HAMILTONIAN FORMULATION OF REDUCED MHD

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Hamiltonian Formulation of Reduced MHD*., R.D. HAZELTINE and P.J. MORRISON, U. of Texas, IFS--The reduced MHD model has become a principal tool for understanding nonlinear processes (e.g., disruptions) in a tokamak discharge. Although analytical treatments of reduced MHD turbulence have been helpful, the model's impressive ability to simulate such phenomena is based primarily on numerical solutions. The present work describes a new analytical approach, not restricted to turbulent regimes, based on Hamiltonian field theory.[1] It has been found that the nonlinear (ideal) reduced MHD system can be expressed in Hamiltonian form: $\dot{\xi}_i = \{H,\xi_i\}$, where $\xi_1 = \psi$, the poloidal flux, and $\xi_2 = \nabla_\perp^2\phi$, with ϕ the electrostatic potential. The Hamiltonian is the usual field energy, $H = 1/2 \int d\underset{\sim}{x}[(\nabla\phi)^2 + (\nabla\psi)^2]$ and the bracket is defined by $\{F,G\} = \int d\underset{\sim}{x}\{\psi([F_1,G_2] + [F_2,G_1]) + \nabla^2\phi[F_2,G_2]\}$, where F and G are functionals of $\underset{\sim}{\xi}$, $F_i \equiv \partial F/\partial\xi_i$, and the inner bracket is given by $[f,g] = \hat{z} \cdot \underset{\sim}{\nabla} f\times\underset{\sim}{\nabla} g$. Discretization and truncation within this Hamiltonian paradigm yields energy conserving approximations. Furthermore we have a Liouville theorem in function space.

[1] P. Morrison and J. Greene, PRL 45, 790(1980); 48, 569 (1982); P. Morrison in "Math. Methods in Hydrodynamics", Ed. M. Tabor(AIP, 1981).

I REDUCED MHD – DERIVATION

Scale ideal MHD

$\vec{v}_\perp$ with $v_p = B_p/\sqrt{\rho}$

$v_\parallel \equiv 0$

t with $\tau_p = a/v_p$ ($a \equiv$ minor radius)

$\vec{B}$ with $B_0 \equiv$ vacuum on axis toridal field

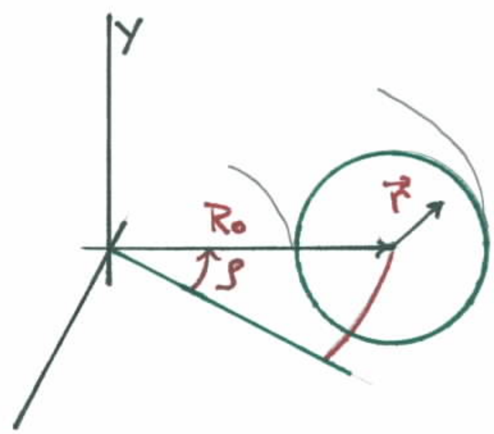
$\nabla_\parallel$ with $R_0 \equiv$ major radius

$\nabla_\perp$ with a

small parameter

$\varepsilon \equiv a/R_0$

$\frac{p}{B_0^2} \sim \varepsilon^2$



$R = R_0 + X$

note $\nabla \cdot \vec{v} \sim \varepsilon$

Scaled ideal MHD

$$\frac{\partial \vec{v}_\perp}{\partial t} + \vec{v}_\perp \cdot \nabla_\perp \vec{v}_\perp = -(\varepsilon \nabla_\parallel + \nabla_\perp) p + \frac{1}{\varepsilon^2}\left[(\varepsilon\nabla_\parallel + \nabla_\perp)\times \vec{B}\right]\times\vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = (\varepsilon \nabla_\parallel + \nabla_\perp)\times(\vec{v}_\perp \times \vec{B})$$

$$\vec{B} = B_T\,\hat{\zeta} + \varepsilon\,\nabla_\parallel \zeta \times \nabla_\perp \psi$$

$$B_T = \frac{1}{1+\varepsilon X} + \varepsilon^2 b$$

$\Longrightarrow$

$$\dot{\psi} = -\frac{\partial \varphi}{\partial \zeta} + \nabla_\parallel \zeta \cdot \nabla_\perp \psi \times \nabla_\perp \varphi$$

$$\dot{U} = -\frac{\partial J}{\partial \zeta} + \nabla_\parallel \zeta \cdot \nabla_\perp \psi \times \nabla_\perp J + \nabla_\parallel \zeta \cdot \nabla_\perp U \times \nabla_\perp \cancel{\zeta}\,\varphi$$

$U = \nabla_\perp^2 \varphi = \hat{\zeta}\cdot \nabla_\perp \times \vec{v}_\perp$ Vorticity

$J = \nabla_\perp^2 \psi$ toridal current

φ = stream function — $\vec{v}_\perp = \hat{\zeta} \times \nabla_\perp \varphi$

II GENERALIZED HAMILTONIAN DYNAMICS

conventional approach

$$\mathcal{L}(q,\dot{q}) \xrightarrow[\text{Legendre}]{} \mathcal{H}(p,q)$$

$$\dot{q}_k = [q_k, \mathcal{H}] \quad \& \quad \dot{p}_k = [p_k, \mathcal{H}] \qquad k = 1, 2, \cdots N$$

$$[f, g] = \sum_k^N \left[\frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k}\frac{\partial f}{\partial p_k} \right]$$

Equivalently

$$\dot{z}^i = [z^i, \mathcal{H}] = J^{ij}\frac{\partial H}{\partial z^j}$$

$$[f, g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} \qquad (J^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

$$z^i = \begin{cases} q_k & k = i = 1, 2, \cdots N \\ p_k & i = k+N = N+1, N+2, \cdots N \end{cases}$$

commutator properties

= $[f, g]$ is bilinear

= $[f, g] = -[g, f] \Leftrightarrow J^{ij} = -J^{ji}$

= $0 = [f, [g, h]] + \circlearrowright \Leftrightarrow$

$$S^{ijk} = J^{il} \frac{\partial J^{jk}}{\partial z^l} + \circlearrowright = 0$$

(Jacobi)

Consider arbitrary coordinate transformation

$$\bar{J}^{st} = \frac{\partial \bar{z}^s}{\partial z^i} J^{ij} \frac{\partial \bar{z}^t}{\partial z^j}$$

$$(\bar{J}^{st}) \neq \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix} \text{ in general !!}$$

But commutator properties are preserved.

<u>Converse Outlook</u>: If J^{ij} has properties, can one find coordinates such that $J^{ij} = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$? Yes! (locally)

<u>Such systems are Hamiltonian</u>

Field Equations

$$\{[F,G]\} = \sum_{k=1}^{M} \int \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right) dz$$

$\frac{\delta F}{\delta \eta}$ is defined by $\left.\frac{dF[\eta+\varepsilon w]}{d\varepsilon}\right|_{\varepsilon=0} = \langle \frac{\delta F}{\delta \eta} | w \rangle$

$$\{[F,G]\} = \langle \frac{\delta F}{\delta u^i} | O^{ij} \frac{\delta G}{\delta u^j} \rangle$$

Systems are Hamiltonian if O^{ij} instills commutator properties

$\{[F,G]\} = -[G,F] \Rightarrow O^{ij}$ anti-self-adjoint

$\{[E,[F,G]]\} + \circlearrowright = 0$ rigid constraint on O^{ij} !

III HAMILTONIAN RMHD

A. Invariants

Use

$$\int d\underset{\sim}{x}\, f[g,h] = \int d\underset{\sim}{x}\, g[h,f] = \int d\underset{\sim}{x}\, h[f,g]$$

to find

$$\frac{dH}{dt} = 0, \quad \frac{dP}{dt} = 0$$

where

$$H = \frac{1}{2}\int d\underset{\sim}{x} \left[(\nabla_\perp \varphi)^2 + (\nabla_\perp \psi)^2 \right]$$

= reduced MHD (kinetic plus magnetic) energy

$$P = \frac{1}{2}\int d\underset{\sim}{x}\, \underset{\sim}{\nabla}_\perp \varphi \cdot \underset{\sim}{\nabla}_\perp \psi$$

= reduced Woltjer invariant

$$\propto \int d\underset{\sim}{x}\, \underset{\sim}{B} \cdot \underset{\sim}{V}$$

Flux conservation, $\frac{d}{dt}\int d\underset{\sim}{x}\, \psi = 0$, is reduced invariance of $\int d\underset{\sim}{x}\, \underset{\sim}{A} \cdot \underset{\sim}{B}$ (Woltjer)

B. Inner brackets

$$(x, y, z) \longrightarrow (r, \theta, \zeta)$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = \zeta$$

Two relevant "inner" brackets:

$$[f, g] \equiv \underset{\sim}{\nabla}\zeta \cdot \underset{\sim}{\nabla} f \times \underset{\sim}{\nabla} g \quad (\text{"toroidal"})$$

$$[f, g]_p \equiv \underset{\sim}{\nabla}\theta \cdot \underset{\sim}{\nabla} f \times \underset{\sim}{\nabla} g \quad (\text{"poloidal"})$$

$$[f, g] = \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial g}{\partial r}\frac{\partial f}{\partial \theta}\right)$$

$$[f, g]_p = \frac{1}{r}\left(\frac{\partial f}{\partial \zeta}\frac{\partial g}{\partial r} - \frac{\partial g}{\partial \zeta}\frac{\partial f}{\partial r}\right)$$

Note: $\left[\frac{r^2}{2}, g\right]_p = -\frac{\partial g}{\partial \zeta}$

Thus

$$\dot{\psi} = [\psi, \varphi] + \left[\frac{r^2}{2}, \varphi\right]_p$$

$$\dot{U} = [\psi, J] + [U, \varphi] + \left[\frac{r^2}{2}, J\right]_p$$

C. Single helicity case

If
$$f(r,\theta,\zeta) = f(r,\theta-\zeta/q_0)$$
then
$$[f,g]_P = \frac{1}{q_0}[f,g]$$

$\therefore$ Equations of motion become
$$\dot{\psi}_h = [\psi_h, \varphi]$$
$$\dot{U} = [\psi_h, J] + [U,\varphi]$$
where
$$\psi_h \equiv \psi + \frac{r^2}{2q_0} = \text{helical Flux}$$

Also
$$H_h \equiv \frac{1}{2}\int d\underset{\sim}{x}\left[(\nabla_\perp\psi_h)^2 + (\nabla_\perp\varphi)^2\right]$$
$$= H + \text{const.} \quad (\text{Flux conservation})$$

$$\therefore \frac{dH_h}{dt} = 0$$

D. General Hamiltonian Functional

$$H = \frac{1}{2}\int d\underset{\sim}{x}\left[(\nabla_\perp \psi)^2 + (\nabla_\perp \varphi)^2\right]$$

$$= -\frac{1}{2}\int d\underset{\sim}{x}\,(\psi J + \varphi U)$$

$$\frac{\delta H}{\delta \psi} = -J, \qquad \frac{\delta H}{\delta U} = -\varphi$$

Equations of motion:

$$\dot{\psi} = \left[\frac{\delta H}{\delta U}, \psi\right] + \left[\frac{\delta H}{\delta U}, \frac{r^2}{2}\right]_P$$

$$\dot{U} = \left[\frac{\delta H}{\delta \psi}, \psi\right] + \left[\frac{\delta H}{\delta U}, U\right] + \left[\frac{\delta H}{\delta \psi}, \frac{r^2}{2}\right]_P$$

Compare to desired form:

(i) $\dot{\psi} = \{\psi, H\}$, $\dot{U} = \{U, H\}$

(ii) $\{F, G\} = -\{G, F\}$ (antisymmetry)

(iii) $\{F, \{G, H\}\} + \text{cyclic} = 0$ (Jacobi)

E. Hamiltonian bracket

$$\{F, G\} \equiv \int d\underset{\sim}{x} \left\{ \psi \left([F_\psi, G_U] + [F_U, G_\psi] \right) \right.$$

$$\left. + U [F_U, G_U] + \frac{r^2}{2} \left([F_\psi, G_U]_p + [F_U, G_\psi]_p \right) \right\}$$

$F_\psi \equiv \delta F / \delta \psi$, etc.

Single helicity: let $\psi \to \psi_h$ and omit last term.

Properties:

(i) $\dot{\psi} = \{\psi, H\}$, $\dot{U} = \{U, H\}$

- straightforward exercise

(ii) Antisymmetry

- obvious from $[f, g] = -[g, f]$

(iii) Jacobi

- lengthy verification.

F. Canonical variables

1. Clebsch Potentials

$$\psi, U \rightarrow P_i, Q_i \qquad i = 1, 2$$

$$\psi = \hat{z} \cdot \underset{\sim}{\nabla} Q_1 \times \underset{\sim}{\nabla} Q_2 = [Q_1, Q_2]$$

$$U = [Q_2, P_2] + [Q_1, P_1]$$

Find

$$\frac{\delta F}{\delta P_1} = \left[\frac{\delta F}{\delta U}, Q_1\right],$$

$$\frac{\delta F}{\delta Q_1} = -\left[\frac{\delta F}{\delta \psi}, Q_2\right] - \left[\frac{\delta F}{\delta U}, P_1\right], \text{ etc.}$$

Thus Find (For single helicity case)

$$\{F, G\} = \sum_i \int d\underset{\sim}{x} \left(\frac{\delta F}{\delta P_i}\frac{\delta G}{\delta Q_i} - \frac{\delta F}{\delta Q_i}\frac{\delta G}{\delta P_i} \right)$$

canonical version of previous bracket.

2. Canonical equations of motion

From $\dot{\psi} = \{\psi, H\}$, Find

$$[\dot{Q}_1, Q_2] + [Q_1, \dot{Q}_2] = \{[Q_1, Q_2], H\}$$

$$= \left[\frac{\delta H}{\delta P_1}, Q_2\right] + \left[Q_1, \frac{\delta H}{\delta P_2}\right]$$

Hence

$$\dot{Q}_i = \frac{\delta H}{\delta P_i} \qquad i = 1, 2$$

Similarly,

$$\dot{P}_i = -\frac{\delta H}{\delta Q_i} \qquad i = 1, 2$$

Explicit Forms: let

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \hat{z} \times \underset{\sim}{\nabla} \varphi \cdot \underset{\sim}{\nabla}, \quad \underset{\sim}{J} = \hat{z}\, \nabla_\perp^2 \varphi$$

Then

$$\frac{dQ_i}{dt} = 0, \quad \frac{dP_1}{dt} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{J} \times \underset{\sim}{\nabla} Q_2), \quad \frac{dP_2}{dt} = \underset{\sim}{\nabla} \cdot (\underset{\sim}{\nabla} Q_1 \times \underset{\sim}{J})$$

3. Interpretation

Can choose

$$\underset{\sim}{V} = -\sum_i P_i \underset{\sim}{\nabla} Q_i$$

provided

$$\sum_i \left(\underset{\sim}{\nabla} P_i \cdot \underset{\sim}{\nabla} Q_i + P_i \nabla^2 Q_i \right) = 0$$

since $\underset{\sim}{\nabla} \cdot (\hat{z} \times \underset{\sim}{\nabla} \varphi) = 0$.

Then

(i) Q_i are "Frozen-in" coordinates

(ii) P_i are covariant components of $\underset{\sim}{V}$, in (Q_1, Q_2, z) system.

(iii) equations of motion become

$$\frac{dP_i}{dt} = -\psi \frac{\partial J}{\partial Q_i}$$

IV APPLICATIONS

A. Liouville Theorem in function space
(equilibrium stat. mech. ?)

B. Automatic Energy Conserving Approx. upon discretization and truncation.
(possible numerical advantage to canonical form?)