II) Trieste Aug 15, 2019 — P. J. Morrison (1)

Review

action: $$S[q] = \int_{t_0}^{t_1} dt \sum_i m_i \frac{\dot{q}_i^2}{2} - V(q)$$

$$= \int_{t_0}^{t_1} dt \int_D d^3a \left[\rho_0 \frac{\dot{q}^2}{2} - V(q, \partial q)\right]$$

Funct. Derivative: Vary & isolate (calc. vs rigor)

$$\delta S = \frac{d}{d\epsilon} S[q + \epsilon \delta q]\Big|_{\epsilon=0} = \left\langle \frac{\delta S}{\delta q}, \delta q \right\rangle$$

Ex

$$H[u] = \int_{\mathbb{R}} \left(\frac{u^3}{6} - \frac{u_x^2}{2}\right) dx$$

$$\delta H = \int \left(\frac{u^2}{2} - \frac{d}{dx} u_x\right) \delta u \, dx = \left\langle \frac{\delta H}{\delta u}, \delta u \right\rangle$$

$\frac{\delta H}{\delta u(x)}$

vs. $\frac{\delta S}{\delta q(a,t)}$ $\Rightarrow$ pairing (inner prdt)

Euler-Lag Map

$$\rho(r,t) = \int_D \rho_0(a)\, d^3a\, \delta(r - q)$$

$(q,p) \Rightarrow \rho, v$ etc

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|}$$

Legendre Transform (Newton $\to$ Hamilton)

Canonical momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}^i}(q, \dot{q})$$

convexity $\det \frac{\partial^2 L}{\partial \dot{q} \partial \dot{q}} \neq 0 \Rightarrow$

$$\dot{q} = \dot{q}(p)$$

$$H = p_i \dot{q}^i - L(q, \dot{q}) \qquad i, j = 1, 2, \ldots N$$

N 2nd order $\longrightarrow$ $2N$ 1st order eqs

$z = (q, p)$

$$\boxed{\dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j}} \qquad (J_c) = \begin{bmatrix} O_N & I_N \\ -I_N & O_N \end{bmatrix}$$

$$\boxed{\dot{z}^i = \{z^i, H\}} \qquad \boxed{\{f, g\} = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j} \quad PB}$$

Infinite DoF

$$p = \frac{\delta L}{\delta \dot{q}}$$

$$H = \int d^3a \left(p\dot{q} - L \right)$$

Poisson Bracket

$$\dot{q}^i = \{ q^i, H \} \qquad \dot{p}_i = \{ p_i, H \} \qquad (*)$$

$$\{F, G\} = \int d^3a \left[\frac{\delta F}{\delta q^i} \frac{\delta G}{\delta p_i} - \frac{\delta F}{\delta p_i} \frac{\delta G}{\delta q^i} \right]$$

$(*)$ equivalent to

$$\rho_0 \ddot{q}^i = \ldots$$

Eulerian Form

$$F[q,p] = \bar{F}[v, \rho, s, B]$$

obtains q, p dep thru

chain rule

$$\delta F = \int d^3a \left(\frac{\delta F}{\delta q} \cdot \delta q + \frac{\delta F}{\delta p} \cdot \delta p \right)$$

$$= \int d^3x \left\{ \frac{\delta \bar{F}}{\delta v} \cdot \delta v + \frac{\delta \bar{F}}{\delta s} \delta s + \frac{\delta \bar{F}}{\delta B} \cdot \delta B + \frac{\delta \bar{F}}{\delta \rho} \delta \rho \right] \qquad (+)$$

$$\delta \rho = \int_D - d^3a \, \rho_0(a) \nabla \delta^3(r-q) \cdot \delta q$$

etc

insert into (+) & equate

$$\int_D d^3a \frac{\delta F}{\delta q} \cdot \delta q + \cdots = \underbrace{\int_D d^3x \frac{\delta \bar{F}}{\delta \rho} (-) \int_D d^3a \, \rho_0(a) \nabla \delta^3 \cdot \delta q}$$

interchange

$$\int d^3a \frac{\delta F}{\delta q} \cdot \delta q = \int d^3a \left(\quad \right) \cdot \delta q$$

$$\frac{\delta F}{\delta q} = \mathcal{O}_\rho \frac{\delta \bar{F}}{\delta \rho} + \mathcal{O}_s \frac{\delta \bar{F}}{\delta s} + \cdots$$

Insert in PB & manipulate $\Rightarrow$

$$
\begin{aligned}
\{F,G\} = -\int \Big\{ & \frac{\delta F}{\delta \underline{v}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta v} \cdot \nabla \frac{\delta F}{\delta v} \\
& + \frac{\nabla \times v}{\rho} \cdot \left(\frac{\delta G}{\delta v} \times \frac{\delta F}{\delta v} \right) \\
& + \frac{\nabla s}{\rho} \cdot \left(\frac{\delta F}{\delta s} \frac{\delta G}{\delta v} - \frac{\delta F}{\delta v} \frac{\delta G}{\delta s} \right) \\
& + B \cdot \left[\frac{1}{\rho} \frac{\delta F}{\delta v} \cdot \nabla \frac{\delta G}{\delta B} - \frac{1}{\rho} \frac{\delta G}{\delta B} \cdot \nabla \frac{\delta F}{\delta v} \right] \\
& + B \cdot \left[\nabla \left(\frac{1}{\rho} \frac{\delta F}{\delta v} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left(\frac{1}{\rho} \frac{\delta G}{\delta v} \right) \cdot \frac{\delta F}{\delta B} \right] \Big\} \, d^3x
\end{aligned}
$$

MG
1980

H from Legendre transform

$$
H = \int \left(\frac{\rho v^2}{2} + \rho\, U(\rho, s) + \frac{B^2}{2} \right) d^3x
$$

EMP (2005). Ham. Fluid Mech

~~[illegible]~~ Can Do for Gyro MHD

What is it?

5B

Use Densities

$\{M, \sigma, \rho, B\} \Rightarrow$

$$\{F, G\} = \int d^3x \; M_i \left(\frac{\delta G}{\delta M_i} \partial_j \frac{\delta F}{\delta M_j} - \right.$$

$$+ \rho \left(\frac{\delta G}{\delta M_i} \partial_i \frac{\delta F}{\delta \rho} - \right)$$

$$+ \sigma \left(\frac{\delta G}{\delta M_i} \partial_i \frac{\delta F}{\delta \rho} - \cdots \right)$$

$$= \left\langle \Psi, \left[\frac{\delta F}{\delta \Psi}, \frac{\delta G}{\delta \Psi} \right] \right\rangle$$

Lie - Poisson form

1980's

Noncanonical Mechanics

What is this strange bracket?

Derived by effecting a noncanonical transformation

$$\dot z^i = J_c^{ij}\frac{\partial H}{\partial z^j} \xrightarrow{z(\bar z)} \dot{\bar z}^i = J^{ij}(\bar z)\frac{\partial \bar H}{\partial \bar z^j}$$

$$H(z) = \bar H(\bar z) \qquad J = \frac{\partial \bar z}{\partial z} J_c \frac{\partial \bar z}{\partial z}$$

Obtain Noncanical Poisson Bracket

$$\{f,g\} = \frac{\partial f}{\partial \bar z^i} J^{ij}(\bar z)\frac{\partial g}{\partial \bar z^j}$$

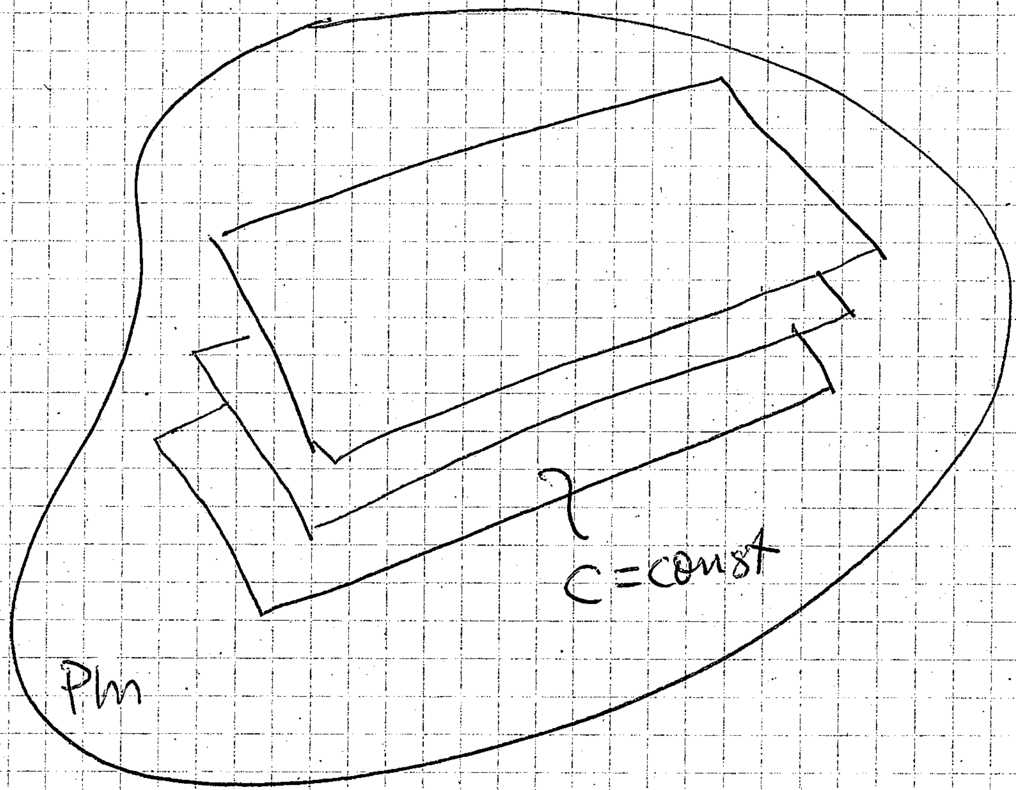
Symplectic Manifold $\longrightarrow$ Poisson Manifold

PM is a manifold w/ a binary operation on functions, where

$$\{\,,\,\} : \mathcal{F}\times\mathcal{F} \to \mathcal{F} \qquad \mathcal{F} : PM \to \mathbb{R}$$

is a Lie enveloping algebra.

such a manifold has a particular geometric structure w/ a foliation by symplectic leaves.



$\{f, C\} = 0 \quad \forall f \quad \Rightarrow \quad C$ a Casimir

$$J\nabla C = 0 \quad \Rightarrow \quad \det J = 0$$

Symplectic leaves $\quad \cap\, C = \text{const.}$

Some Recent Work

Tassi et al. PPCF 50 085004 (2008)

Waelbroeck et al. PoP 16 032109 (2009)

Tassi, et al. PoP (2009)

Tassi et al. Nuc. Fusion in press