Sherwood 1980

Tucson

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RELATED WORK

NONCANONICAL VARIABLES - FINITE SYSTEMS

R. LITTLEJOHN, J. MATH. PHYS., 20, 2445 (1979).

KORTEWEG-DE VRIES BRACKET

C. GARDNER, J. MATH. PHYS., 12, 1548 (1971).

V. ZAKHAROV AND L. FADDEEV, FUNC. ANAL. APPLIC., 5 (1972).

IDEAL MHD – EULERIAN VARIABLES

$$\frac{\partial \underline{U}}{\partial t} = -\underline{\nabla}\left(\frac{U^2}{2}\right) + \underline{U} \times (\underline{\nabla} \times \underline{U}) - \frac{1}{\rho}\underline{\nabla}\left(\rho^2 \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho}(\underline{\nabla} \times \underline{B}) \times \underline{B}$$

$$\frac{\partial \rho}{\partial t} = -\underline{\nabla} \cdot (\rho \underline{U})$$

$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{U} \times \underline{B})$$

$$\frac{\partial S}{\partial t} = -\underline{U} \cdot \underline{\nabla} S$$

$$U(\rho, S) = \frac{\text{INTERNAL ENERGY}}{\text{MASS}}$$

$$S = \frac{\text{ENTROPY}}{\text{MASS}}$$

$$p = \rho^2 \frac{\partial U}{\partial \rho}$$

$$T = \frac{\partial U}{\partial S}$$

QUASILINEAR – SYMMETRIC

HYPERBOLIC

FORMULATION

2 HAMILTONIAN DENSITY $\quad H = \frac{1}{2}\rho v^2 + \frac{B^2}{2} + \rho U(\rho, s)$

$$\hat{H}\{\rho, s, \underline{v}, \underline{B}\} = \int_V H(\rho, s, \underline{v}, \underline{B})\, d\tau$$

DESIRE POISSON BRACKET SUCH THAT THE MHD EQS. ABOVE CAN BE PUT IN THE FORM

$$\frac{\partial \bar{\psi}^i}{\partial t} = [\bar{\psi}^i, \hat{H}] \qquad i = 0, 1, 2 \cdots 7$$

THE $\bar{\psi}^i$'s ARE FUNCTIONAL DYNAMICAL VARIABLES DEFINED BELOW.

STRUCTURE (LIE ALGEBRA)

3

CONSIDER VECTOR SPACE $\mathcal{V}$, OVER $\mathbb{R}$, WITH ELEMENTS OF THE FORM

$$\hat{F}\{\underline{\chi}\} = \int_V F\left(\underline{x}, t; \underline{\chi}, \frac{\partial \underline{\chi}}{\partial x_\alpha}, \frac{\partial^2 \underline{\chi}}{\partial x_\alpha \partial x_\beta}, \dots\right) d\tau$$

$\underline{\chi}$ IS AN N-TUPLE OF $C^\infty(V)$ FUNCTIONS $\chi^i(\underline{x},t)$

(IN PARTICULAR, $\chi^0 \equiv \rho$, $\chi^1 \equiv s$, $\chi^{2,3,4} \equiv \underline{v}$, & $\chi^{5,6,7} \equiv \underline{B}$)

DEFINE BRACKET ON VECTOR SPACE WHICH IS A BILINEAR FUNCTION THAT MAPS $\mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$. ALSO

(I) $[\hat{F}, \hat{F}] = 0 \quad \forall\, \hat{F} \in \mathcal{V}$ FOR $\mathcal{V}$ OVER $\mathbb{R}$ $\Longleftrightarrow$

$[\hat{F}, \hat{G}] = -[\hat{G}, \hat{F}]$ FOR $\hat{F}, \hat{G} \in \mathcal{V}$

(II) JACOBI IDENTITY

$$[\hat{E},[\hat{F},\hat{G}]] + [\hat{F},[\hat{G},\hat{E}]] + [\hat{G},[\hat{E},\hat{F}]] = 0$$

DYNAMICAL VARIABLES AND EQUATIONS OF MOTION

4

SET OF DYNAMICAL VARIABLES $\mathcal{D} \subset \mathcal{V}$ ~~HAS~~ HAS ELEMENTS OF THE FORM

$$\bar{\xi}^{i}\{\xi^{i}\} = \int_V f_i(\underline{x})\, \xi^{i}(\underline{x},t)\, d\tau \qquad \text{for } i = 0,1,2 \cdots 7$$

where $\xi^{i} \in C^{\infty}(V)$ AND THE f_i ARE ARBITRARY.

SUBSTITUTING $\bar{\xi}^{0} = \bar{\rho}$ YIELDS THE EQ. OF MOTION

$$\frac{\partial \bar{\xi}^{0}}{\partial t} - [\bar{\xi}^{0}, \hat{H}] = \int_V f_0(\underline{x}) \left[\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) \right] d\tau = 0$$

FROM WHICH WE OBTAIN THE EQUATION OF MASS CONSERVATION.

BRACKET

$$[\hat F, \hat G] = -\int_V \Big\{ \Big[\frac{\delta \hat F}{\delta \rho} \nabla \cdot \frac{\delta \hat G}{\delta \underline{v}} + \frac{\delta \hat F}{\delta \underline{v}} \cdot \nabla \frac{\delta \hat G}{\delta \rho} \Big]_I$$

$$+ \Big[\frac{1}{\rho} \frac{\delta \hat F}{\delta \underline{v}} \cdot (\nabla \times \underline{v}) \times \frac{\delta \hat G}{\delta \underline{v}} \Big]_R + \Big[\frac{\nabla s}{\rho} \cdot \Big(\frac{\delta \hat F}{\delta s} \frac{\delta \hat G}{\delta \underline{v}} - \frac{\delta \hat G}{\delta s} \frac{\delta \hat F}{\delta \underline{v}} \Big) \Big]_S$$

$$+ \Big[\Big[\frac{1}{\rho} \frac{\delta \hat F}{\delta \underline{v}} \cdot \underline{B} \times \Big(\nabla \times \frac{\delta \hat G}{\delta \underline{B}} \Big) \Big] + \frac{\delta \hat F}{\delta \underline{B}} \cdot \nabla \times \Big(\underline{B} \times \frac{1}{\rho} \frac{\delta \hat G}{\delta \underline{v}} \Big) \Big]_B \Big\} d\tau$$

$$= \int_V \frac{\delta \hat F}{\delta \chi^i} O^{ij} \frac{\delta \hat G}{\delta \chi^j} d\tau$$

WHERE $$\frac{\delta \hat F}{\delta \chi} = \frac{\partial F}{\partial \chi} - \sum_i \frac{d}{dx^i} \frac{\partial F}{\partial \left(\frac{\partial \chi}{\partial x_i} \right)} + \cdots$$

COMPARE WITH $$[\hat F, \hat G] = \frac{\partial \hat F}{\partial z^i} J^{ij} \frac{\partial \hat G}{\partial z^j}$$

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$ AND $$z^i \in \{ q_1, \cdots q_N ; p_1, \cdots p_N \}$$

ONE-TIME COMMUTATION RELATIONS

6

If $\quad f_i(\underline{x}) = \delta(\underline{x} - \underline{x}_i)$

$$[v_i, v_j] = (\hat{i}\times\hat{j})\cdot(\underline{\nabla}\times\underline{v})\,\delta(\underline{x}_i - \underline{x}_j)\,\rho^{-1}$$

$$[s, \rho] = 0 \qquad\qquad [\rho, B_i] = 0$$

$$[s, B_i] = 0$$

$$[v_i, s] = \rho^{-1}\,\hat{i}\cdot\underline{\nabla} s\;\delta(\underline{x}_1 - \underline{x}_i)$$

$$[v_i, \rho] = -\hat{i}\cdot\underline{\nabla}\,\delta(\underline{x}_i - \underline{x}_0)$$

$$[v_i, B_j] = \rho^{-1}\,\hat{j}\cdot\left[(\underline{B}\times\hat{i})\times\underline{\nabla}\,\delta(\underline{x}_i - \underline{x}_j)\right]$$

7

CLEANER BRACKET

$$\{\rho, s, \underline{v}, \underline{B}\} \longrightarrow \{\rho, \sigma, \underline{M}, \underline{B}\}$$

where $\sigma = \rho s$, $\underline{M} = \rho \underline{v}$.

IN THESE VARIABLES GET EIGHT CONSERVATION EQS. WITH BRACKET

$$[\hat{F}, \hat{G}] = -\int_V d\tau \Big\{ \rho \Big[\frac{\delta \hat{F}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{G}}{\delta \rho} - \frac{\delta \hat{G}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{F}}{\delta \rho} \Big] + \underline{M} \cdot \Big[\frac{\delta \hat{F}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{G}}{\delta \underline{M}} - \frac{\delta \hat{G}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{F}}{\delta \underline{M}} \Big]$$

$$+ \sigma \Big[\frac{\delta \hat{F}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{G}}{\delta \sigma} - \frac{\delta \hat{G}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{F}}{\delta \sigma} \Big] + \underline{B} \cdot \Big[\frac{\delta \hat{F}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{G}}{\delta \underline{B}} - \frac{\delta \hat{G}}{\delta \underline{M}} \cdot \underline{\nabla} \frac{\delta \hat{F}}{\partial \underline{M}} \Big]$$

$$+ \underline{B} \cdot \Big[\underline{\nabla} \frac{\delta \hat{F}}{\delta \underline{B}} \cdot \frac{\delta \hat{G}}{\delta \underline{M}} - \underline{\nabla} \frac{\delta \hat{G}}{\delta \underline{B}} \cdot \frac{\delta \hat{F}}{\delta \underline{M}} \Big] \Big\}$$

FOURIER TRANSFORM

8

LET V BE A UNIT CUBE. ADOPT PERIODIC B.C.

$$\rho = \sum_{\underline{k}} \rho_{\underline{k}}(t) \exp(2\pi i \, \underline{k} \cdot \underline{x}) \qquad \underline{k} \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$$

$$\frac{\delta \hat{F}}{\delta \rho} = \sum_{\underline{k}} \frac{\partial \hat{F}}{\partial \rho_{\underline{k}}} \exp(-2\pi i \, \underline{k} \cdot \underline{x})$$

BRACKET BECOMES

$$[\hat{F}, \hat{G}] = \sum_{\underline{k}, \underline{l}} \frac{\partial \hat{F}}{\partial \underline{Z}_{\underline{k}}} \cdot \underline{\underline{O}}_{\underline{k}, \underline{l}} \cdot \frac{\partial \hat{G}}{\partial \underline{Z}_{\underline{l}}}$$

WHERE $\underline{Z}_{\underline{k}} = (\rho_{\underline{k}}, \sigma_{\underline{k}}, \underline{M}_{\underline{k}}, \underline{B}_{\underline{k}})$ AND

$$\underline{\underline{O}}_{\underline{k}, \underline{l}} = \left[\begin{array}{cc|c|ccc} 0 & 0 & -\rho_{\underline{l}+\underline{k}} \, \underline{k} & 0 & 0 & 0 \\ 0 & 0 & -\sigma_{\underline{l}+\underline{k}} \, \underline{k} & 0 & 0 & 0 \\ \hline \rho_{\underline{l}+\underline{k}} \, \underline{l} & \sigma_{\underline{l}+\underline{k}} \, \underline{l} & \underline{l} \, \underline{M}_{\underline{l}+\underline{k}} - \underline{M}_{\underline{l}+\underline{k}} \, \underline{k} & \multicolumn{3}{c}{\underline{l} \, \underline{B}_{\underline{l}+\underline{k}} - \underline{l} \cdot \underline{B}_{\underline{l}+\underline{k}} \, \underline{\underline{I}}'} \\ \hline 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -\underline{k} \, \underline{B}_{\underline{l}+\underline{k}} + \underline{k} \cdot \underline{B}_{\underline{l}+\underline{k}} \, \underline{\underline{I}}' & 0 & 0 & 0 \\ 0 & 0 & & 0 & 0 & 0 \end{array} \right]$$

NOTICE $\underline{\underline{O}}_{\underline{k},\underline{l}} = - \tilde{\underline{\underline{O}}}_{\underline{l},\underline{k}}$ (TILDE $\Rightarrow$ TRANSPOSE)

BY MAKING USE OF ANY ONE-ONE ONTO MAP

$$f : S \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z} \qquad \text{where } S = \{0,1,2,3,4,5,6,7\}$$

WE OBTAIN THE FORM

$$[F, G] = \frac{\partial F}{\partial z^i} J^{ij} \frac{\partial G}{\partial z^j} \qquad i, j \in \{0, \pm 1, \pm 2, \ldots\}$$

WHERE $z^i \in \{ \rho_{\underline{k}}, \sigma_{\underline{k}}, \underline{M}_{\underline{k}}, \underline{B}_{\underline{k}} \mid \underline{k} \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \}$,

AND $J^{ij} = - J^{ji}$.

$\blacksquare$

POSSIBILITIES

— APPROXIMATIONS WHICH PRESERVE HAMILTONIAN CHARACTER?

DISCRETIZATION ⟶ TRUNCATION

— INTEGRAL INVARIANTS COMMUTE w/ $\hat{H}$.

HOW MANY?