d\vec{v} q\delta f_1\right)^2 \leq \left[\frac{R4\pi}{k^2}\sum\int d\vec{v}\frac{f_0}{T}q^2\right]\cdot\sum\int d\vec{v}\delta f_1^2\frac{T}{f_0} = \sum\int d\vec{v}\delta f_1^2\frac{T}{f_0}. \tag{20}$$

In the second step the factor in brackets is unity, using Eq. (16). Combining Eqs. (19) and (20) gives $\delta^2 R \geq 0$; whence, for $k\lambda_D \ll 1$ and $R \ll 1$, we obtain $\delta^2\epsilon < 0$ and hence ϵ is a maximum, as claimed. Introducing Eq.(17) into Eq.(2) gives, for $k\lambda_D \ll 1$,

$$\gamma < \frac{1}{2}\,\mathrm{Max}\left(\frac{1}{H}\frac{dH}{dt}\right) < \sum\frac{\langle u\rangle}{\lambda_D}\frac{1}{\epsilon^{1/2}} = \sum\frac{k\langle u\rangle}{\sqrt{2}}. \tag{21}$$

In the following sections we shall identify this family of bounds, corresponding to maxima in ϵ, with the drift waves.

4. Comparison with Dispersion Relations

To help us interpret the bound, it is useful to compare our extremal solution, Eq. (14), with the formal solution for an eigenmode of the form $\exp i(\vec{k}\cdot\vec{x}-\omega t)$(again assuming the local approximation for simplicity). Then

$$f_1(\omega,\vec{k}) = f_0\frac{q\phi_1}{T}\alpha \tag{22}$$

$$\alpha = -1 + \int_{-\infty}^{t} dt'(\vec{u}\cdot\nabla + \partial/\partial t)' \exp\left[i\vec{k}\cdot(x'-x) - i\omega(t'-t)\right] \tag{23}$$

where ω, $\vec{k}$ satisfy the dispersion relation obtained by inserting f_1 into Poisson's equation. Given an eigensolution, we could choose to construct $\gamma = \mathrm{Im}\omega$ from a quadratic form similar to Eq. (2):

$$\gamma = \frac{1}{2}\frac{1}{\bar{H}}\frac{d\bar{H}}{dt} = Re\frac{\sum\int d\vec{v}qf_1^* i\vec{k}\cdot\vec{u}\phi_1}{\sum\int d\vec{v}f_1^*f_1T/f_0 + \sum\int d\vec{v}qf_1^*\phi_1} = \frac{Re\sum\overline{\alpha^* i\vec{k}\cdot\vec{u}}}{\sum\overline{\alpha^2} + \sum\bar{\alpha}} \tag{24}$$

where $\bar{H}$ is just H in Hermitian form ($f_1^2 \rightarrow f_1^* f_1$ etc.), and other bars denote various velocity averages. This expression is exact if f_1 is an eigensolution. From quasineutrality $\sum\int d\vec{v}qf_1 \propto \sum\bar{\alpha} \simeq 0$, we may conclude

$$\gamma \simeq \frac{Re\sum\overline{\alpha^* i\vec{k}\cdot\vec{u}}}{\sum\overline{|\alpha^2|}} \lesssim \frac{k\bar{u}}{|\bar{\alpha}|} \equiv \frac{\bar{u}}{\lambda_D}\frac{1}{\epsilon_\alpha^{1/2}} \tag{25}$$

where the final term has been rewritten in the form of our earlier bound, Eq. (2). Here the dielectric constant ϵ_α is given by

$$\epsilon_\alpha = \frac{|\alpha|^2}{k^2\lambda_D^2}. \tag{26}$$

For concreteness, consider a plasma column in a magnetic field with ion and electron distributions of the form $f_0 = n(p_\theta)\exp\left(\frac{-mv^2}{2T}\right)$ and $p_\theta = mrv_\theta + \frac{q}{c}rA_\theta$. In more familiar notation,

$$\vec{k}\cdot\vec{u} = k_\theta\frac{T}{n}\frac{\partial n}{\partial r}\frac{r}{(\frac{\partial p_\theta}{\partial r})} \cong \omega_* \tag{27}$$

to lowest order in larmor radius(i.e. dropping the v_θ term in $\frac{\partial p_\theta}{\partial r}$). Drift waves have $\omega < k_\| v_\|$ for the bulk of electrons, thus the adiabatic leading term of α dominates and $|\alpha_i| \sim |\alpha_e| \sim 1$. Whether instability occurs depends on other terms that may or may not cause the real part of the numerator in Eq.(25) to vanish. But if there is instability, the rate will not exceed $\sim \omega_*$, by Eqs. (25), (26) and (27) with $|\alpha| \simeq 1$.

With $|\alpha| = 1$, Eq. (25) is essentially our bound in Eq. (21) found by extremizing ϵ. By Eq. (26), $|\alpha| = 1$ corresponds to $\epsilon_\alpha \simeq \epsilon = 2/k^2\lambda_D^2$, the maximum value for a given k. In some circumstances instability also occurs at the minimum value, $\epsilon \gtrsim 4$, but these high—frequency "drift—cyclotron" instabilities are of little interest in toroidal devices. Of greater importance are the instabilities at intermediate values of ϵ_α, lying between the extremes. Examples are the flute modes of MHD theory, which are stabilized by proper magnetic design and beta values below MHD ballooning limits. For flute modes $\omega > k_\| v_\|$ for both ions and electrons, in which case the adiabatic term in α is almost canceled by the $\frac{\partial}{\partial t}$ term. Thus $|\alpha| \sim |\omega_*/\omega|$ with $\omega = i\gamma$, if unstable, and Eq. (25) becomes an identity $\gamma = \gamma$. The actual value of γ, determined by smaller terms in α, is $\gamma^2 = \omega_*\omega_D/k^2\rho_i^2$, where ρ_i is the ion gyroradius and ω_D is the curvature drift frequency; and $|\alpha| \sim (\omega_D/\omega_*)^{1/2} k\rho_i \lesssim k\rho_i \ll 1$. For $|\alpha| \simeq k\rho_i$ in Eq. (26), the dielectric constant becomes $\epsilon_\alpha = \rho_i^2/\lambda_{Di}^2$, appropriate for $E \times B$ motion $(nm_i(c\frac{E}{B})^2 = \rho_i^2/\lambda_{Di}^2(E^2/8\pi))$.

These examples are summarized in Table 1, from which we see that the spectrum of electrostatic instabilities forms a hierarchy of increasing values of ϵ_α and decreasing growth constants. The similarity of these qualitative results to our bound, Eq. (2), suggests a similar hierarchy for the bound with increasing ϵ, wherein drift waves are the lowest-lying family of γ's corresponding to the maximum values of ϵ. Thus we are led to conjecture that extremizing ϵ selects out the drift waves, and that this is the appropriate limit to set bounds on growth rates and transport in MHD—stable plasma confinement devices.

Table 1. Comparison of bounds and actual values for electrostatic growth constants

Class	ϵ_α	γ bound, Eq. (25)	Actual Max. γ
Drift—cyclotron (uninteresting in toroidal devices)	$\gtrsim 4$	$\frac{\lvert\vec{u}\rvert}{\lambda_D \epsilon_\alpha^{1/2}} \simeq \frac{\omega_*}{k\lambda_D}$	$(k_{\rho i})^{-1/2} \frac{\omega_*}{k\lambda_D}$ (a)
Flute modes (stabilized by MHD Criteria)	$\frac{\rho_i^2}{\lambda_{Di}^2}$	$\frac{\lvert\vec{u}\rvert}{\lambda_D \epsilon_\alpha^{1/2}} \simeq \frac{\omega_*}{k\rho_i}$	$\frac{(\omega_* \omega_D)^{1/2}}{k\rho_i}$
Drift Waves	$\frac{1}{k^2\lambda_D^2}$	$\frac{\lvert\vec{u}\rvert}{\lambda_D \epsilon_\alpha^{1/2}} \simeq \omega_*$	$\simeq \omega_*$

(a) A.B. Mikhailovskii and A.V. Timofeev, Soviet Phys. JETP (Engl. Transl.) 17, 626 (1963). Note that $k\rho_i > 1$ for this mode.

5. Application to Tokamaks

For a tokamak, $\vec{u}$ in Eq.(3) has components parallel and perpendicular to the field lines. To lowest order in $k\rho_i$:

$$u_{\|} = \frac{J_{\|}}{2ne} = \frac{cT}{qB}\frac{1}{n}\frac{\partial n}{\partial x}\left\{\frac{\partial \ln B_\theta}{\partial \ln n}\frac{B}{B_\theta}\frac{1}{\beta_\theta}\right\} \tag{28}$$

$$u_\perp = \frac{cT}{qB}\left\{\frac{1}{n}\frac{\partial n}{\partial x} + \frac{1}{T}\frac{\partial T}{\partial x}\left(-\frac{3}{2} + \frac{mv^2}{2T}\right)\right\} \tag{29}$$

for $f_0 = n(m/2\pi T)^{3/2}\exp(-\frac{mv^2}{2T})$. Here $\beta_\theta = 2nT/(B_\theta^2/8\pi)$ and B_θ is the poloidal field. For cases of interest ($\beta_\theta = \mathcal{O}(1)$), $u_{\|}$ exceeds $u_\perp$ by a factor of order $B/B_\theta = \mathcal{O}(10)$. If $u_{\|} > v_i$, as can occur in tokamaks at low density, $u_{\|}$ can drive beam—like instabilities at high frequency and $\epsilon \gtrsim 4$ (its minimum value, by Eq.(4)). However, for low frequency modes, $E_{\|} \ll E_\perp$ and

$$|u_{\|}E_{\|}| \ll |u_\perp E_\perp|, \tag{30}$$

even though $u_{\|} > u_\perp$. Consequently, referring back to the derivation of our bound in which each component of $\vec{u}\cdot\vec{E}$ was bounded separately (Eq.(7)), we conclude that we should drop $u_{\|}$ in bounding low frequency drift waves and take

$$\langle u\rangle \simeq \langle u_\perp\rangle \tag{31}$$

in Eq. (21).

Eqs. (28) and (29) assume the pressure gradient of each charge species is supported by its own $\vec{j}\times\vec{B}$ force. Actually ion collisional viscosity tends to stop ion rotation but such effects only create an ambipolar potential that transfers the pressure of one charge species to the $\vec{j}\times\vec{B}$ force of the other, which must then carry twice the current assumed in Eqs. (29)-(30). Thus in all cases

$$\sum k\langle u\rangle \simeq 2k\langle u_\perp\rangle_{one\ species} = 2\omega_*\left(1+\frac{3}{\sqrt{2}}\eta\right), \tag{32}$$

where we have now carried out the velocity average; $\eta = \partial\ln T/\partial\ln n$; and we take $T_i = T_e$ for simplicity.

Finally, as is shown in the Appendix, we are at liberty to add an arbitrary toroidal velocity of the form $a = r\omega_0$ to $\vec{u}$, with a perpendicular component $a_\perp = a(B_\theta/B)$. In principle, optimizing ω_0 would improve the bound. However, since u_e and u_i have opposite signs through q, for $T_i = T_e$ the bound has the form

$$k|u_{\perp i} + a_\perp| + k|u_{\perp e} + a_\perp| = k\big|(|u_{\perp i}| + a_\perp)\big| + k\big|(-|u_{\perp i}| + a_\perp)\big|$$

which has the same value for any $-|u_{\perp i}| < a_\perp < |u_{\perp i}|$ and is larger for $a_\perp$ outside this range. If $T_i \neq T_e$, optimizing $a_\perp$ does improve the bound somewhat.

From these considerations, we conjecture that, given MHD stability, the growth constant of low frequency waves in tokamaks is given by extremizing Eq. (2) with $\langle u \rangle \simeq \langle u_\perp \rangle$, which selects out drift waves.

A virtue of our approach is that the extremal eigenvalue problem, Eq. (15), is easier to solve than the full dispersion relation. Qualitatively, the solution is Eq. (16), which yields, from Eqs. (21) and (32),

$$\gamma < \sqrt{2}\omega_* \left(1 + \frac{3}{\sqrt{2}}\eta\right), \tag{33}$$

where again we take $T_i = T_e$ for simplicity. Using this value of γ, diffusion coefficients of the form γ/k^2 are essentially the same as those that have recently been shown to provide a good fit to global confinement times in tokamaks.[5]

A defect of the method is that the unstable range of k is not determined. For transport calculations, it is the smallest unstable k that is of interest, since $\gamma/k^2 \propto \omega^*/k^2 \propto 1/k$ is greatest for small k. Qualitatively, consistent with identifying the extremization of ϵ with drift waves, we should require $\omega \sim \omega_* \gtrsim k_\parallel v_i$. Then, for $k_\parallel \gtrsim (qR)^{-1}$ one finds[2]

$$k_\perp \rho_i \gtrsim \frac{B_\theta}{B} \tag{34}$$

where $q \propto \left(\frac{B}{B_\theta}\frac{a}{R}\right)$ is the tokamak safety factor. Generally, however, the minimum k must be specified independently (e.g. $k\rho_i \sim \frac{1}{3}$ in Ref. 5). Since $k_{min}^{-1} \ll$ plasma radius, we expect γ and the transport coefficients to vary with radial position. To apply this idea, we could solve the eigenvalue problem, Eq. (15), locally over a restricted volume, for example, a shell of thickness $\lambda \sim k_{min}^{-1}$ lying between two flux surfaces. Again periodic boundary conditions should be chosen, corresponding to zero free energy flow into or out of the volume. This boundary condition is the essence of the local approximation, as discussed in Ref. 1.

6. Discussion

Using a variational technique, we have developed a bound on the growth constant γ versus wavenumber k for drift waves, Eq. (21). Calculating the bound is easier than searching the full dispersion relation, and therefore the bound may be useful to estimate transport coefficients of the form γ/k^2. Moreover, results agree closely with those from dispersion relations, and they are of such general nature as to give greater assurance that nothing has been missed.

The underlying premise associating the variational method with drift waves remains conjectural. The basic idea, developed from examples in Section 4, is that the instabilities allowed by the Vlasov equation under various circumstances form a hierarchy of which the lowest lying family of γ's at a given k is the drift waves, which are the only surviving low frequency, electrostatic modes in an MHD—stable system. As an alternative to detailed calculations, one could select out the drift waves by forming a bound, as we have done, and search for the lowest lying family of bounds. Even if one accepts this much of the conjecture, associating extrema of the bounds with the corresponding extreme family of γ's is inexact. Implicitly we are saying that, if the extremizing function f_1 falls near an eigenfunction f_n with growth constant $\gamma_n \doteq \text{Im}\ \omega_n$, we could claim:

$$\gamma_n \leq \left(\frac{1}{2}\frac{1}{H}\frac{\partial H}{\partial t}\right)_{at\ f_1} < \frac{\sum k\langle u\rangle}{(\epsilon^{1/2})_{at\ f_1}}. \tag{35}$$

However, the left—hand inequality may not hold (even though the exact equality would hold for an exact eigenfunction, as noted in Eq. (24)). The operator $\dot{H}$ is symmetric (see Eq.(36), below), so that $\dot{H}/H$ does bound its eigenvalues (as in the Ritz method of quantum mechanics). But eigenvalues of $\dot{H}$ are not necessarily equal to the real part (γ_n) of eigenvalues of the Vlasov operator. The inequality does hold approximately, to the extent that the linearized Vlasov equation maps the set of perturbations in a given range of $k_\perp$ into itself (see Eq. (11)).

Probably the success of our extremum process rests on the insensitivity of γ to the exact form of $f_1(\vec{v})$. Our choice of H has correctly evoked the main feature that $f_1 \propto f_0$, as can be seen by comparing our extremal function, Eq. (14), with the eigenfunction, Eq. (22). Beyond that, the coefficients R and α are quite different. To satisfy Poisson's equation, $R = \frac{1}{2}k^2\lambda_D^2$ since the terms are additive, while for the actual eigenfunction Poisson's equation yields $\alpha_i = -\alpha_e = 1$

(quasineutrality). Yet, Eq. (24) gives qualitatively the same result for either solution:

$$\frac{\dot{H}}{H} \rightarrow \frac{\sum \alpha}{\sum \alpha^2 + \sum \alpha} ku \xrightarrow[\sum \alpha \ll \alpha^2]{} \frac{1}{\alpha} ku \xrightarrow[\alpha=1]{} ku$$

$$\frac{\dot{H}}{H} \rightarrow \frac{\sum R}{\sum R^2 + \sum R} ku \xrightarrow[R^2 \ll R]{} \frac{\sum R}{\sum R} ku \rightarrow ku$$

and similarly for the bound, Eq. (21),

$$\frac{u}{\lambda_D} \frac{1}{\epsilon^{1/2}} = \frac{u}{\lambda_D} \frac{\sqrt{R}}{R+1} \rightarrow \frac{u}{\lambda_D} \sqrt{R} \xrightarrow[R=\frac{1}{2}k^2\lambda_D^2]{} ku.$$

Besides evoking $f_1 \propto f_0$, H as an operator has important transformation properties that sharpen the bound by eliminating the energy derivative of f_0. Define operators V and $\mathcal{H}$ by $\frac{\partial f_1}{\partial t} = V f_1$ (Vlasov operator) and $H = (f, \mathcal{H} f)$ (Hermitian product). Then

$$\dot{H} = (f_1, (V^\dagger \mathcal{H} + \mathcal{H} V) f_1) = Re \sum_\ell \int d\vec{x} d\vec{v} f_{1\ell}^* q_\ell \vec{u}_\ell \cdot \nabla \phi_1. \tag{36}$$

The Hermitian symmetry of this expression may be verified directly by expressing $\nabla\phi$ in terms of a Green's function solution integrated over the charge $\sum_m \int d\vec{v}' q_m f_{1m}$ and interchanging labels $\vec{x}$, $\vec{v}$; $\vec{x}'$ $\vec{v}'$ and also the charge species labels l, m. The simplicity of this result follows from the fact that $\mathcal{H}V$ is almost anti—Hermitian. If it were exactly anti—Hermitian, $\dot{H} = 0$ and H would be a constant of the linearized equations of motion. It can be shown that for any such constant there must exist a transformation making the Vlasov operator anti—Hermitian, and $\mathcal{H}V$ is such a transformation if $f_0 = \exp(-\varepsilon/T)$ (Maxwellian).[6] The Hermitian residual, represented by $\vec{u}$, results from the momentum dependence of f_0 that allows the plasma to carry the currents required for magnetic confinement, as in our example, Eq. (27).

The transformation properties of H derive from its relationship to a non—linear constant of the motion, namely the Helmholtz free energy given by:

$$A = \sum \int d\vec{x} d\vec{v} (T f \ln f + f\varepsilon) + \Phi \tag{37}$$

where $\varepsilon = \frac{1}{2} m v^2$. Minimizing A on f gives, to lowest order in $f_1 = f - f_0$, a quadratic form like H with $f_0 = \exp(-\varepsilon/T)$. For this $f_0, \dot{H} = 0$ exactly for the linearized Vlasov equation.

The utility of $\dot{H}/H$ as a bound on γ lies in the fact that $\dot{H}$ and the bound will be small if ion and electron f_0's are almost Maxwellian.

The free energy driving the drift instabilities is expansion energy given by[1,2]

$$\frac{\epsilon E^2}{8\pi} \cong \frac{1}{k^2 a^2} nT \tag{38}$$

where $a = n/(dn/dx)$. Substituting $\epsilon = 2/k^2 \lambda_D^2$ we obtain $E = (T/ea)$ and

$$\gamma \sim k \frac{cE}{B} \sim \frac{kcT}{eB} \frac{1}{n} \frac{dn}{dx} = \omega_*. \tag{39}$$

Finally, we note that our results, derived for electrostatic modes, appear to have broader validity. The same bound, $\gamma < \omega_*$, is derived for magnetostatic perturbations in the Appendix.

Appendix
Magnetostatic Modes and Reference Frame Change

For magnetostatic modes, in $\dot{H}$ we focus on the term

$$\sum \int d\vec{x} d\vec{v} q f_1 u_i \left(\frac{\vec{v}}{c} \times \vec{B}_1\right)_i \leq 2 \sum \frac{\langle u_i^2 \frac{v^2}{c^2}\rangle^{1/2}}{\lambda_D} K^{1/2} \Phi^{1/2} \tag{A1}$$

by analogy with Eq. (7), where $v^2 = T/m$, and

$$\Phi = \frac{1}{8\pi} \int d\vec{x} B_1^2 = \frac{1}{8\pi} \int d\vec{x} (\nabla \times A_1)^2 = \frac{1}{2} \int d\vec{x} \, \frac{\vec{j}_1}{c} \cdot \vec{A}_1 \tag{A2}$$

$$-\nabla^2 \vec{A}_1 = \frac{4\pi}{c} \vec{j}_1 = \frac{4\pi}{c} \sum \int d\vec{v} q \vec{v} f_1. \tag{A3}$$

Then, with the previous notation $R = K/\Phi$,

$$\gamma < \frac{\sum \langle u \rangle v/c}{\lambda_D} \left(\frac{R}{(R+1)^2}\right)^{1/2}. \tag{A4}$$

By analogy to Eq. (13), extremizing gives

$$f_1 = \frac{f_0}{T} \frac{q}{c} \vec{v} \cdot \vec{A}_1 R \tag{A5}$$

$$-\nabla^2 \vec{A}_1 = \frac{4\pi}{c^2} \sum \int d\vec{v} q \frac{f_0}{T} R(\vec{v} \cdot \vec{A}_1) q \vec{v} = \sum \frac{1}{\lambda_D^2} \left[\frac{v^2}{c^2} \vec{A}_1 + \frac{\vec{u}_0}{c} \left(\frac{\vec{u}_0}{c} \cdot \vec{A}_1\right) R\right] \tag{A5}$$

where $\vec{u}_0 = \int f_0 \vec{v} d\vec{v} / \int f_0 d\vec{v}$. Dropping the second term, we obtain

$$-\nabla^2 \vec{A}_1 = \sum \frac{v^2}{\lambda_D^2 c^2} \vec{A}_1 R \tag{A7}$$

as the eigenvalue problem in R. Approximately, since $v_e \gg v_i$,

$$R = k^2 \lambda_D^2 \frac{c^2}{v_e^2} \tag{A8}$$

and for $R \ll 1 (kc/\omega_{pe} \ll 1)$

$$\epsilon = \frac{(R+1)^2}{R} \rightarrow \frac{1}{k^2 \lambda_D^2} \frac{v_e^2}{c^2}. \tag{A9}$$

Substituting into Eq. (A4) gives as the extremal bound

$$\gamma < \frac{\sum \langle u \rangle v/c}{\lambda_D} \frac{1}{\epsilon^{1/2}} \cong \frac{k \langle u_e \rangle}{\sqrt{2}}, \tag{A10}$$

which is essentially the same as Eq.(21) for electrostatic modes. Again we assume $u_\perp$ to be dominant, in which case $\gamma \lesssim \omega_*$, as for electrostatic modes. As derived here the magnetostatic and electrostatic bounds would be additive. However, by carrying out the variation on ϕ_1 and $\vec{A}_1$ simultaneously, one finds that the extremal solutions approximately separate, indicating that these are independent modes and are not additive. So $\gamma < \omega_*$ by either process. In reaching this conclusion we have dropped the $\partial\vec{A}/\partial t$ term, which is important in $E_\parallel$ but is not important here since $u_\parallel E_\parallel \ll u_\perp E_\perp$.

Next we consider a change of reference frame accomplished by adding a momentum term to Φ:[1]

$$\Phi' = \Phi + \frac{c}{4\pi}\int d\vec{x}\vec{a}\cdot\vec{E}_1 \times \vec{B}_1 \tag{A11}$$

For certain classes of $\vec{a}$, the time derivative of Φ' has the form

$$\frac{d\Phi'}{dt} = -\int d\vec{x}\,\vec{j}_1\cdot\vec{E}_1 - \int d\vec{x}\vec{a}\cdot\left(\rho_1\vec{E}_1 + \frac{1}{c}\vec{j}_1\times\vec{B}_1\right) \tag{A12}$$

where again boundary conditions are chosen so that surface terms vanish. Adding this to

$$\frac{dK}{dt} = \int d\vec{x}\,\vec{j}_1\cdot\vec{E}_1 - \sum\int d\vec{x}d\vec{v}\,qf_1\vec{u}\cdot\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}_1\right) \tag{A13}$$

we obtain

$$\frac{dH}{dt} = -\sum\int d\vec{x}d\vec{v}qf_1(\vec{u}+\vec{a})\cdot\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}_1\right), \tag{A14}$$

which is the generalization of Eq. (6) in a reference frame moving at velocity $\vec{a}$. Choices of $\vec{a}$ for which the above is true include $\vec{a}$ = constant (from overall linear momentum conservation); and $\vec{a} = \vec{\omega_0}\times\vec{r}$ (from conservation of angular momentum about an axis that could be the tokamak axis), as applied in Section 5. These results hold independent of symmetry properties of f_0. However, a wise choice would be $\vec{a}$ that makes f_0 for the ions and electrons be as nearly Maxwellian in a common reference frame as possible.

Though not quite relativistically correct, Φ' is essentially the field energy in the moving frame. It is a positive definite function, so that the fundamental inequality $\Phi' < H$ in Eq. (11) still holds. To see this, consider the case $\vec{a} = a\hat{z}$ and define

$$\vec{E}_1^* = (1-\frac{a^2}{c^2})^{-1/2}\left[\hat{x}\left(E_x + \frac{a}{c}B_y\right) + \hat{y}\left(E_y - \frac{a}{c}B_x\right)\right] + \hat{z}E_z$$

$$\vec{B}_1^* = (1 - \frac{a^2}{c^2})^{-1/2} \left[\hat{x}\left(B_x - \frac{a}{c}E_y\right) + \hat{y}\left(B_y + \frac{a}{c}E_x\right)\right] + \hat{z}B_z. \tag{A15}$$

Then

$$\Phi' = \frac{1}{2}\left(1 - \frac{a^2}{c^2}\right)(\Phi + \Phi^*) + \frac{a^2}{c^2}\left(E_z^2 + B_z^2\right) \tag{A16}$$

where Φ^* is Φ with $\vec{E}_1$ and $\vec{B}_1$ replaced by $\vec{E}_1^*$ and $\vec{B}_1^*$.

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