DQ 33

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WAVE Action,

Action-Angle Variables,

& Adiabatic Invariance

for the

Continuous Spectrum of Vlasov Poisson

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WHY WAVE Action?

It is useful for describing wave propagation in inhomogeneous time varying media, for example. For slowly varying media wave action is nearly constant, but the energy & frequency can change a lot.

WHENCE WAVE Action?

* wave quanta, e.g. plasmons; analogy w/ q. mechanics. (where there?)
* Average Lagrangian techniques – Whitham e.g.
* fooling around
* Hamiltonian idea: Wave Action = Action Variable = Adiabatic Invariant

Fundamental?

LIMITATIONS

* quantum analogy is silly – where there?
* fooling around is good method for smart people
* Lagrangian techniques: 1) $\exists$ strange action principles e.g. $\mathcal{L}\psi = 0$ has

Where is F=ma?

$$S_1[\psi] = \int (\mathcal{L}\psi)^2$$
$$S_2[\psi] = \int \psi^+ \mathcal{L}\psi \quad \text{etc}$$

2) Must develop methods of proof; beyond all orders etc. — lack Hamiltonian intuition

3) $\nexists$ methods for continuous spectrum?

These tools exist for Hamiltonian approach! The fundamental degrees of freedom are not "waves"

4) Problem is easy for fluid theories w/o wave-ptle resonance

Vlasov-Poisson = Hamiltonian Field Theory

$$\frac{\partial f}{\partial t}(x,v,t) + [\mathcal{E}, f] = 0 \qquad [f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial x}$$

$$H[f] = \int \frac{mv^2}{2} f \, dx\, dv + \frac{1}{8\pi}\int E^2 dx$$

$$\frac{\delta H}{\delta f} = \mathcal{E} = \frac{m}{2}v^2 + \phi$$

$$\frac{\partial f}{\partial t} = \{f, H\}$$

like:

$$\psi_t = \{\psi, H\} = \delta H/\delta \pi$$

$$\pi_t = \{\pi, H\} = -\delta H/\delta \psi$$

Klein-Gordon etc.

$$\{F, G\} = \int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] dx\, dv$$

Noncanonical Poisson Bracket

$$\{F, G\} = \int \left(\frac{\delta F}{\delta \psi}\frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi}\frac{\delta G}{\delta \psi}\right)$$

has Casimirs

$$C[f] = \int \mathcal{C}(f)\, dx\, dv \qquad \{C, F\} = 0 \quad \forall F$$

Linear Theory (about homogeneous equilibria)

$$f = f_0(v) + \sum_k \frac{1}{2} f_k(v,t)\, e^{ikx}$$

$$\{F,G\}_L = \frac{4i}{mV} \sum_{k=1}^{\infty} k \int_{-\infty}^{\infty} dv\, \frac{\partial f_0}{\partial v} \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta F}{\delta f_{-k}}\frac{\delta G}{\delta f_k} \right)$$

$$\boxed{\frac{\partial f_k}{\partial t} = \{f_k, H_L\}_L}$$

$\Rightarrow$ linear dynamics

$$H_L = \delta^2 F = \delta^2 H + \delta^2 C$$

Kruskal-Oberman energy

Canonization

$$q_k(v) = \frac{mV}{4ik\, f_0'(v)} f_k(v)$$

$$p_k(v) = f_{-k}$$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} dv \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta F}{\delta p_k}\frac{\delta G}{\delta q_k} \right)$$

$$\delta^2 F = -\frac{m}{4} \int_{-\infty}^{\infty} \sum_k \frac{v\,|f_k|^2}{\frac{\partial f_0}{\partial v}}\, dv + \frac{V}{16\pi} \sum_k |E_k|^2$$

$$= \iint \sum_k f_k(v)\, \mathcal{O}(v|v')\, f_k(v')\, dv\, dv'$$

Not Diagonal

Diagonalization

Mixed Variable Generating Function"(al)":

$$\boxed{F[\underset{\text{New}}{P_k}, \underset{\text{Old}}{q_k}] = \sum_k \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \; q_k(v)\, P_k(u)\, \mathcal{G}_k(u,v)}$$

$$\mathcal{G}_k(u,v) = \epsilon_I(k,v)\,\frac{1}{\pi}\,\mathbb{P}\frac{1}{u-v} + \epsilon_R(k,v)\,\delta(u-v)$$

$$\frac{\delta F}{\delta q_k(v)} = p_k(v) = \int_{-\infty}^{\infty} P_k(u)\, \mathcal{G}_k(u,v)\, du$$

$$\frac{\delta F}{\delta P_k(u)} = Q_k(u) = \int_{-\infty}^{\infty} q_k(v)\, \mathcal{G}_k(u,v)\, dv$$

Basic Identity: $\displaystyle\int_{-\infty}^{\infty} \tilde{\mathcal{G}}_k(u,v')\, \mathcal{G}_k(u,v)\, du = \delta(v-v')$

$$\mathcal{G}_k(u,v) = \frac{1}{|\epsilon|^2(u)} \left\{ \epsilon_I(k,u) \frac{1}{\pi} \mathbb{P} \frac{1}{u-v} + \epsilon_R(k,u)\, \delta(u-v) \right\}$$

Inserting $q_k(v)$ & $p_k(v)$ into into H_L

$\longrightarrow$ diagonal form

$$H[Q_k, P_k] = \int du \sum_k (-i k u)\, Q_k P_k$$

Action-Angle Variables

$$Q_k = \sqrt{J_k}\, e^{-i\theta_k(u)} \qquad P_k = i\sqrt{J_k(u)}\, e^{i\theta_k(u)}$$

$\Longrightarrow$

$$\boxed{H[J_k, \theta_k] = \int du \sum_k (k u)\, J_k(u)}$$

$\left(\text{like SHO } \sum_k \omega_k J_k \right)$

Adiabatic Invariance for Continuous Spectra?

Suppose $f_0(v, \epsilon t)$ has a slow time dependence $\Rightarrow$

slow

$(q_k, p_k) \rightleftarrows (Q_k, P_k)$ is an explicitely time dependent transformation (Since $\epsilon_{I,R}(k, v, \epsilon t)$) $\Rightarrow$

$$\boxed{H(Q_k, P_k) = \int du \sum_k (-iku)\, Q_k P_k + \frac{\partial F}{\partial t}}$$

New Term

$$\frac{\partial F}{\partial t} = \int du \int dv \int du'\; Q_k(u') P_k(u)\, \tilde{g}(u', v)\, \frac{\partial g}{\partial t}(u, v)$$

$$\frac{\partial F}{\partial t} = \int du \int dv \int du'\; \sqrt{J_k(u')}\, e^{-i\theta_k(u')}\, e^{i\theta_k(u)} \sqrt{J_k(u)}\; \tilde{g}(u', v)\, \frac{\partial g}{\partial t}(u, v)$$

$$\boxed{\dot{J}_k = -\frac{\delta H}{\delta \theta_k} = -\frac{\partial(\partial F/\partial t)}{\partial \theta_k} = \mathcal{O}(\epsilon)}$$

small

Consider an arbitrary time T & integrate $\Longrightarrow$

$$\Delta J_{kT} := \int_0^T \dot{J}_k \, dt = -\int_0^T \frac{\delta}{\delta \theta_k} \frac{\partial F}{\partial t} \, dt \qquad \text{W.T.S.}$$

$$\boxed{|\Delta J_{kT}| \leq \frac{C \epsilon}{|k u|} \sim \frac{\Delta f_0}{T}}$$

C must be indep. of T!

With this, the action remains arbitrarily small with large changes in Δf_0 if T is large, i.e. the change in f_0 is made very slowly over a long time.

Question

Can other wave actions (Kaufman, Brizard, Tracy, Crawford, ...) that are functions of x & t be represented as a sum over these continuum action variables?