Hamiltonian and Action Principle Formulations of Plasma Physics

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Review HAP formulations with plasma applications.

"Hamiltonian systems .... are the basis of physics." M. Gutzwiller

Thanks: mentors, colleagues, students ....

Finalized Course Overview

1. Review of Basics (finite $\rightarrow$ infinite)

2. Ideal Fluids and Magnetofluids A

3. Ideal Fluids and Magnetofluids B

4. Ideal Fluids and Magnetofluids C

5. Kinetic Theory – Canonization & Diagonalization, Continuous Spectra, Krein-like Theorems

6. Metriplecticism: relaxation paradigms for computation and derivation

General References

Numbers refer to items on my web page: `http://www.ph.utexas.edu/∼morrison/` where all can be obtained under ‘Publications’.

177. G. I. Hagstrom and P. J. Morrison, “Continuum Hamiltonian Hopf Bifurcation II, Spectral Analysis, Stability and Bifurcations in Nonlinear Physical Systems, eds. O. Kirillov and D. Pelinovsky (Wiley, 2013). Constraints on how unstable modes emerge from the continuous spectrum are explored.

176. P. J. Morrison and G. I. Hagstrom, “Continuum Hamiltonian Hopf Bifurcation I, Spectral Analysis, Stability and Bifurcations in Nonlinear Physical Systems, eds. O. Kirillov and D. Pelinovsky (Wiley, 2013). See 177.

138. P. J. Morrison, “On Hamiltonian and Action Principle Formulations of Plasma Dynamics, in New Developments in Nonlinear Plasma Physics: Proceedings for the 2009 ICTP College on Plasma Physics, eds. B. Eliasson and P. Shukla, American Institute of Physics Conference Proceedings No.1188 (American Institute of Physics, New York, 2009) pp. 329–344. Describes the procedure for building actions for magnetofluids etc.

103. N. J. Balmforth and P. J. Morrison: “Hamiltonian Description of Shear Flow,” in Large-Scale Atmosphere-Ocean Dynamics II, eds. J. Norbury and I. Roulstone (Cambridge, Cambridge, 2002) pp. 117–142. Diagonalization of shear flow is performed, which amounts to completion of Arnold’s program.

91. P. J. Morrison, “Hamiltonian Description of the Ideal Fluid, Reviews of Modern Physics 70, 467–521 (1998). A comprehensive survey of Hamiltonian and action principles for fluids.

9. P. J. Morrison, “Poisson Brackets for Fluids and Plasmas, in Mathematical Methods in Hydrodynamics and Integrability in Dynamical Systems, eds. M. Tabor and Y. Treve, American Institute of Physics Conference Proceedings No. 88 (American Institute of Physics, New York, 1982) pp. 13–46. A review of the Hamiltonian structure of many plasma models.

HAP Formulations of PP: I Basics

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Action Principle

Hero of Alexandria (75 AD) $\longrightarrow$ Fermat (1600's) $\longrightarrow$

$\boxed{\text{Hamilton's Principle (1800's)}}$

The Procedure:

- Configuration Space Q: $\quad q^i(t)\,, \qquad i = 1, 2, \ldots, N \quad \longleftarrow$ #DOF

- Kinetic - Potential: $\quad L = T - V : TQ \times \mathbb{R} \to \mathbb{R}$

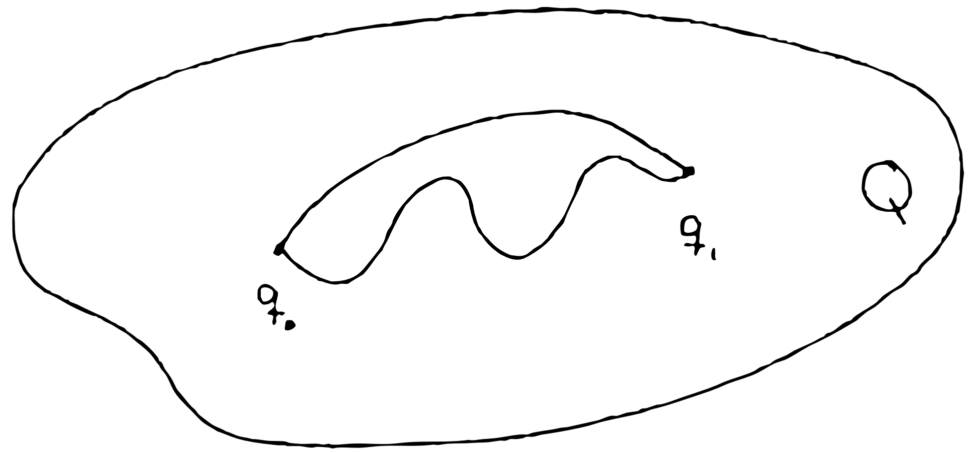
- Action Functional: paths $\to \mathbb{R}$

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\,, \quad \delta q(t_0) = \delta q(t_1) = 0$$

$\boxed{\text{Extremal path} \Longrightarrow \text{Lagrange's equations}}$

Variation Over Paths

$S[q_{\text{path}}] = $ number



Functional Derivative: $\Leftrightarrow$ vanishing first variation

$$\frac{\delta S[q]}{\delta q^i} = 0 \qquad \Longrightarrow$$

Lagrange's Equations:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0\,.$$

Hamilton's Equations

Canonical Momentum: $p_i = \frac{\partial L}{\partial \dot{q}^i}$

Legendre Transform: $H(q,p) = p_i \dot{q}^i - L$

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,}$$

Failure of LT (not convex) $\Longrightarrow$ Dirac constraint theory

Phase Space Coordinates: $z = (q,p)$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j}\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

symplectic 2-form = (cosymplectic form)$^{-1}$: $\omega^c_{ij} J_c^{jk} = \delta_i^k$,

Phase-Space Action

Gives Hamilton's equations directly

$$S[q,p] = \int_{t_0}^{t_1} dt \, \Big(p_i \dot{q}^i - H(q,p)\Big)$$

Defined on paths γ in phase space $\mathcal{P}$ (e.g. T^*Q) parameterized by time, t, i.e., $z_\gamma(t) = (q_\gamma(t), p_\gamma(t))$. Then $S : \mathcal{P} \to \mathbb{R}$. Domain of S any smooth path $\gamma \in \mathcal{P}$.

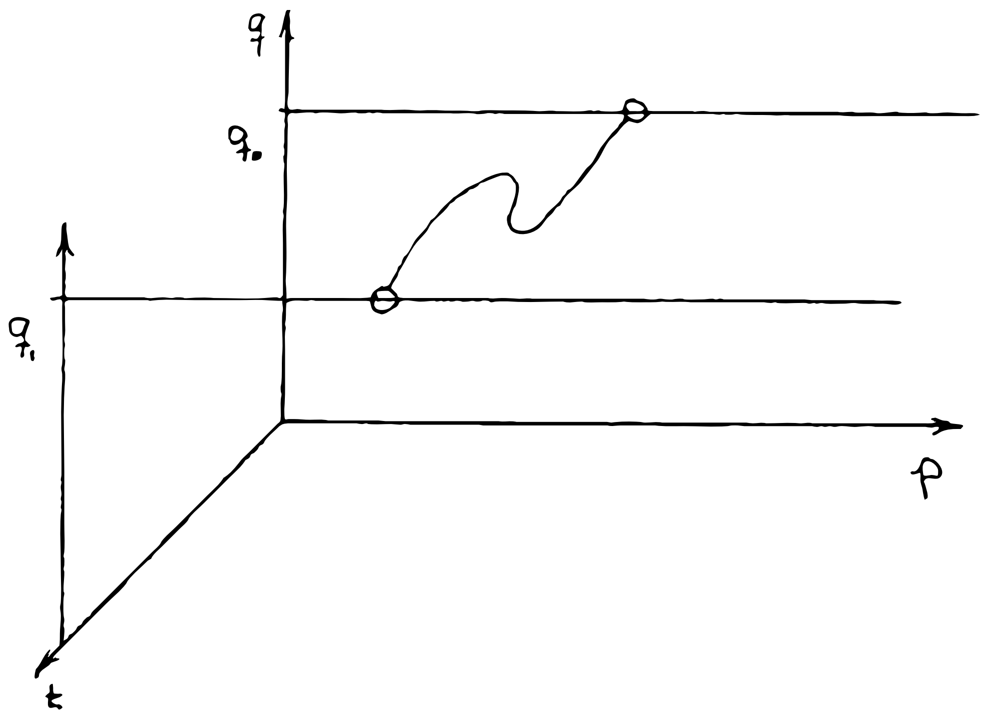
Law of nature, set Fréchet or functional derivative,to zero. Varying S by perturbing path, $\delta z_\gamma(t)$, gives

$$\delta S[z_\gamma; \delta z_\gamma] = \int_{t_0}^{t_1} dt \left[\delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) - \delta q^i \left(\dot{p}_i + \frac{\partial H}{\partial q^i}\right) + \frac{d}{dt}\Big(p_i \delta q^i\Big)\right] .$$

Under the assumption $\delta q(t_0) = \delta q(t_1) \equiv 0$, with no restriction on δp, boundary term vanishes.

Admissible paths in $\mathcal{P}$ have 'clothesline' boundary conditions.

Phase-Space Action Continued



$$\delta S \equiv 0 \quad \Rightarrow \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \quad i = 1, 2, \ldots, N\,,$$

Thus, extremal paths satisfy Hamilton's equations.

Alternatives

Rewrite action S as follows:

$$S[z] = \int_{t_0}^{t_1} dt \left(\frac{1}{2} \omega^c_{\alpha\beta} z^\alpha \dot{z}^\beta - H(z) \right) =: \int_\gamma (d\theta - H dt)$$

where $d\theta$ is a differential one-form.

Exercise: What are boundary conditions. General θ?

Exercise: Particle motion in given electromagnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$

$$S[\mathbf{r}, \mathbf{p}] = \int_{t_0}^{t_1} dt \left[\mathbf{p} \cdot \dot{\mathbf{r}} - \frac{1}{2m} \left| \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, t) \right|^2 - e\phi(\mathbf{r}, t) \right] .$$

Show Lorentz force law arises from S.

Generalized Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980)....

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad\qquad [A,B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\quad [A,B] = -[B,A]\,,$

Jacobi identity $\longrightarrow$ $\quad [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

G. Darboux: $det J \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus <u>Casimirs</u>

Matter models in Eulerian variables: $J^{ij} = c_k^{ij} z^k \leftarrow$ Lie − Poisson Brackets

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{P}$ is differentiable manifold with bracket $[\,,\,] : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ st $C^\infty(\mathcal{P})$ with $[\,,\,]$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

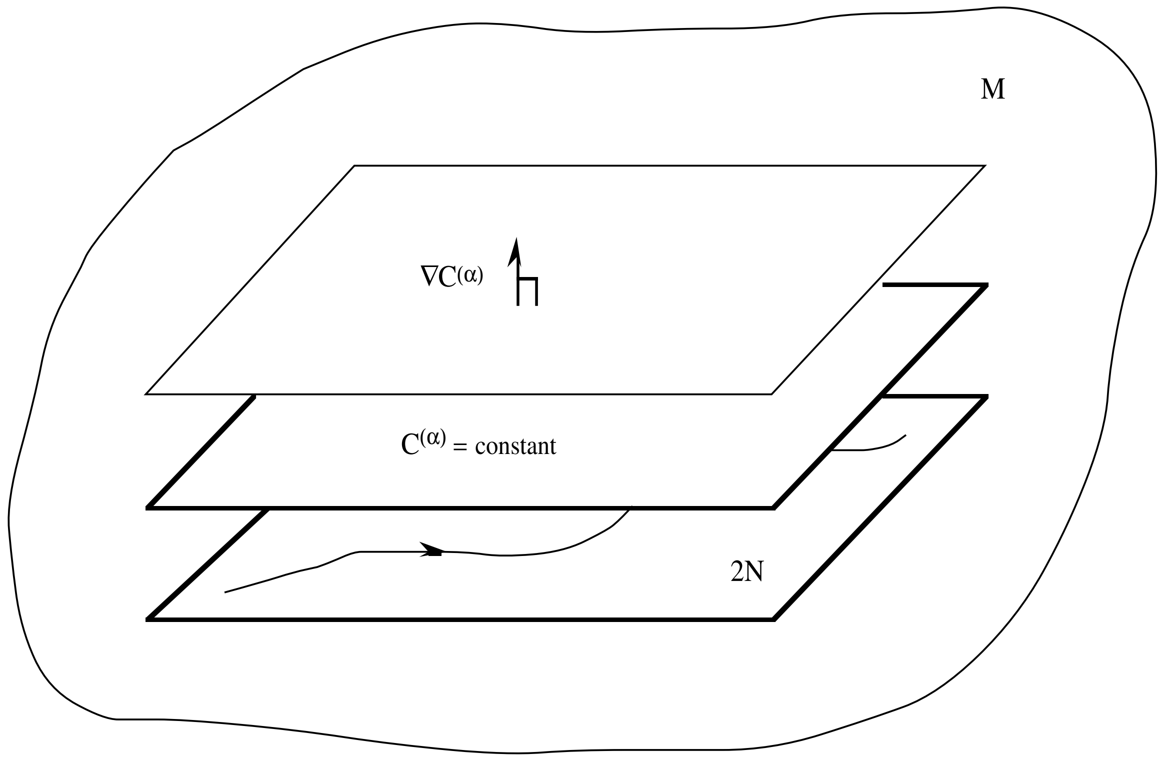
Because of degeneracy, $\exists$ functions C st $[f, C] = 0$ for all $f \in C^\infty(\mathcal{P})$. Called Casimir invariants (Lie's distinguished functions.)

Poisson Manifold $\mathcal{P}$ Cartoon

Degeneracy in $J \Rightarrow$ Casimirs:

$$[f, C] = 0 \quad \forall\ f : \mathcal{P} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Hamiltonian Reduction

Bracket Reduction:

Reduced set of variables $(q,p) \mapsto w(q,p)$

Bracket Closure:

$$[w,w] = c(w) \qquad\qquad f(q,p) = \widehat{f} \circ w = \widehat{f}(w(q,p))$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket

Hamiltonian Closure:

$$H(q,p) = \widehat{H}(w)$$

Example: Eulerian fluid variables are noncanonical variables

(pjm & John Greene 1980)

Reduction Examples/Exercises

- Let $q \in Q = \mathbb{R}^3$ and define the angular momenta $L_i = \epsilon_{ijk} q_j p_k$, with $i, j, k = 1, 2, 3$. Show $[L_i, L_j] = f_{ij}(L)$. What is f_{ij}?

- Given $w_k = L^i_k(q) p_i$, with $i = 1, 2, \ldots N$, find a nontrivial condition on L^i_k that ensures reduction.

Why Action/Hamiltonian?

- Beauty, Teleology, . . . : Still a good reason!
- 20th Century framework for physics: Plasma models too.
- Symmetries and Conservation Laws: energy-momentum . . . .
- Generality: do one problem $\Rightarrow$ do all.
- Approximation: pert theory, averaging, . . . one function.
- Stability: built-in principle, Lagrange-Dirichlet, δW, ...
- Beacon: motivation, e.g. $\exists$ ∞-dim KAM theorem? . . . .
- Numerical Methods: structure preserving algorithms:

  symplectic/conservative integrators, ...
- Statistical Mechanics: energy and measure.

Functionals

Functions: number $\longmapsto$ number e.g. $f : \mathbb{R}^n \to \mathbb{R}$

example

Generalized Coordinate: $q(t) = A\cos(\omega t + \phi)$ e.g. SHO

Functionals: function $\longmapsto$ number e.g. $F : \mathbb{L}^2 \to \mathbb{R}$

examples

General: $F[u] = \int \mathcal{F}(u, u_x, u_{xx}, \dots)\, dx\,.$

Hamilton's Principle: $S[q] = \frac{1}{2}\int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\,.$

Vlasov Energy: $H[f] = \frac{m}{2}\int f v^2\, dx dv + \frac{1}{2}\int E^2\, dx.$

Functional Differentiation

First variation of function:

$$\delta f(z;\delta z) = \sum_{i=1}^{n} \frac{\partial f(z)}{\partial z_i}\,\delta z_i =: \nabla f \cdot \delta z\,, \qquad f(z) = f(z_1, z_2, \ldots, z_n)\,.$$

First variation of functional:

$$\delta F[u;\delta u] = \frac{d}{d\epsilon} F[u + \epsilon\,\delta u]\Big|_{\epsilon=0} = \int_{x_0}^{x_1} \delta u\,\frac{\delta F}{\delta u(x)}\,dx =: \left\langle \frac{\delta F}{\delta u}, \delta u \right\rangle\,.$$

dot product	$\cdot$	$\Longleftrightarrow$	scalar product	$<,>$
index	i	$\Longleftrightarrow$	integration variable	x
gradient	$\frac{\partial f(z)}{\partial z_i}$	$\Longleftrightarrow$	functional derivative	$\frac{\delta F[u]}{\delta u(x)}$

$\boxed{\text{Vary and Isolate} \longrightarrow \text{Functional Derivative}}$

Functional Differentiation Examples/Exercises

- Given

$$H[u] = \int_{\mathbb{T}} dx \left(\frac{u^3}{6} - \frac{u_x^2}{2} \right) , \quad u : \mathbb{T} \to \mathbb{R}$$

  What is $\delta u / \delta x$?

- Given

$$\mathcal{E}[\mathbf{E}] = \frac{1}{2} \int_{\mathbb{R}} d^3x \, |\mathbf{E}|^2$$

  What is $\delta \mathcal{E} / \delta \mathbf{E}$? For $\mathbf{E} = -\nabla \phi$, how are $\delta \mathcal{E} / \delta \mathbf{E}$ and $\delta \mathcal{E} / \delta \phi$ related?

Relativistic N-Particle Action

Dynamical Variables: $q_i(t), \phi(x,t), A(x,t)$

$$S[q,\phi,A] \;=\; -\sum_{i=1}^{N}\int dt\;\; mc^2\sqrt{1-\frac{\dot{q}_i^2}{c^2}} \qquad \longleftarrow \text{ ptle kinetic energy}$$

$$\text{coupling} \longrightarrow \qquad -e\int dt\sum_{i=1}^{N}\int d^3x\left[\phi(x,t)+\frac{\dot{q}_i}{c}\cdot A(x,t)\right]\delta\left(x-q_i(t)\right)$$

$$\text{field `energy'} \longrightarrow \qquad +\frac{1}{8\pi}\int dt\int d^3x\left[E^2(x,t)-B^2(x,t)\right].$$

Variation:

$$\frac{\delta S}{\delta q^i(t)}=0 \quad \Longrightarrow \quad \text{EOM \& Fields},$$

$$\frac{\delta S}{\delta\phi(x,t)}=0\,,\quad \frac{\delta S}{\delta A(x,t)}=0 \quad \Longrightarrow \quad \text{ME \& Sources}$$

All done?

Irrelevant Information

Reductions, Approximations, Mutilations, . . . :

$\Longrightarrow$ Constraints (explicit or implicit) $\Longrightarrow$ Interesting!

<u>Finite Systems</u>

B-lines, ptle orbits, self-consistent models, . . .

<u>Infinite Systems</u>

kinetic theories, fluid models, mixed . . .

Lagrangian (material) or Eulerian (spacial) variables

Big Actions to Little Actions

Hamiltonian B-lines: Set $\phi = 0$, specify B, let $r_G \to 0$

$$S[r] = \int A \cdot dr \qquad \text{Kruskal (52)}$$

Hamiltonian ptle orbits: Specify ϕ and B non-selfconsistent

Standard ptle orbit action $\Longrightarrow$ tools

Hamiltonian self-consistent models: Specify ϕ and B partly

Single-Wave Model: OWM(71), Kaufman & Mynick (79), Tennyson et al.(94), Balmforth et al. (2013), . . .

Multi-Wave Model: Cary & Doxas, Escande, del-Castillo, Finn, . . . Evstatiev (2004)

Moment Models: Kida, Chanell, Meacham et al. (95), Shadwick . . . , Perin et al. (2014).

Finite DOF Hamiltonian Vocabulary

Integrable 1 DOF

Poincare Section 1.5 DOF

KAM integrable limit

Invariant Tori good surfaces

Island Overlap broken surfaces

Chirkov-Taylor Map chaos

Greene's Criterion tori far from integrable

Renormalization universality

Spectra no asymptotic stability

Stability Lagrange $\delta^2 W$, Dirichlet $\delta^2 H$, Energy-Casimir $\delta^2 F$,. . .

Normal Forms stable $\Rightarrow H = \sum \omega(q^2 + p^2)/2$, linear/nonlinear

Infinite DOF Hamiltonian Vocabulary

Integrable KdV, . . . , rare, Greene and Kruskal

KAM active area in mathematics

Spectra discrete, continuous

Stability $\delta^2 W$, $\delta^2 H$, $\delta^2 F$

Normal Forms linear/nonlinear perturbation theories

Action Reduction direct method of calculus of variations

Noether's Thm energy-momentum tensor only believable way

Hamiltonian Reduction little systems from big, exact/approximate

HAP Formulations of PP: II Magnetofluids A

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Magnetofluid References

Numbers refer to items on my web page: `http://www.ph.utexas.edu/~morrison/` where all can be obtained under ‘Publications’.

187. M. Lingam and P. J. Morrison, “The Action Principle for Generalized Fluid Motion Including Gyroviscosity,” submitted (2014). Discusion of general models with gyroscopic Lagrangians, including Eulerian variations.

186. I. Keramidas Charidakos, M. Lingam, P. J. Morrison, R. L. White, and A. Wurm, “Action Principles for Extended MHD Models,” Physics of Plasmas (2014). Discussion of two-fluid action and extended MHD. A complicated Lagrange to Euler map is described.

185. K. Kimura and P. J. Morrison, ”On Energy Conservation in Extended Magnetohydrodynamics,” Physics of Plasmas 21, 082101 (2014). A discussion of how ‘ordinary’ derivations can lead to incorrect results.

184. P. J. Morrison, M. Lingam, and R. Acevedo, “Hamiltonian and Action Formalisms for Two-Dimensional Gyroviscous MHD,” Physics of Plasmas 21, 082102 (2014). Derivation of Braginskii MHD from an action principle, derivation of the gyromap, and Hamiltonian reduction.

182. C. Tronci, E. Tassi, E. Camporeale, and P. J. Morrison, “Hybrid Vlasov-MHD Models: Hamiltonian vs. Non-Hamiltonian,” Plasma Physics and Controlled Fusion 56, 095008 (2014). Like item 185, models can be non-energy conserving. It is shown that lead non-Hamiltonian models in the literature have false instabilities.

138. P. J. Morrison, “On Hamiltonian and Action Principle Formulations of Plasma Dynamics, in New Developments in Nonlinear Plasma Physics: Proceedings for the 2009 ICTP College on Plasma Physics, eds. B. Eliasson and P. Shukla, American Institute of Physics Conference Proceedings No.1188 (American Institute of Physics, New York, 2009) pp. 329–344. Describes the procedure for building actions for magnetofluids etc.

91. P. J. Morrison, “Hamiltonian Description of the Ideal Fluid, Reviews of Modern Physics 70, 467–521 (1998). A comprehensive survey of Hamiltonian and action principles for fluids.

General Method for Building Actions Applied to Magnetofluids

Ex Post Facto Discovery vs. Ab Initio Construction

Senior Progeny

Computability and Intuition

Reductions $\Longrightarrow$

Vlasov-Maxwell, two-fluid theory, MHD, ...

Neglect clearly identifiable dissipation $\Longrightarrow$

Action principles and Hamiltonian structure

identified *ex post facto*

Simplifications: Reduced Fluid Models

Approximations:

asymptotic expansions, systematic ordering

Model Building:

Mutilations, put it what one this is important, closures etc.

Other Progeny:

Gyrokinetics, guiding-center kinetics, gyrofluids, ...

Hamiltonian? Action?

Building Action Principles Ab Initio

Step 1: Select Domain

For fluid a spatial domain; for kinetic theory a phase space

Step 2: Select Attributes – Eulerian Variables (Observables)

L to E, map e.g. MHD $\{v, \rho, s, B\}$. Builds in constraints!

Step 3: Eulerian Closure Principle

Terms of action must be 'Eulerianizable' $\Rightarrow$ EOMs are!

Step 4: Symmetries

Traditional. Rotation, etc. via Noerther $\rightarrow$ invariants

Closure Principle

If closure principle is satisfied, then

i) Equations of motion obtained by variation are ‘Eulerianizable’.

ii) There exists a noncanonical Hamiltonian description.

Ideal Fluid and MHD

Fluid Action Kinematics

Giuseppe Luigi Lagrange, Mécanique analytique (1788)

Lagrangian Variables:

Fluid occupies domain D e.g. (x, y, z) or (x, y)

Fluid particle position $q(a, t)$, $q_t : D \to D$
bijective, smooth, diffeomorphism, . . .

Particle label: a e.g. $q(a, 0) = a$.

Deformation: $\frac{\partial q^i}{\partial a^j} = q^i_{,j}$

Determinant: $\mathcal{J} = \det(q^i_{,j}) \neq 0 \Rightarrow a(q, t)$

Identity: $q^i_{,k} a^k_{,j} = \delta^i_j$

Volume: $d^3q = \mathcal{J} d^3a$

Area: $(d^2q)_i = \mathcal{J} a^j_{,i}(d^2a)_j$

Line: $(dq)_i = q^i_{,j}(da)_j$

<u>Eulerian Variables:</u>

Observation point: r

Velocity field: $v(r,t) = ?$ Probe sees $\dot{q}(a,t)$ for some a.

What is a ? $r = q(a,t) \quad \Rightarrow \quad a = q^{-1}(r,t)$

$$v(r,t) = \dot{q}(a,t)|_{a=q^{-1}(r,t)}$$

IDEAL MHD

Attributes:

Entropy (1-form):

$$s(r,t) = s_0|_{a=a(r,t)} \ ,$$

Mass (3-form):

$$\rho d^3x = \rho_0 d^3a \quad \Rightarrow \quad \rho(r,t) = \left.\frac{\rho_0}{\mathcal{J}}\right|_{a=a(r,t)} \ .$$

B-Flux (2-form):

$$B \cdot d^2x = B_0 \cdot d^2a \quad \Rightarrow \quad B^i(r,t) = \left.\frac{q^i_{,j} B^j_0}{\mathcal{J}}\right|_{a=a(r,t)} \ .$$

Kinetic Potential

Kinetic Energy:

$$K[q]=\frac{1}{2}\int_D d^3a\,\rho_0|\dot{q}|^2=\frac{1}{2}\int_D d^3x\,\rho|v|^2$$

Potential Energy:

$$\begin{aligned}V[q] &= \int_D d^3a\,\rho_0\mathcal{V}(\rho_0/\mathcal{J},s_0,|q^i_{,j}B^j_0|/\mathcal{J})=\frac{1}{2}\int_D d^3x\,\rho\mathcal{V}(\rho,s,|B|)\\ &= \int_D d^3a\,\rho_0\mathcal{U}(\rho_0/\mathcal{J},s_0)+\frac{1}{2}\frac{|q^i_{,j}B^j_0|^2}{\mathcal{J}^2}\end{aligned}$$

Action:

$$S[q]=\int dt\,(K-V)\,,\qquad \delta S=0 \quad\Rightarrow\quad \text{Ideal MHD}$$

Alternative: Lagrangian variations induce constrained Eulerian variations $\Rightarrow$ Serrin, Newcomb, Euler-Poincaré, ...

Stability: δW, Lagrangian, Eulerian, dynamical accessible, Andreussi, Pegoraro, pjm. (2010 – 2014)

Equations of Motion and Eulerianization

Hamiltonian Structure

Legendre Transformation:

$$p = \frac{\delta L}{\delta \dot{q}} = \rho_0 \, \dot{q} \qquad\qquad L \to H$$

$$H = \frac{1}{2}\int_D d^3a \, |p|^2/\rho_0 + \int_D d^3a \, \left(\rho_0 \mathcal{U}(\rho_0/\mathcal{J}, s_0) + \frac{1}{2}\frac{|q^i_{,j} B_0^j|^2}{\mathcal{J}^2} \right)$$

Poisson Bracket:

$$\{F, G\} = \int_D d^3a \left(\frac{\delta F}{\delta q^i}\frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i}\frac{\delta F}{\delta p_i} \right)$$

EOM:

$$\dot{q} = \{q, H\} = p/\rho_0 \qquad\qquad \dot{p} = \{p, H\} = \rho_0 \ddot{q} = \ldots.$$

Complicated pde for $q(a, t)$. Exercise. Derive it.

Eulerianization

Momentum:

$$\rho \frac{\partial v}{\partial t} = -\rho v \cdot \nabla v - \nabla p + \frac{1}{c} J \times B$$

Attributes:

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\nabla \cdot (\rho v) \\
\frac{\partial s}{\partial t} &= -v \cdot \nabla s \\
\frac{\partial B}{\partial t} &= -\nabla \times E = \nabla \times (v \times B)
\end{aligned}$$

Thermodynamics:

$$p = \rho^2 \frac{\partial U}{\partial \rho} \qquad\qquad s = \frac{\partial U}{\partial s}$$

Infinite-Dimensional Hamiltonian Structure

Field Variables: $\psi(\mu, t)$ e.g. $\mu = x$, $\mu = (x, v)$, . . .

Poisson Bracket:

$$\{A, B\} = \int \frac{\delta A}{\delta \psi} \mathcal{J}(\psi) \frac{\delta A}{\delta \psi} \, d\mu$$

Lie-Poisson Bracket:

$$\{A, B\} = \left\langle \psi, \left[\frac{\delta A}{\delta \psi}, \frac{\delta A}{\delta \psi} \right] \right\rangle$$

Cosymplectic Operator:

$$\mathcal{J} \, \cdot \; \sim [\psi, \; \cdot \;]$$

Form for <u>Eulerian theories</u>: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

Whence?

Eulerian Reduction

$$F[q,p] = \widehat{F}[v,\rho,s,B]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket in terms of Eulerian variables (pjm & John Greene 1980)

It is an algorithmic process. Manipulations like calculus.

Hamiltonian Closure:

$$H = \int_D d^3x \, \left(\rho|v|^2/2 + \rho U(\rho,s) + |B|^2/2\right)$$

Chain rule to density Eulerian variables, $\{\rho, \sigma, M, B\}$

$$
\begin{aligned}
\{F,G\} \;=\; & -\int_D d^3r \Bigg[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \\
& + \; \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
& + \; B \cdot \left[\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
& + \; + B \cdot \left[\nabla \left(\frac{\delta F}{\delta M} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left(\frac{\delta G}{\delta M} \right) \cdot \frac{\delta F}{\delta B} \right] \Bigg] ,
\end{aligned}
$$

Eulerian Hamiltonian form:

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}\,, \quad \frac{\partial s}{\partial t} = \{s, H\}\,, \quad \frac{\partial v}{\partial t} = \{v, H\}\,, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}\,.$$

Densities:

$$M = \rho v \qquad\qquad \sigma = \rho s$$

HAP Formulations of PP: III Magnetofluids B

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Magnetofluid B Overview

- Complete MHD
- Other magnetofluids. More ab initio construction

Infinite-Dimensional Hamiltonian Structure

Field Variables: $\psi(\mu, t)$ e.g. $\mu = x$, $\mu = (x, v)$, . . .

Poisson Bracket:

$$\{A, B\} = \int \frac{\delta A}{\delta \psi} \mathcal{J}(\psi) \frac{\delta A}{\delta \psi} \, d\mu$$

Lie-Poisson Bracket:

$$\{A, B\} = \left\langle \psi, \left[\frac{\delta A}{\delta \psi}, \frac{\delta A}{\delta \psi} \right] \right\rangle$$

Cosymplectic Operator:

$$\mathcal{J} \, \cdot \, \sim [\psi, \, \cdot \,]$$

Form for <u>Eulerian theories</u>: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

Whence?

Eulerian Reduction

$$F[q,p] = \widehat{F}[v,\rho,s,B]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket in terms of Eulerian variables (pjm & John Greene 1980)

It is an algorithmic process. Manipulations like calculus.

Hamiltonian Closure:

$$H = \int_D d^3x \, \left(\rho|v|^2/2 + \rho U(\rho,s) + |B|^2/2\right)$$

Chain rule to density Eulerian variables, $\{\rho, \sigma = \rho s, M = \rho v, B\}$

$$
\begin{aligned}
\{F, G\} &= -\int_D d^3r \Bigg[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \\
&+ \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
&+ B \cdot \left[\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
&+ +B \cdot \left[\nabla \left(\frac{\delta F}{\delta M} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left(\frac{\delta G}{\delta M} \right) \cdot \frac{\delta F}{\delta B} \right] \Bigg],
\end{aligned}
$$

Eulerian Hamiltonian form

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, H\}, \quad \frac{\partial M}{\partial t} = \{M, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.$$

What is

$$\frac{\delta \rho(x)}{\delta \rho(x')} = \ ?$$

Chain rule to density Eulerian variables, $\{\rho, \sigma, M, B\}$

$$
\begin{aligned}
\{F, G\} \;&=\; -\int_D d^3r \Bigg[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \\
&+\; \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
&+\; B \cdot \left[\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
&+\; +B \cdot \left[\nabla \left(\frac{\delta F}{\delta M} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left(\frac{\delta G}{\delta M} \right) \cdot \frac{\delta F}{\delta B} \right] \Bigg],
\end{aligned}
$$

Eulerian Hamiltonian form

$$
\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, H\}, \quad \frac{\partial M}{\partial t} = \{M, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.
$$

What is

$$
\frac{\delta \rho(x)}{\delta \rho(x')} = \; \delta(x - x') \text{ ?}
$$

Chain rule to density Eulerian variables, $\{\rho, \sigma, M, B\}$

$$
\begin{aligned}
\{F, G\} &= -\int_D d^3r \left[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \right. \\
&+ \rho \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left(\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
&+ B \cdot \left[\frac{\delta F}{\delta M} \cdot \nabla \frac{\delta G}{\delta B} - \frac{\delta G}{\delta M} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
&+ \left. + B \cdot \left[\nabla \left(\frac{\delta F}{\delta M} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left(\frac{\delta G}{\delta M} \right) \cdot \frac{\delta F}{\delta B} \right] \right],
\end{aligned}
$$

Eulerian Hamiltonian form

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial \sigma}{\partial t} = \{\sigma, H\}, \quad \frac{\partial M}{\partial t} = \{M, H\}, \text{ and } \frac{\partial B}{\partial t} = \{B, H\}.$$

What is

$$\frac{\delta \rho(x)}{\delta \rho(x')} = \delta(x - x') \qquad \frac{\partial q^i}{\partial q^j} = \delta^i_j$$

Explicit Eulerian Reduction

Reduce Lagrangian Hamiltonian description to Eulerian Hamiltonian description.

Recall.

Hamiltonian:

$$H = \frac{1}{2}\int_D d^3a\,|p|^2/\rho_0 + \int_D d^3a\,\left(\rho_0\mathcal{U}(\rho_0/\mathcal{J}, s_0) + \frac{1}{2}\frac{|q^i_{,j}B^j_0|^2}{\mathcal{J}^2}\right)$$

Poisson Bracket:

$$\{F,G\} = \int_D d^3a\left(\frac{\delta F}{\delta q^i}\frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i}\frac{\delta F}{\delta p_i}\right)$$

EOM:

$$\dot{q} = \{q,H\} = p/\rho_0 \qquad \dot{p} = \{p,H\} = \rho_0\ddot{q} = -\frac{\delta V}{\delta q}$$

Functional Chain Rule

Suppose functionals F and G are restricted to Eulerian variables

$$F[q,p] = \widehat{F}[\rho, s, v, B]\,.$$

Then, variation gives

$$\begin{aligned}
\delta F \;&=\; \int_D d^3a\, \left(\frac{\delta F}{\delta q}\cdot \delta q + \frac{\delta F}{\delta p}\cdot \delta p\right) = \delta \widehat{F} \\
&=\; \int_D d^3x\, \left(\frac{\delta \widehat{F}}{\delta \rho}\delta\rho + \frac{\delta \widehat{F}}{\delta s}\delta s + \frac{\delta \widehat{F}}{\delta v}\cdot \delta v + \frac{\delta \widehat{F}}{\delta B}\cdot \delta B\right)\,.
\end{aligned}$$

Here, $\{\delta\rho, \delta s, \delta v, \delta B\}$ induced by $(\delta q, \delta p)$. How?

Recall

$$\rho(r,t) = \frac{\rho_0}{\mathcal{J}}\bigg|_{a=a(r,t)} = \int_D d^3a\, \rho_0(a)\, \delta\left(r - q\left(a,t\right)\right)\,.$$

Thus

$$\delta\rho = -\int_D d^3a\, \rho_0 \nabla \delta(r-q)\cdot \delta q\,, \qquad \delta s\,, \delta B\,, \delta v = \ldots$$

Insertion of $\delta\rho$ etc. gives

$$\int_D d^3a \left(\frac{\delta F}{\delta q}\cdot\delta q + \frac{\delta F}{\delta p}\cdot\delta p\right) = -\int_D d^3x\,\frac{\delta \widehat{F}}{\delta\rho}\int_D d^3a\,\rho_0\nabla\delta(r-q)\cdot\delta q + \dots\,.$$

Interchange integration order, remove $\int_D d^3a$ since δq arbitrary gives

$$\frac{\delta F}{\delta q} = \mathcal{O}_\rho\frac{\delta\widehat{F}}{\delta\rho} + \mathcal{O}_s\frac{\delta\widehat{F}}{\delta s} + \mathcal{O}_v\frac{\delta\widehat{F}}{\delta v} + \mathcal{O}_B\frac{\delta\widehat{F}}{\delta B}\,,$$

where the $\mathcal{O}$'s are operators involving integration over d^3x and Dirac delta functions. Upon insertion with similar expression for $\delta F/\delta p$, doing some rearrangement, and dropping the hats, yields $\longrightarrow$

$$
\begin{aligned}
\{F, G\} &= -\int_D d^3x \left[\left(\frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta v} - \frac{\delta G}{\delta \rho} \nabla \cdot \frac{\delta F}{\delta v} \right) \right. \\
&+ \left(\frac{\nabla \times v}{\rho} \cdot \frac{\delta G}{\delta v} \times \frac{\delta F}{\delta v} \right) + \frac{\nabla s}{\rho} \cdot \left(\frac{\delta F}{\delta s} \frac{\delta G}{\delta v} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta v} \right) \\
&+ B \cdot \left[\frac{1}{\rho} \frac{\delta F}{\delta v} \cdot \nabla \frac{\delta G}{\delta B} - \frac{1}{\rho} \frac{\delta G}{\delta v} \cdot \nabla \frac{\delta F}{\delta B} \right] \\
&+ \left. B \cdot \left[\nabla \left(\frac{1}{\rho} \frac{\delta F}{\delta v} \right) \cdot \frac{\delta G}{\delta B} - \nabla \left(\frac{1}{\rho} \frac{\delta G}{\delta v} \right) \cdot \frac{\delta F}{\delta B} \right] \right].
\end{aligned}
$$

Then $M = \rho v$ and $\sigma = \rho s$ gives Lie-Poisson form.

Other Magnetofluids

Braginskii MHD

$$\rho\left(v_t + v\cdot\nabla v\right) = -\nabla p + J\times B + \nabla\cdot\Pi$$

Gyroviscosity Tensor: $\Pi_{ij} = \frac{p}{B} N_{jsik} \frac{\partial v_s}{\partial x_k}$

Action:

$$S[q] = \int dt\,(K + G - V)\,,$$

Gyroscopic Term:

$$G[q] = \int_D d^3a\,\Pi^*\cdot\dot{q} = \int_D d^3x\,M^*\cdot v$$

where

$$\Pi^* = \nabla\times L^* = \frac{m}{2e}\mathcal{J}\,\hat{b}\times\nabla\left(\frac{p}{B}\right)$$

$\delta S[q] = 0 \quad \Rightarrow \quad$ Braginskii MHD

pjm, Lingam, Acevedo, Wurm 2014

Inertial MHD (Tassi)

Basic Idea: Can 'freeze-in' anything one likes! (2-form attribute)

Choose:

$$\mathbf{B}_e = \mathbf{B} + d_e^2 \nabla \times \mathbf{J},$$

Action:

$$S = \int dt \int d^3x \left(\rho \frac{v^2}{2} - \rho U(\rho, s) - \mathbf{B}_e \cdot \mathbf{B} \right).$$

Attributes:

$$\rho d^3x = \rho_0 d^3a, \qquad B_e^i = \frac{B_{e0}^j}{\mathcal{J}} \frac{\partial q^i}{\partial a_j}$$

$\delta S[q] = 0 \quad \Rightarrow \quad$ IMHD

Eulerian Reduction

$$F[q,p] = \widehat{F}[\omega,\psi]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket in terms of Eulerian variables (ω,ψ)

It is an algorithmic process.

Example: 2D IMHD

$$\{F,G\} = -\int d^3x \Big(\omega[F_\omega, G_\omega] + \psi_e([F_\omega, G_{\psi_e}] - [G_\omega, F_{\psi_e}])\Big)$$

$$H = \int d^2x (d_e^2 (\nabla^2\psi)^2 + |\nabla\psi|^2 + |\nabla\varphi|^2)$$

Produces 2D incompressible IMHD (Ottaviani-Porcelli model)!

Above, $F_\omega := \delta F/\delta\omega$, $[f,g] := f_x g_y - g_y g_x]$, $\omega = \hat{z}\cdot\nabla\times v$, $B = \hat{z}\times\nabla\psi$.

Two-Fluid Action

Keramidas Charidakos, Lingam, pjm, R. White and A. Wurm

$$
\begin{aligned}
S[q_s, A, \phi] = \int dt \ \int d^3x \, \Bigg[\left| - \frac{1}{c} \frac{\partial A(x,t)}{\partial t} - \nabla \phi(x,t) \right|^2 & \\
- \left| \nabla \times A(x,t) \right|^2 \Bigg] \frac{1}{8\pi} & \qquad (1) \\
+ \sum_s \int d^3a \, n_{s0}(a) \int d^3x \, \delta\left(x - q_s(a,t)\right) & \\
\times \left[\frac{e_s}{c} \, \dot{q}_s \cdot A(x,t) - e_s \phi(x,t) \right] & \qquad (2) \\
+ \sum_s \int d^3a \, n_{s0}(a) \Bigg[\frac{m_s}{2} \, |\dot{q}_s|^2 & \\
- m_s U_s \left(m_s n_{s0}(a) / \mathcal{J}_s, s_{s0} \right) \Bigg] . & \qquad (3)
\end{aligned}
$$

Eulerian Observables:

$$\{n_\pm, v_\pm, A, \phi\}$$

Reduced Variables

New Lagrangian Variables:

$$Q(a,t) = \frac{1}{\rho_{m0}(a)}\left(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)\right)$$

$$D(a,t) = e\left(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)\right)$$

$$\rho_{m0}(a) = m_i n_{i0}(a) + m_e n_{e0}(a)$$

$$\rho_{q0}(a) = e\left(n_{i0}(a) - n_{e0}(a)\right) \; .$$

Consistent Expansion:

$$\frac{v_A}{c} << 1\,, \qquad \frac{m_e}{m_i} << 1 \qquad \Rightarrow \qquad \text{quasineutrality}$$

Eulerian Closure:

$$\{n, s, s_e, v, J\}$$

Extended MHD

Ohm's Law:

$$E + \frac{v \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv)\right) - \frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en}\,.$$

Momentum:

$$nm\left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v\right) = -\nabla p + \frac{J \times B}{c} - \frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)\,.$$

Extended MHD

Ohm's Law:

$$E + \frac{v \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv)\right) - \frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en}\,.$$

Momentum:

$$\begin{aligned} nm\left(\frac{\partial v}{\partial t} + (V \cdot \nabla)v\right) &= -\nabla p + \frac{J \times B}{c} \\ &\quad -\frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)\,. \end{aligned}$$

Consistent with an ordering of Lüst (1958)

Extended MHD

Ohm's Law:

$$E + \frac{v \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv)\right) - \frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en} \, .$$

Momentum:

$$nm\left(\frac{\partial v}{\partial t} + (v \cdot \nabla) v\right) = -\nabla p + \frac{J \times B}{c} - \frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right) \, .$$

Consistent with an ordering of Lüst (1958)

Extended MHD

Ohm's Law:

$$E + \frac{v \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv)\right)$$
$$-\frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en}\,.$$

Momentum:

$$nm\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) \;=\; -\nabla p + \frac{J \times B}{c}$$
$$-\frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)\,.$$

Extended MHD

Ohm's Law:

$$E + \frac{v \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + \nabla \cdot (vJ + Jv)\right) - \frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en}.$$

Momentum:

$$nm\left(\frac{\partial v}{\partial t} + (v \cdot \nabla)v\right) = -\nabla p + \frac{J \times B}{c} - \frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right).$$

Consistent with an ordering of Lüst (1958)

Noether $\rightarrow$ Energy Conservation

Energy:

$$H = \int d^3x \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|v|^2}{2} + \frac{m_e}{ne^2}\frac{|J|^2}{2} \right]$$

Energy conservation requires

$$\frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)$$

in momentum equation. Otherwise inconsistent.

Physical dissipation is real. Fake dissipation is troublesome, particularly for reconnection studies. Kimura and pjm (2014).

HAP Formulations of PP: V Kinetic Theory – Canonization & Diagonalization, Continuous Spectra, Krein-like Theorems

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Overview

- Solve stable linearized Vlasov-Poisson as a Hamiltonian system.

- Normal Form:
$$H = \sum_i^N \frac{\omega_i}{2}\left(p_i^2 + q_i^2\right) = \sum_i^N \omega_i\, J_i \rightarrow \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \omega_k(u) \left(P_k^2(u) + Q_k^2(u)\right)$$
When stable $\exists$ a canonical transformation to this form. NEMs and Krein-Moser.

- Continuous Spectrum: Transform $G[f]$ (generalization of Hilbert transform) that diagonalizes Vlasov.

- General Diagonalization: General transform for a large class of Hamiltonian systems.

- Continuous spectra and Krein bifurcations.

Vlasov-Poisson

Phase space density $f(x, v, t)$ (1 + 1 + 1 field theory):

$$f : X \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{\geq 0}$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi[x, t; f]}{\partial x}\frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\phi_{xx} = -4\pi \left[e \int_{\mathbb{R}} f(x, v, t)\, dv - \rho_B \right]$$

Energy:

$$H = \frac{m}{2} \int_{\mathbb{T}} \int_{\mathbb{R}} v^2 f\, dx dv + \frac{1}{8\pi} \int_{\mathbb{T}} (\phi_x)^2\, dx$$

Boundary Conditions:

$$\text{periodic} \qquad \Longleftrightarrow \qquad X = \mathbb{T} := [0, 2\pi)$$

Linear Vlasov-Poisson

Linearization:

$$f = f_0(v) + \delta f(x, v, t)$$

Linearized EOM:

$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial x} - \frac{e}{m} \frac{\partial \delta \phi[x, t; \delta f]}{\partial x} \frac{\partial f_0}{\partial v} = 0$$

$$\delta \phi_{xx} = -4\pi e \int_{\mathbb{R}} \delta f(x, v, t)\, dv$$

Linearized Energy (Kruskal and Oberman, 1958):

$$H_L = -\frac{m}{2} \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{v\,(\delta f)^2}{f_0'}\, dv dx + \frac{1}{8\pi} \int_{\mathbb{T}} (\delta \phi_x)^2\, dx$$

Solution of Linear VP by Transform

Assume

$$\delta f = \sum_k f_k(v,t) e^{ikx}\,, \qquad \delta\phi = \sum_k \phi_k(t) e^{ikx}$$

Linearized EOM:

$$\begin{aligned} \frac{\partial f_k}{\partial t} \;&+\; ikvf_k - ik\phi_k \frac{e}{m}\frac{\partial f_0}{\partial v} = 0 \\ k^2\phi_k \;&=\; 4\pi e \int_{\mathbb{R}} f_k(v,t)\, dv \end{aligned} \qquad \text{(LVP)}$$

Three methods:

- Laplace Transforms (Landau and others 1946)
- Normal Modes (Van Kampen, Case,... 1955)
- Coordinate Change $\Longleftrightarrow$ Integral Transform (pjm, Pfirsch, Shadwick, Balmforth, Hagstrom, 1992 $\longrightarrow$ 2013)

Summary

The Transform,

$$G[g](v) := \epsilon_R(v)\, g(v) + \epsilon_I(v)\, H[g](v)\,,$$

where H is the Hilbert transform and $\epsilon_{R,I}$ are functions that depend on f_0, has an inverse $\widehat{G}$ that maps (LVP) into

$$\frac{\partial g_k}{\partial t} + iku\, g_k = 0$$

whence

$$f_k(v,t) = G\left[\widehat{G}[\mathring{f}_k]e^{-ikut}\right]\,,$$

where

$$\mathring{f}_k(v) := f_k(v, t=0)$$

.

Good Equilibria f_0 and Initial Conditions $\overset{\circ}{f}_k$

Definition (VP1). *A function $f_0(v)$ is a good equilibrium if $f_0'(v)$ satisfies*

(i) $f_0' \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, $1 < q < \infty$ and $0 < \alpha < 1$,

(ii) $\exists\, v* > 0$ *st* $|f_0'(v)| < A|v|^{-\mu}\ \forall\, |v| > v*$, *where* $A, \mu > 0$, *and*

(iii) $f_0'/v < 0 \ \ \forall\, v \in \mathbb{R}$ *or* f_0 *is Penrose stable. Assume* $f_0'(0) = 0$.

Definition (VP2). *A function, $\overset{\circ}{f}_k(v)$, is a good initial condition if it satisfies*

(i) $\overset{\circ}{f}_k(v), v\overset{\circ}{f}_k(v) \in L^p(\mathbb{R})$,

(ii) $\int_{\mathbb{R}} \overset{\circ}{f}_k(v)\, dv < \infty$.

Hilbert Transform

Definition

$$H[g](x) := \frac{1}{\pi} \fint_{\mathbb{R}} \frac{g(t)}{t-x}\, dt\,,$$

$\fint_{\mathbb{R}}$ denotes Cauchy principal value.

$\exists$ theorems about Hilbert transforms in L^p and $C^{0,\alpha}$. Plemelj, M. Riesz, Zygmund, and Titchmarsh $\cdots$ (Can be extracted from Calderón-Zygmund theory.) Recent tome by King.

Hilbert Transform Theorms

Theorem (H1).

(ii) *$H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator:*

$$\|H[g]\|_p \leq A_p \, \|g\|_p \,,$$

where A_p depends only on p,

(ii) *H has an inverse on $L^p(\mathbb{R})$, given by*

$$H[H[g]] = -g \,,$$

(iii) *$H\colon L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}) \to L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$.*

Theorem (H2). *If* $g_1 \in L^p(\mathbb{R})$ *and* $g_2 \in L^q(\mathbb{R})$ *with* $\frac{1}{p} + \frac{1}{q} < 1$, *then*

$$H[g_1 H[g_2] + g_2 H[g_1]] = H[g_1] H[g_2] - g_1 g_2 \,.$$

Proof : Based on the Hardy-Poincaré-Bertrand theorem, Tricomi.

Lemma (H3). *If* $vg \in L^p(\mathbb{R})$, *then*

$$H[vg](u) = u\, H[g](u) + \frac{1}{\pi} \int_{\mathbb{R}} g \, dv \,.$$

Proof : $\frac{v}{v-u} = \frac{u+v-u}{v-u} = \frac{u}{v-u} + 1$ □

The Transform

Definition (G1). *The transform is defined by*

$$\begin{aligned} f(v) &= G[g](v) \\ &:= \epsilon_R(v)\, g(v) + \epsilon_I(v)\, H[g](v)\,, \end{aligned}$$

where

$$\epsilon_I(v) = -\pi\omega_p^2 f_0'(v)/k^2\,, \qquad \epsilon_R(v) = 1 + H[\epsilon_I](v)\,.$$

Remarks

1. We suppress the dependence of ϵ on k throughout. Note, $\omega_p^2 := 4\pi n_0 e^2/m$ is the plasma frequency corresponding to an equilibrium of number density n_0.

2. $\epsilon = \epsilon_R + i\epsilon_I$ (complex extended, appropriately) is the plasma dispersion relation whose vanishing $\Rightarrow$ discrete normal eigenmodes. When $\epsilon \neq 0$ $\exists$ only continuous spectrum; there is no dispersion relation.

Transform Theorems

Theorem (G2). *$G\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator:*

$$\|G[g]\|_p \leq B_p \, \|g\|_p \,,$$

where B_p depends only on p.

Theorem (G3). *If f_0 is a good equilibrium, then $G[g]$ has an inverse,*

$$\widehat{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \widehat{G}[f](u) \\ &:= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} \, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} \, H[f](u) \,, \end{aligned}$$

where $|\epsilon|^2 := \epsilon_R^2 + \epsilon_I^2$.

Proof : First we show $g \in L^p(\mathbb{R})$, then $g = \widehat{G}[G[g]]$.

If $\epsilon_R(u)/|\epsilon(u)|^2$ and $\epsilon_I(u)/|\epsilon(u)|^2$ are bounded, then clearly $g \in L^p(\mathbb{R})$. For good equilibria the numerators are bounded and everything is Hölder, so it is only necessary to show that $|\epsilon|$ is bounded away from zero. Either of the conditions of (VP1)(iii) assures this. Consider the first (monotonicity) condition,

$|f_0'| > 0$ for $v \neq 0$ and $f_0'(0) = 0$. We need only look at $v = 0$ and $v = \infty$. At $v = 0$

$$\epsilon_R(0) = 1 - \frac{\omega_P^2}{k^2}\int_{\mathbb{R}} \frac{f_0'}{v} dv > 1 > 0\,,$$

while as $v \to \infty$, $\epsilon_R \to 1$.

That $\widehat{G}$ is the inverse follows directly upon inserting $G[g]$ of (G1) into $g = \widehat{G}[G[g]]$, and using (H2) and $\epsilon_R(v) = 1 + H[\epsilon_I]$. $\square$

That $\widehat{G}$ is the inverse follows directly upon inserting $G[g]$ of (G1) into $g = \widehat{G}[G[g]]$, and using (H2) and $\epsilon_R(v) = 1 + H[\epsilon_I]$.

$$\begin{aligned}
g(u) &= \widehat{G}[f](u) = \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u) \\
&= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, [\epsilon_R(u)\, g(u) + \epsilon_I(u)\, H[g](u)] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H\left[\epsilon_R(u')\, g(u') + \epsilon_I(u')\, H[g](u')\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H\left[H[\epsilon_I]\, g + \epsilon_I\, H[g]\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} \left[H[\epsilon_I](u) H[g](u) - g(u)\, \epsilon_I(u)\right] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[\epsilon_I] H[g] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u)\, [1 + H[\epsilon_I](u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \epsilon_R(u) = g(u)
\end{aligned}$$

□

Lemma (G4). *If ϵ_I and ϵ_R are as above, then*

(i) *for $vf \in L^p(\mathbb{R})$,*

$$\widehat{G}[vf](u) = u\,\widehat{G}[f](u) - \frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{\mathbb{R}} f\,dv\,,$$

(ii) $\widehat{G}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$

(iii) *and if $f(u,t)$ and $g(v,t)$ are strongly differentiable in t; i.e. the mapping $t \mapsto f(t) = f(t,\cdot) \in L^p(\mathbb{R})$ is differentiable, (the usual difference quotient converges in the L^p sense), then*

a) $\widehat{G}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial \widehat{G}[f]}{\partial t} = \frac{\partial g}{\partial t}$,

b) $G\left[\frac{\partial g}{\partial t}\right] = \frac{\partial G[g]}{\partial t} = \frac{\partial f}{\partial t}$.

Proof : *(i) goes through like (H3), (ii) follows from $\epsilon_R = 1 + H[\epsilon_I]$, and (iii) follows because G is bounded and linear.* $\square$

Solution

Solve like Fourier transforms: operate on EOM with $\widehat{G} \Rightarrow$,

$$\frac{\partial g_k}{\partial t} + iku\, g_k = 0$$

and so

$$g_k(u,t) = \mathring{g}_k(u) e^{-ikut}$$

Using $\mathring{g}_k = \widehat{G}[\mathring{f}_k]$ we obtain the solution

$$\begin{aligned} f_k(v,t) &= G[g_k(u,t)] \\ &= G\left[\mathring{g}_k(u) e^{-ikut}\right] = G\left[\widehat{G}[\mathring{f}_k] e^{-ikut}\right] \end{aligned}$$

Theorem (S1). *For good initital conditions and equilibria,*

$$f_k(v,t) = G\left[\widehat{G}[\mathring{f}_k] e^{-ikut}\right]$$

is an $L^p(\mathbb{R})$ solution of (LVP).

VP Hamiltonian Structure

Energy is quadratic $\Rightarrow$ SHO? However, V-P equation is quadratically nonlinear. Canonically conjugate variables?

Noncanonical Poisson Bracket (pjm 1980):

$$\{F,G\} = \int f\left[\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right] dxdv$$

F and G are functionals. VP $\Longleftrightarrow$

$$\frac{\partial f}{\partial t} = \{f,H\} = [f,\mathcal{E}].$$

where $\mathcal{E} = mv^2/2 + e\phi$ and

$$[f,\mathcal{E}] = \frac{1}{m}\left(\frac{\partial f}{\partial x}\frac{\partial \mathcal{E}}{\partial v} - \frac{\partial \mathcal{E}}{\partial x}\frac{\partial f}{\partial v}\right)$$

Organizes: VP, Euler, QG, Defect Dyn, Benny-Dirac, ...

Linear Hamiltonian Structure

Linearization:

$$f = f_0(v) + \delta f$$

where $f_0(v)$ assumed stable (NEMs ok) $\Longrightarrow$

$$\{F, G\}_L = \int f_0 \left[\frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f} \right] dx\, dv \ ,$$

which with the Kruskal and Oberman energy,

$$H_L = -\frac{m}{2} \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{v\, (\delta f)^2}{f_0'}\, dv dx + \frac{1}{8\pi} \int_{\mathbb{T}} (\delta \phi_x)^2\, dx \,,$$

LVP $\Longleftrightarrow$

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L \ .$$

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{\mathbb{R}} f_0' \left(\frac{\delta F}{\delta f_k} \frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k} \frac{\delta F}{\delta f_{-k}} \right) dv \,.$$

Linear Hamiltonian:

$$\begin{aligned} H_L &= -\frac{m}{2} \sum_k \int_{\mathbb{R}} \frac{v}{f_0'} \, |f_k|^2 \, dv + \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2 \\ &= \sum_{k,k'} \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(v) \, \mathcal{O}_{k,k'}(v|v') \, f_{k'}(v') \, dv dv' \end{aligned}$$

Canonize:

$$q_k(v,t) = \frac{m}{ikf_0'} f_k(v,t) \,, \qquad p_k(v,t) = f_{-k}(v,t)$$

$\Longrightarrow$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta q_k} \frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k} \frac{\delta F}{\delta p_k} \right) dv \,.$$

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, \mathcal{G}[P'_k](v)\, dv$$

Canonical Coordinate Change $(q, p) \longleftrightarrow (Q', P')$:

$$p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = \mathcal{G}[P_k](v)\,, \qquad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = \mathcal{G}^\dagger[q_k](u)$$

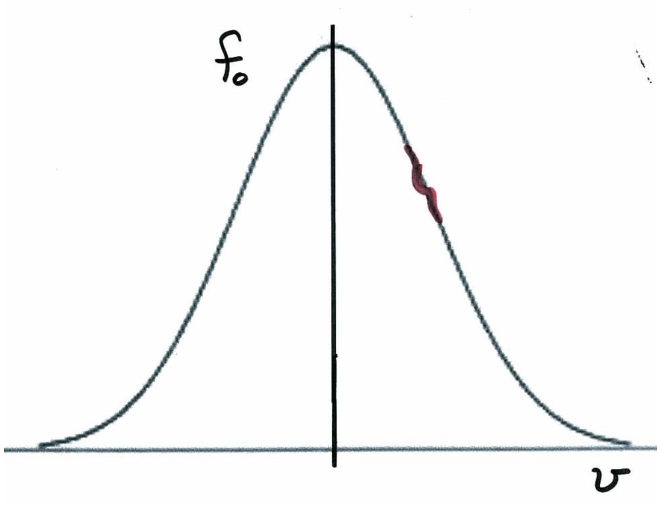
New Hamiltonian:

$$H_L = \frac{1}{2} \sum_{k=1}^{\infty} \int_{\mathbb{R}} du\, \sigma_k(u) \omega_k(u) \left[Q_k^2(u) + P_k^2(u) \right]$$

where $\omega_k(u) = |ku|$ and the signature is

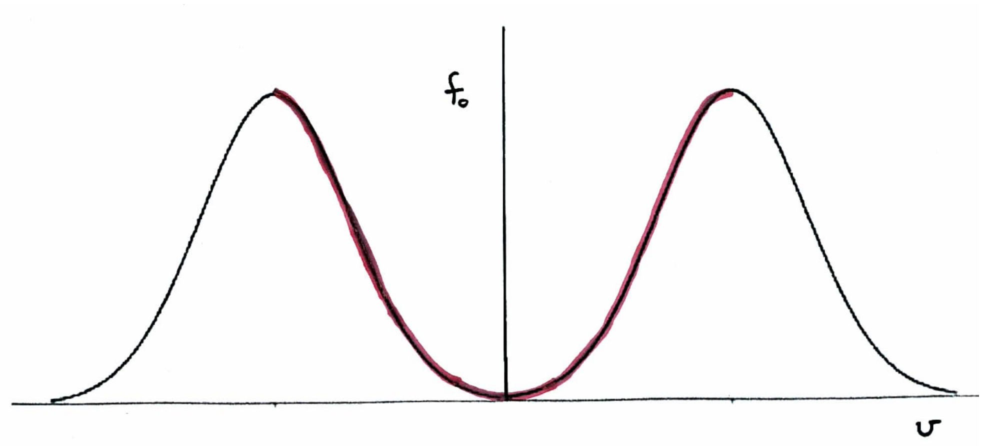
$$\boxed{\sigma_k(v) := -\,\mathrm{sgn}(v f'_0(v))}$$

Sample Homogeneous Equilibria



← Maxwellian

BiMaxwellian →



Hamiltonian Spectrum

Hamiltonian Operator:

$$\partial_t f_k = -ikv f_k + \frac{if_0'}{k} \int_{\mathbb{R}} d\bar{v}\, f_k(\bar{v}, t) =: \mathcal{H}_k f_k\,,$$

Complete System:

$$\partial_t f_k = \mathcal{H}_k f_k \qquad \text{and} \qquad \partial_t f_{-k} = \mathcal{H}_{-k} f_{-k}\,, \qquad\qquad k \in \mathbb{R}^+$$

Lemma *If λ is an eigenvalue of the Vlasov equation linearized about the equilibrium $f_0'(v)$, then so are $-\lambda$ and λ^* . Thus if $\lambda = \gamma + i\omega$, then eigenvalues occur in the pairs, $\pm\gamma$ and $\pm i\omega$, for purely real and imaginary cases, respectively, or quartets, $\lambda = \pm\gamma \pm i\omega$, for complex eigenvalues.*

Spectral Stability

Definition The dynamics of a Hamiltonian system linearized around some equilibrium solution, with the phase space of solutions in some Banach space $\mathcal{B}$, is spectrally stable if the spectrum $\sigma(\mathcal{H})$ of the time evolution operator $\mathcal{H}$ is purely imaginary.

Theorem *If for some $k \in \mathbb{R}^+$ and $u = \omega/k$ in the upper half plane the plasma dispersion relation,*

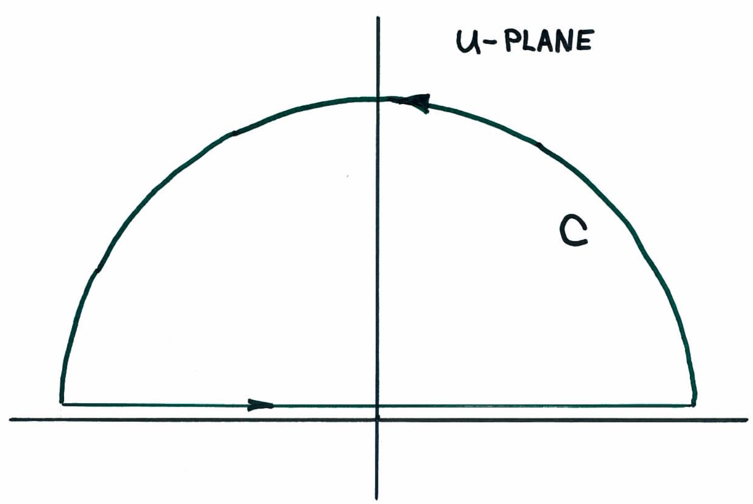
$$\varepsilon(k,u) := 1 - k^{-2}\int_{\mathbb{R}} dv \frac{f_0'}{u-v} = 0\,,$$

then the system with equilibrium f_0 is spectrally unstable. Otherwise it is spectrally stable.

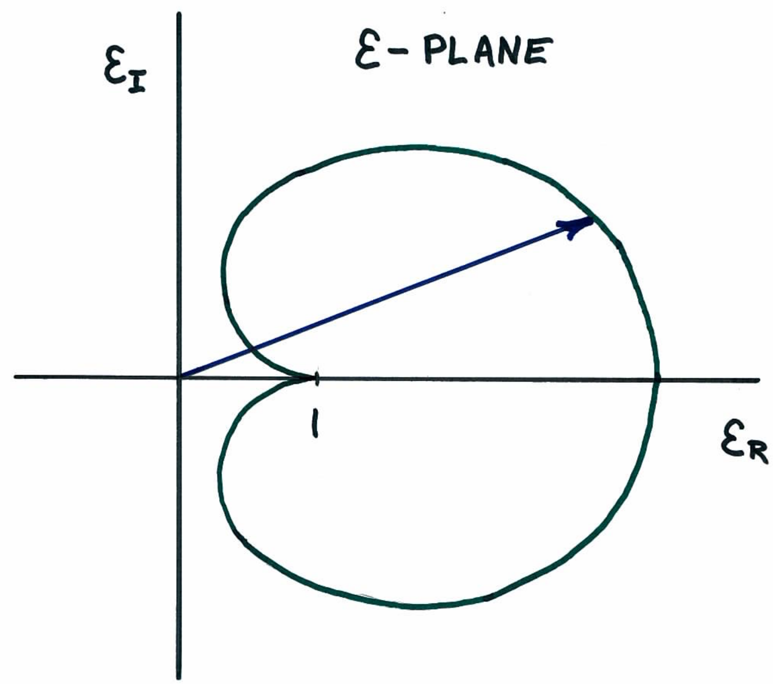
Nyquist Method

$$f_0' \in C^{0,\alpha}(\mathbb{R}) \Rightarrow \varepsilon \in C^{\omega}(uhp).$$

Therefore, Argument Principle $\Rightarrow$ winding # = # zeros of ε



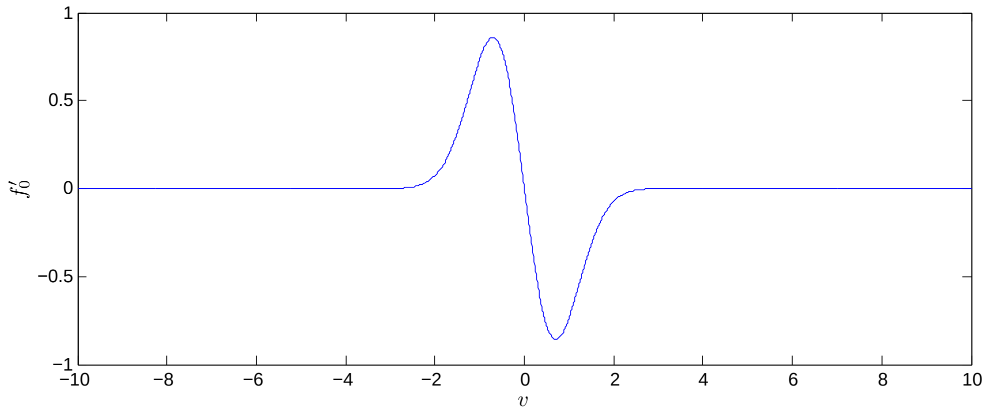
Stable $\rightarrow$

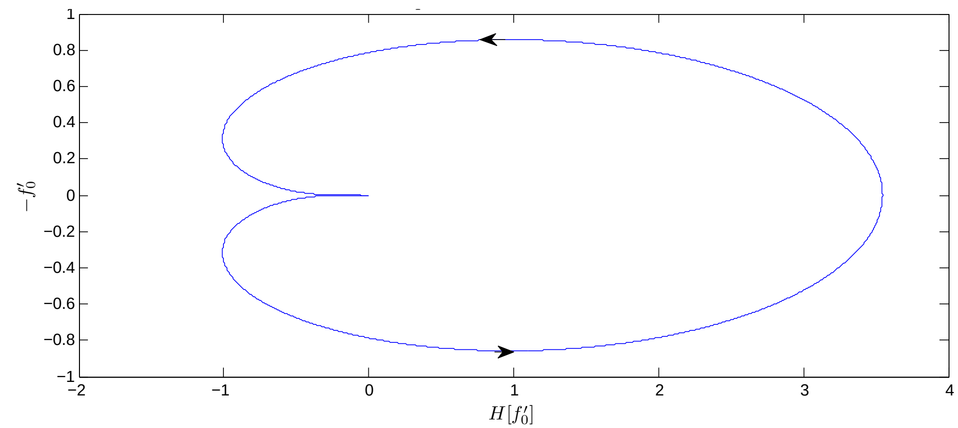


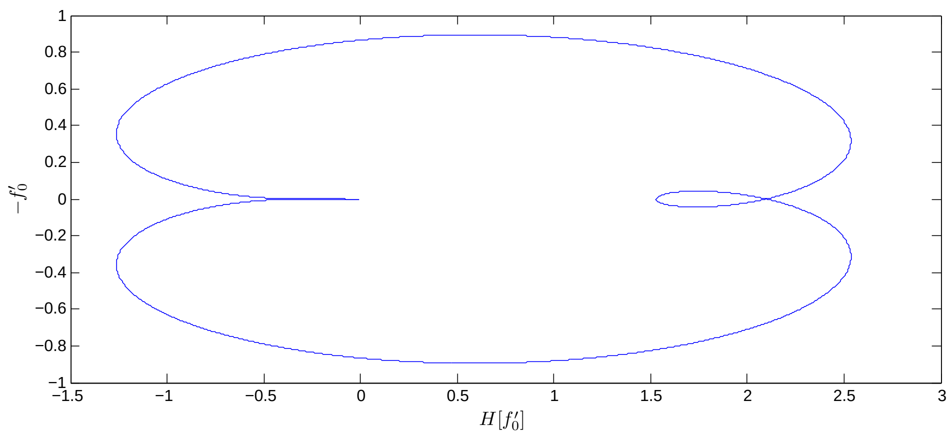
Nyquist Method Examples

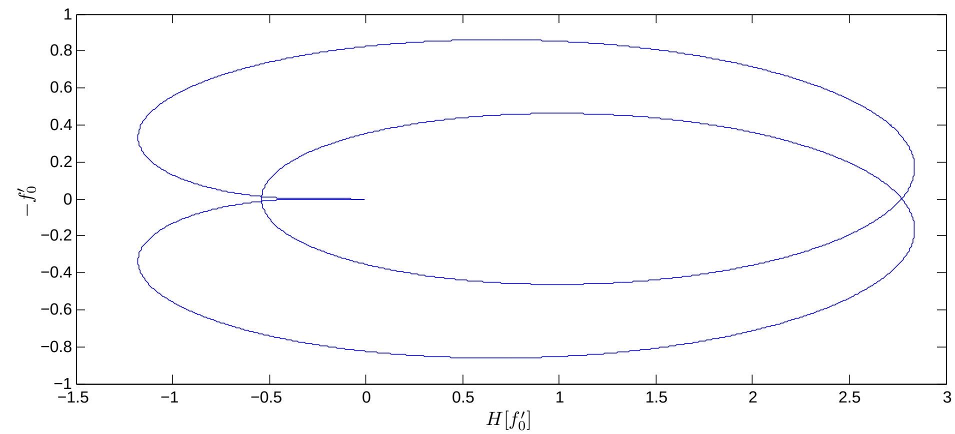
Winding number of $u \in \mathbb{R} \mapsto \varepsilon$, or

$$\lim_{u \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} dv \, \frac{f_0'}{v-u} = H[f_0'](u) - i f_0'(u) \,,$$









Spectral Theorem

Set $k = 1$ and consider $\mathcal{H}\colon f \mapsto ivf - if_0' \int f$ in the space $W^{1,1}(\mathbb{R})$.

$W^{1,1}(\mathbb{R})$ is Sobolev space containing closure of functions $\|f\|_{1,1} = \|f\|_1 + \|f'\|_1 = \int_{\mathbb{R}} dv(|f| + |f'|)$. Contains all functions in $L^1(\mathbb{R})$ with weak derivatives in $L^1(\mathbb{R})$. $\mathcal{H}$ is densely defined, closed, etc.

Definition Resolvent of $\mathcal{H}$ is $R(\mathcal{H}, \lambda) = (\mathcal{H} - \lambda I)^{-1}$ and $\lambda \in \sigma(\mathcal{H})$. (i) λ in point spectrum, $\sigma_p(\mathcal{H})$, if $R(\mathcal{H}, \lambda)$ not injective. (ii) λ in residual spectrum, $\sigma_r(\mathcal{H})$, if $R(\mathcal{H}, \lambda)$ exists but not densely defined. (iii) λ in continuous spectrum, $\sigma_c(\mathcal{H})$, if $R(\mathcal{H}, \lambda)$ exists, densely defined but not bounded.

Theorem *Let* $\lambda = iu$. *(i)* $\sigma_p(\mathcal{H})$ *consists of all points* $iu \in \mathbb{C}$, *where* $\varepsilon = 1 - k^{-2} \int_{\mathbb{R}} dv\, f_0'/(u - v) = 0$. *(ii)* $\sigma_c(\mathcal{H})$ *consists of all* $\lambda = iu$ *with* $u \in \mathbb{R} \setminus (-i\sigma_p(\mathcal{H}) \cap \mathbb{R})$. *(iii)* $\sigma_r(\mathcal{H})$ *contains all the points* $\lambda = iu$ *in the complement of* $\sigma_p(\mathcal{H})$ *that satisfy* $f_0'(u) = 0$.

cf. e.g. P. Degond (1986). Similar but different.

Structural Stability

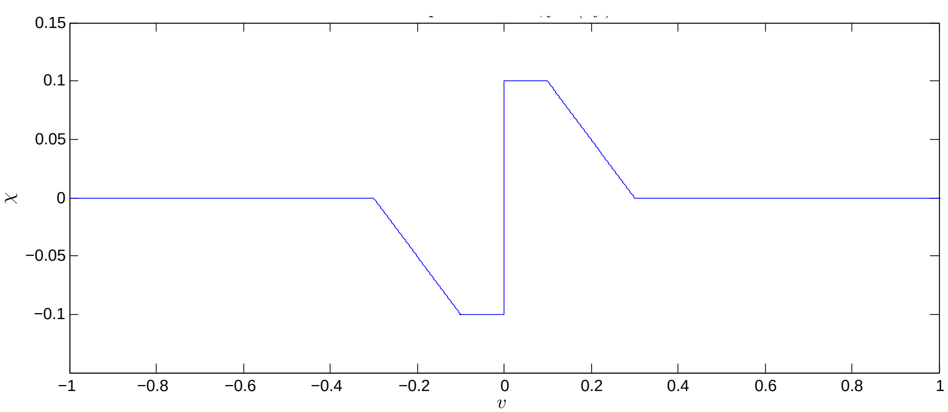
Definition Consider an equilibrium solution of a Hamiltonian system and the corresponding time evolution operator $\mathcal{H}$ for the linearized dynamics. Let the phase space for the linearized dynamics be some Banach space $\mathcal{B}$. Suppose that $\mathcal{H}$ is spectrally stable. Consider perturbations $\delta\mathcal{H}$ of $\mathcal{H}$ and define a norm on the space of such perturbations. Then we say that the equilibrium is <u>structurally stable</u> under this norm if there is some $\delta > 0$ such that for every $\|\delta\mathcal{H}\| < \delta$ the operator $\mathcal{H} + \delta\mathcal{H}$ is spectrally stable. Otherwise the system is <u>structurally unstable</u>.

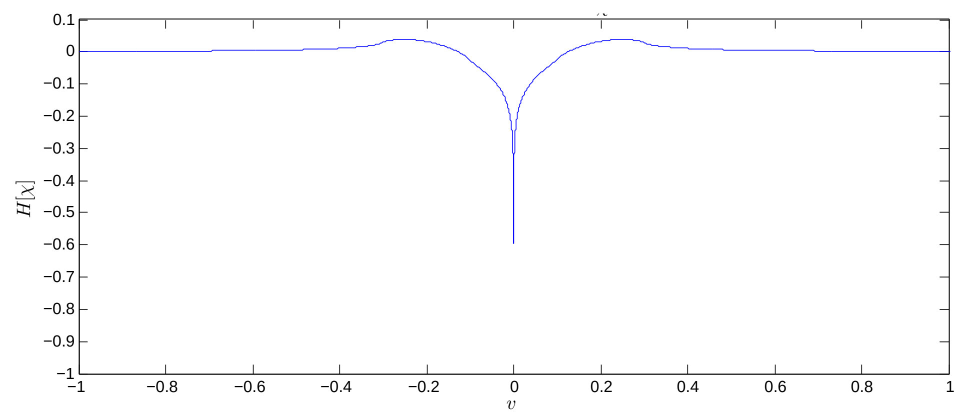
Definition Consider the formulation of the linearized Vlasov-Poisson equation in the Banach space $W^{1,1}(\mathbb{R})$ with a spectrally stable homogeneous equilibrium function f_0. Let $\mathcal{H}_{f_0+\delta f_0}$ be the time evolution operator corresponding to the linearized dynamics around the distribution function $f_0 + \delta f_0$. If there exists some ϵ depending only on f_0 such that $\mathcal{H}_{f_0+\delta f_0}$ is spectrally stable whenever $\|\mathcal{H}_{f_0} - \mathcal{H}_{f_0+\delta f_0}\| < \epsilon$, then the equilibrium f_0 is structurally stable under perturbations of f_0.

All f_0 are Structurally Unstable in $W^{1,1}$

True in space where Hilbert transform unbounded, e.g. $W^{1,1}$. Small perturbation $\Rightarrow$ big jump in Penrose plot.

Theorem *A stable equilibrium distribution is structurally unstable under perturbations of f_0' in the Banach spaces $W^{1,1}$ and $L^1 \cap C_0$.*





Easy to make ‘bumps’ in f_0 that are small in norm. What to do?

Krein-Like Theorem for VP

Theorem *Let f_0 be a stable equilibrium distribution function for the Vlasov equation. Then f_0 is structurally stable under dynamically accessible perturbations in $W^{1,1}$, if there is only one solution of $f_0'(v) = 0$. If there are multiple solutions, f_0 is structurally unstable and the unstable modes come from the roots of f_0' that satisfy $f_0''(v) < 0$.*

Remark A change in the signature of the continuous spectrum is a necessary and sufficient condition for structural instability. The bifurcations do not occur at all points where the signature changes, however. Only those that represent valleys of the distribution can give birth to unstable modes.

Summary – Conclusions

- Described the Vlasov-Poisson system.
- Described G transform and its properties.
- Canonized, diagonalized, and defined signature for σ_c.
- Variety of Krein-like theorems, e.g. valley theorem.

HAP Formulations of PP: VI Metriplecticism: relaxation paradigms for computation and derivation

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Ravello, September 27, 2014

Goal: Describe formal structures for dissipation and their use as a guide for deriving models and for calculating stationary

Metriiplictic References

Numbers refer to items on my web page: `http://www.ph.utexas.edu/∼morrison/` where all can be obtained under 'Publications'.

161. Anthony M. Bloch, Philip J. Morrison, and Tudor S. Ratiu, Gradient Flows in the Normal and Kaehler Metrics and Triple Bracket Generated Metriplectic Systems, in Recent Trends in Dynamical Systems, eds. A. Johann et al., Springer Proceedings in Mathematics & Statistics 35, DOI 10.1007/978-3-0348-0451-615, (2013) pp. 365–408. A wide spectrum of symplectic and dissipative dynamics is considered, including dissipative formalisms, metriplectic dynamics, and hybrid dynamics. Underlying geometry is investigated and general triple bracket constructions are given with many examples.

144. G. R. Flierl and P. J. Morrison, Hamiltonian-Dirac Simulated Annealing: Application to the Calculation of Vortex States, Physica D 240, 212232 (2011). The double bracket formalism is generalized by introducing a general from for infinite-dimensional systems, introducing a metric, and incorporating Dirac constraint theory. The formalism is used to numerically obtain a variety of vortex states.

134. P. J. Morrison, Thoughts on Brackets and Dissipation: Old and New, Journal of Physics: Conference Series 169, 012006 (12pp) (2009). Brackets are revisited and ideas about open and closed systems are discussed. Attempts are made to algebraic couple the symplectic and gradient parts of the flow.

30. P. J. Morrison, A Paradigm for Joined Hamiltonian and Dissipative Systems, Physica D 18, 410419 (1986). Metriplectic dynamics is further developed with finite and infinite dimensional examples given. The name metriplectic is introduced.

13. P. J. Morrison, Bracket Formulation for Irreversible Classical Fields, Physics Letters A 100, 423427 (1984). The first reference where the full axioms of metriplectic dynamics are given. Here the idea that the sum of symplectic and symmetric brackets can effect the equilibrium variational principle is introduced. Triple brackets are introduced for the construction of the dynamics.

Overview

1. Dissipative Structures
   (a) Rayleigh, Cahn-Hilliard
   (b) Hamilton Preliminaries
   (c) Hamiltonian Based Dissipative Structures
      i. Metriplectic Dynamics
      ii. Double Bracket Dynamics
2. Computations
   (a) XXXX Contour Dynamics
   (b) 2D Euler Vortex States

Rayleigh Dissipation Function

Introduced for study of vibrations, stable linear oscillations, in 1873 (see e.g. Rayleigh, Theory of Sound, Chap. IV §81)

Linear friction law for n-bodies, $\mathbf{F}_i = -b_i(\mathbf{r}_i)\mathbf{v}_i$, with $\mathbf{r}_i \in \mathbb{R}^3$. Rayleigh was interested in linear vibrations, $\mathcal{F} = \sum_i b_i \, ||\mathbf{v}_i||^2/2$.

Coordinates $\mathbf{r}_i \to q_\nu$ etc. $\Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\nu}\right) - \left(\frac{\partial \mathcal{L}}{\partial q_\nu}\right) + \left(\frac{\partial \mathcal{F}}{\partial \dot{q}_\nu}\right) = 0$$

Ad hoc, phenomenological, yet is generalizable, geometrizable (e.g. Bloch et al.,...)

Cahn-Hilliard Equation

Models phase separation, nonlinear diffusive dissipation, in binary fluid with 'concentrations' n, $n = 1$ one kind $n = -1$ the other

$$\frac{\partial n}{\partial t} = \nabla^2 \frac{\delta F}{\delta n} = \nabla^2 \left(n^3 - n - \nabla^2 n \right)$$

Lyapunov Functional

$$F[n] = \int d^3x \left[\frac{1}{4} \left(n^2 - 1 \right)^2 + \frac{1}{2} |\nabla n|^2 \right]$$

$$\frac{dF}{dt} = \int d^3x \frac{\delta F}{\delta n} \frac{\partial n}{\partial t} = \int d^3x \frac{\delta F}{\delta n} \nabla^2 \frac{\delta F}{\delta n} = -\int d^3x \left| \nabla \frac{\delta F}{\delta n} \right|^2 \leq 0$$

For example in 1D

$$\lim_{t \to \infty} n(x,t) = \tanh(x/\sqrt{2})$$

Ad hoc, phenomenological, yet generalizable and very important (Otto, Ricci Flows, Poincarè conjecture on S^3, ...)

Hamiltonian Preliminaries

Finite $\longrightarrow$ Infinite degrees of freedom

Canonical Hamiltonian Dynamics

Hamilton's Equations:

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,$$

Phase Space Coordinates: $\quad z = (q, p)$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j}\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

Symplectic Manifold $\mathcal{Z}_s$:

$$\dot{z} = Z_H = [z, H]$$

with Hamiltonian vector field generated by Poisson bracket

$$[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}$$

symplectic 2-form = (cosymplectic form)$^{-1}$: $\quad \omega^c_{ij} J_c^{jk} = \delta_i^k,$

Noncanonical Hamiltonian Dynamics

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad\qquad [A,B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\quad [A,B] = -[B,A]\,,$

Jacobi identity $\longrightarrow$ $\quad [A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus <u>Casimirs</u>

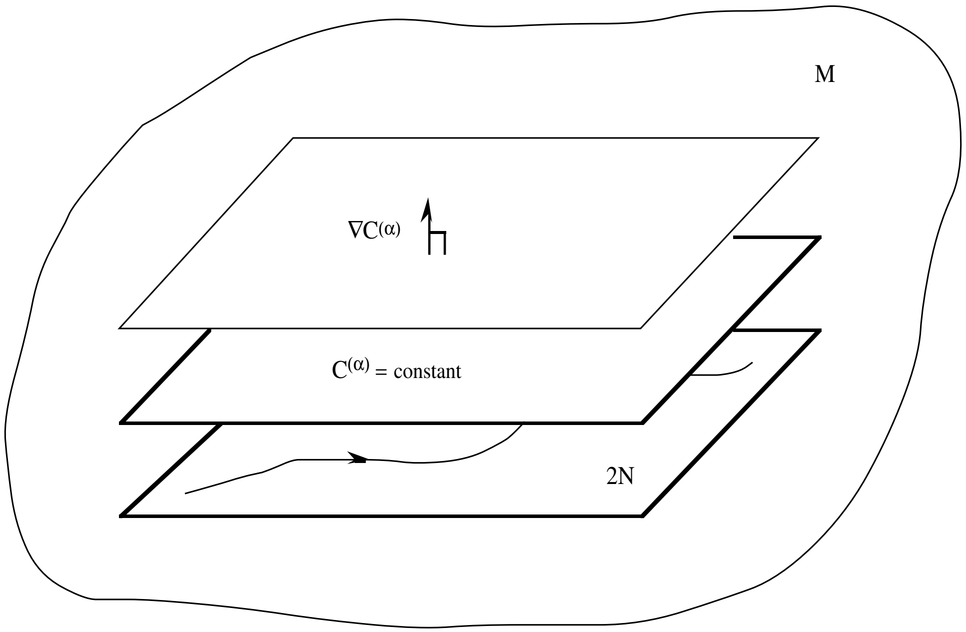
Eulerian Media: $\quad J^{ij} = c_k^{ij} z^k \qquad \longleftarrow$ Lie – Poisson Brackets

Poisson Manifold $\mathcal{Z}_p$

Degeneracy $\Rightarrow$ Casimir Invariants:

$$[C, g] = 0 \quad \forall g : \mathcal{Z}_p \to \mathbb{R}$$

Foliation by Casimir Invariants:



Leaf Hamiltonian vector fields:

$$Z_f^p = [z, f]$$

Example 2D Euler

Noncanonical Poisson Brackets:

$$\{F, G\} = \int dxdy\, \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right] = -\int dxdy\, \frac{\delta F}{\delta \zeta}\, [\zeta, \cdot]\, \frac{\delta G}{\delta \zeta}$$

ζ = vorticity, $\psi = \triangle^{-1}\zeta$ =streamfunction

$$[f, g] = J(f, g) = f_x g_y - f_y g_x = \frac{\partial(f, g)}{\partial(x, y)}$$

Hamiltonian:

$$H[\zeta] = \frac{1}{2}\int d\mathbf{x}\, v^2 = \frac{1}{2}\int d\mathbf{x}\, |\nabla\psi|^2$$

Equation of Motion:

$$\zeta_t = \{\zeta, H\}$$

PJM (1981) and P. Olver (1982)

Dirac Constrained Hamiltonian Dynamics

Ingredients:

Two functions $D_{1,2} : \mathcal{Z} \to \mathbb{R}$ and good Poisson bracket

Generalized Dirac:

$$[f,g]_D = \frac{1}{[D_1,D_2]}\Big([D_1,D_2][f,g] - [f,D_1][g,D_2] + [g,D_1][f,D_2]\Big)$$

Degeneracy $\Rightarrow$ D's are Casimir Invariants:

$$[D_{1,2},g]_D = 0 \quad \forall\ g\colon \mathcal{Z}_p \to \mathbb{R}$$

Foliation again and Dirac Hamiltonian vector fields:

$$Z^d_f = [z,f]_D$$

Hamiltonian Based Dissipation

Metriplectic Dynamics

A <u>dynamical</u> model of thermodynamics that ‘captures’:

- First Law: conservation of energy

- Second Law: entropy production

pjm (1984,1986,...), Ottinger GENERIC (1997)

Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of $[,]$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator: $\mathcal{F} = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = [H, \mathcal{F}] + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0$$

  Foliate $\mathcal{Z}$ by level sets of H with $(H, f) = 0\ \forall\ f \in C^\infty(M)$.

- 2nd Law: entropy production

$$\dot{S} = [S, \mathcal{F}] + (S, \mathcal{F}) = (S, S) \geq 0$$

  Lyapunov relaxation to the equilbrium state: $\delta\mathcal{F} = 0$.

Metriplectic Dynamics

Natural hybrid Hamiltonian and dissipative flow on that embodies the first and second laws of thermodynamics;

$$\dot{z} = (z, S) + [z, H]$$

where Hamiltonian, H, is the energy and entropy, S, is a Casimir.

Degeneracies:

$$(H, g) \equiv 0 \quad \text{and} \quad [S, g] \equiv 0 \quad \forall\ g$$

First and Second Laws:

$$\frac{dH}{dt} = 0 \qquad \text{and} \qquad \frac{dS}{dt} \geq 0$$

Seeks equilibria $\equiv$ extermination of Free Energy $F = H + S$:

$$\delta F = 0$$

Examples

- Finite dimensional theories, rigid body, etc.

- Kinetic theories: Boltzmann equation, Lenard-Balescu equation, ...

- Fluid flows: various nonideal fluids, Navier-Stokes, MHD, etc.

5. Relaxing free rigid body

In order to illustrate the formalism outlined in the previous section we treat an example. We begin by considering the motion of a rigid body with fixed center of mass under no torques. The motion of such a free rigid body is governed by Euler's equations

$$\begin{aligned} \dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3), \\ \dot{\omega}_2 &= \omega_3\omega_1(I_3 - I_1), \\ \dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2). \end{aligned} \tag{27}$$

Here we have done some scaling, but the dynamical variables ω_i, $i = 1,2,3$, are related to the three principal axis components of the angular velocity, while the constants I_i, $i = 1,2,3$, are related to the three principal moments of inertia.

This system conserves the following expressions for rotational kinetic energy and squared magnitude of the angular momentum:

$$H = \tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2), \tag{28a}$$

$$l^2 = \omega_1^2 + \omega_2^2 + \omega_3^2. \tag{28b}$$

The quantity H can be used to cast eqs. (27) into Hamiltonian form in terms of a noncanonical Poisson Bracket [4] that involves the three dynamical variables, ω_i. The matrix (J^{ij}) introduced in section 3 has a null eigenvector that is given by $\partial l^2/\omega_i$; i.e. l^2 is a Casimir. The noncanonical Poisson bracket is

$$[f, g] = \frac{\partial f}{\partial \omega_i}\omega_k\epsilon_{ijk}\frac{\partial g}{\partial \omega_j}, \quad i, j, k = 1,2,3, \tag{29}$$

where ϵ_{ijk} is the Levi-Civita symbol. Evidently eqs. (27) are equivalent to

$$\dot{\omega}_i = [\omega_i, H], \quad i = 1,2,3, \tag{30}$$

and we have for an arbitrary function $S(l^2), [S, f] = 0$ for all f.

So far we have endowed the phase space, which has coordinates ω_i, with a cosymplectic form. Let us now add to this a metric component. In this case a dynamical constraint manifold corresponds to a surface of constant energy, i.e. an ellipsoid. We wish to construct a (g^{ij}) that has $\partial H/\omega_i$ as a null eigenvector, while possessing two nonzero eigenvalues of the same sign. This is conveniently done by defining the bracket $(,)$ in terms of a projection matrix; i.e.

$$(f, h) = -\lambda\left[\left(\frac{\partial H}{\partial \omega_i}\frac{\partial H}{\partial \omega_j} - \delta_{ij}\frac{\partial H}{\partial \omega_l}\frac{\partial H}{\partial \omega_l}\right)\frac{\partial f}{\partial \omega_i}\frac{\partial h}{\partial \omega_j}\right]. \tag{31}$$

For now we take λ to be constant, but it could depend upon ω. Explicitly the (g^{ij}) is given by

$$(g^{ij}) = \lambda\begin{bmatrix} I_2^2\omega_2^2 + I_3^2\omega_3^2 & -I_1I_2\omega_1\omega_2 & -I_1I_3\omega_1\omega_3 \\ -I_1I_2\omega_1\omega_2 & I_1^2\omega_1^2 + I_3^2\omega_3^2 & -I_2I_3\omega_1\omega_3 \\ -I_1I_3\omega_1\omega_3 & -I_2I_3\omega_1\omega_3 & I_1^2\omega_1^2 + I_2^2\omega_2^2 \end{bmatrix}. \tag{32}$$

We are now in a position to display a class of metriplectic flows for the rigid body; i.e.

$$\begin{aligned} \dot{\omega}_i &= \{\omega_i, F\} = [\omega_i, F] + (\omega_i, F) \\ &= J^{ij}\frac{\partial H}{\partial \omega_j} + g^{ij}\frac{\partial S}{\partial \omega_j}, \quad i = 1,2,3, \end{aligned} \tag{33}$$

where $F = H - S$, H is given by eq. (28a) and S is an arbitrary function of l^2. For the case $i = 1$ we have

$$\begin{aligned} \dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3) + 2\lambda S'(l^2)\omega_1 \\ &\quad\times[I_2(I_2 - I_1)\omega_2^2 + I_3(I_3 - I_1)\omega_3^2]. \end{aligned} \tag{34}$$

The other two equations are obtained upon cyclic permutation of the indices. By design this system conserves energy but produces the generalized entropy $S(l^2)$ if $\lambda > 0$, which could be chosen to correspond to angular momentum.

It is well known that equilibria of Euler's equations composed of pure rotation about either of the principal axes corresponding to the largest and smallest principal moments of inertia are stable. If we suppose that $I_1 < I_2 < I_3$, then stability of an equilibrium defined by $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$

Generalized Vlasov-Lenard-Balescu

GVLB equation:

$$\frac{\partial f}{\partial t}(x,v,t) = -v \cdot \nabla f + \nabla\phi(x;f) \cdot \frac{\partial f}{\partial v} + \frac{\partial f}{\partial t}(x,v,t)\Big|_c$$

Energy Entropy:

$$H = \frac{1}{2}\int dxdv\, m|v|^2 + \frac{1}{2}\int dx\,|E|^2 \qquad S = \int\int dxdv\, s(f)$$

Symmetric Bracket:

$$(A,B) = -\int dxdv \int dx'dv' \left[\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f} - \frac{\partial}{\partial v_i'}\frac{\delta A}{\delta f'}\right] T_{ij} \left[\frac{\partial}{\partial v_i}\frac{\delta B}{\delta f} - \frac{\partial}{\partial v_i'}\frac{\delta B}{\delta f'}\right]$$

Entropy Matching:

$$T_{ij} = w_{ij}(x,v,x',v)M(f)M(f')/2 \quad \text{with} \quad M\frac{\partial^2 s}{\partial f^2} = 1$$

Collision Operator

Two counting dichotomies:

- Exclusion vs. Nonexclusion
- Distinguishability vs. Indistinguishability

$\Rightarrow$ 4 possibilities

$$\begin{array}{rcl} \mathrm{IE} & \rightarrow & \mathrm{F-D} \\ \mathrm{IN} & \rightarrow & \mathrm{B-E} \\ \mathrm{DN} & \rightarrow & \mathrm{M-B} \\ \mathrm{DE} & \rightarrow & ? \end{array}$$

Collision Operator

Two counting dichotomies:

- Exclusion vs. Nonexclusion

- Distinguishability vs. Indistinguishability

$\Rightarrow$ 4 possibilities

$$\begin{aligned} \mathrm{IE} &\rightarrow \mathrm{F-D} \\ \mathrm{IN} &\rightarrow \mathrm{B-E} \\ \mathrm{DN} &\rightarrow \mathrm{M-B} \\ \mathrm{DE} &\rightarrow \mathrm{L-B} \end{aligned}$$

Lynden-Bell (1967) proposed this for stars which are distinguishable.

Collision Operator

Kadomstev and Pogutse (1970) collision operator
with formal H-theorm to F-D ?

Metriplectic formalism $\rightarrow$ can do for *any* monotonic distribution

Conservation (mass,momentum,energy) and Lyapunov:

$$w_{ij}(z,z') = w_{ji}(z,z') \quad w_{ij}(z,z') = w_{ij}(z',z) \quad g_i w_{ij} = 0\,,$$

where $z = (x,v)$ and $g_i = v_i - v_i'$.

'Entropy' Compatibility:

$$S[f] = \int dz\, s(f) \quad \Rightarrow \quad M\frac{d^2 s}{df^2} = 1$$

Collision Operator Examples

Landau kernel:

$$w_{ij}^{(L)} = (\delta_{ij} - g_i g_j / g^2)\delta(\mathbf{x} - \mathbf{x}')/g$$

Landau Entropy Compatibility

$$S[f] = \int dz\, f \ln f \qquad \Rightarrow \qquad M \frac{d^2 s}{df^2} = 1 \Rightarrow M = f$$

Lynden-Bell Entropy Compatibility

$$S[f] = \int dz\, s(f) \qquad \Rightarrow \qquad M \frac{d^2 s}{df^2} = 1 \Rightarrow M = f(1 - f)$$

Good Dissipative Models are Metriplectic!

Double Brackets

Double Brackets and Simulated Annealing

Good Idea:

Vallis, Carnevale, and Young; Shepherd, (1989)

'Simulated Annealing' Bracket:

$$((f,g)) = [f, z^\ell][z^\ell, g] = \frac{\partial f}{\partial z^i} J^{i\ell} J^{\ell j} \frac{\partial g}{\partial z^j}\,,$$

Use bracket dynamics to do extremization $\Rightarrow$ Relaxing Rearrangement

$$\frac{d\mathcal{F}}{dt} = ((\mathcal{F}, H)) = ((\mathcal{F}, \mathcal{F})) \geq 0$$

Lyapunov function, $\mathcal{F}$, yields asymptotic stability to rearranged equilibrium.

- Maximizing energy at fixed Casimir: Works fine sometimes, but limited to circular vortex states ....

Generalized Simulated Annealing

'Simulated Annealing' Bracket:

$$((f,g))_D = [f, z^m]_D \, g_{mn} \, [z^n, g]_D = \frac{\partial f}{\partial z^i} J_D^{in} \, g_{mn} \, J_D^{nj} \frac{\partial g}{\partial z^j} \,,$$

Relaxation Property: $\frac{dH}{dt} = ((H,H))_D \geq 0$ at constant Casimirs

General Geometric Construction:

Suppose manifold $\mathcal{Z}$ has both Riemannian and Symplectic structure: Given two vector fields $Z_{1,2}$ the following is defined:

$$\mathbf{g}(Z_1, Z_2)$$

If the two vector fields are Hamiltonian, e.g., Z_f, then we have the bracket

$$((f,g)) = \mathbf{g}(Z_f, Z_g)$$

which produces a 'relaxing' flow. Such flows exist for Kähler manifolds.

Contour Dynamics Calculations

Calculation of V-States in Contour Dynamics

Goal:

CD/Waterbag Hamiltonian Reduction:

vorticity, $\omega(x,y,t) \longrightarrow \mathbf{X}(\sigma)$, vortex patch boundary

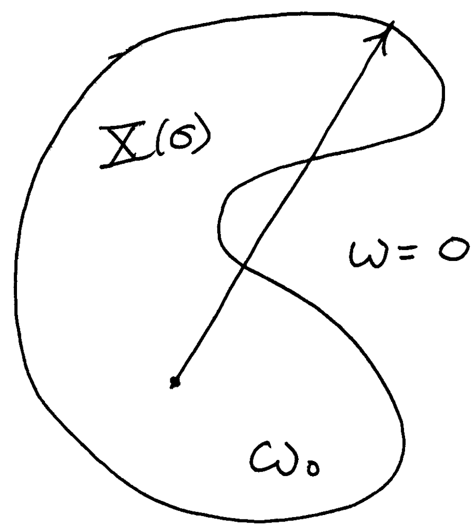
Calculation:

V-States by simulated annealing

Contour Dynamics/Waterbags

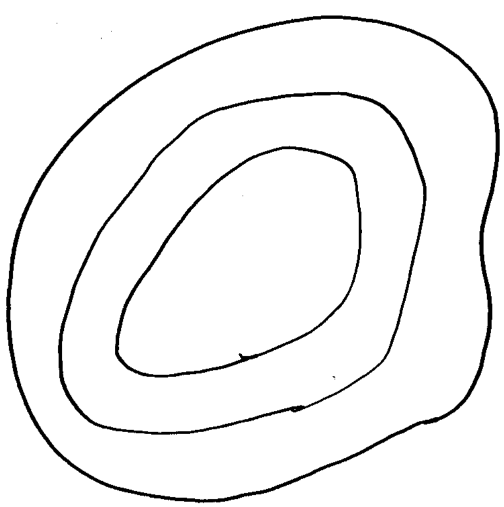
Plane Curve:

$$\mathbf{X}(\sigma) = (X(\sigma), Y(\sigma))$$



parameter σ arbitrary

(arc length not conserved)

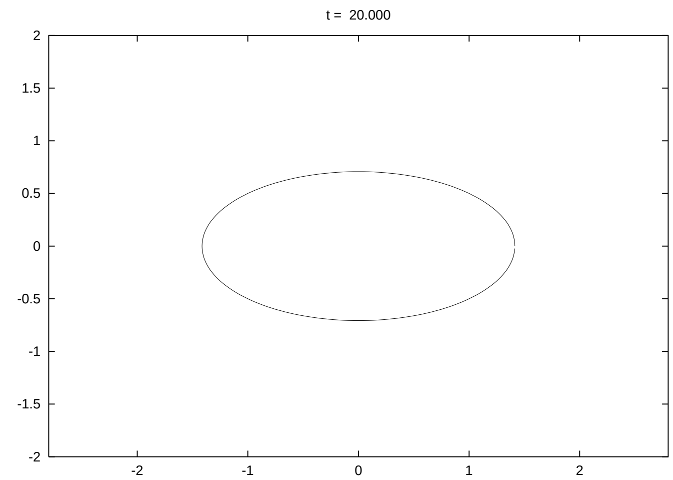


$\longleftarrow$ The Onion

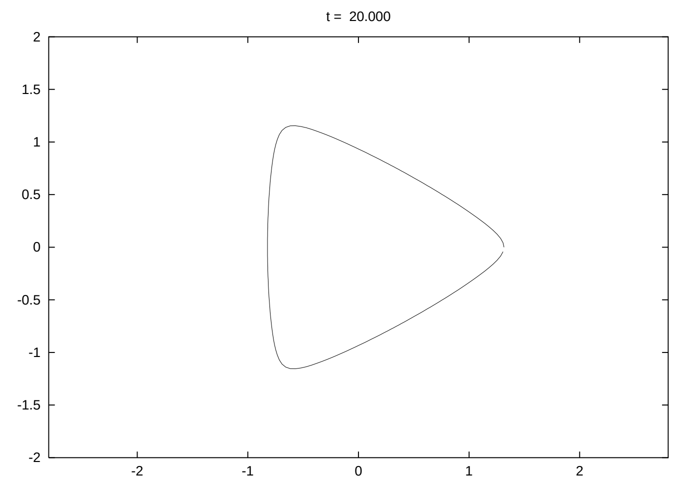
V-States

$\longrightarrow$ Equilibria in rotation frame; $\delta(H + \Omega L) = 0$

Kirchoff Ellipse:



3-fold:



Hamiltonian Form

Observables are Parameterization Invariance Functionals:

$$F[X,Y] = \oint d\sigma\, \mathcal{F}(X,Y,X_\sigma,Y_\sigma,Y_{\sigma\sigma},X_{\sigma\sigma},\ldots)$$

Invariance (equivalence relation): $X_\sigma := \partial/\partial\sigma$, etc.

$$\oint d\sigma \mathcal{F}(X,Y,X_\sigma,Y_\sigma,Y_{\sigma\sigma},X_{\sigma\sigma},\ldots)$$
$$= \oint d\tau\, \mathcal{F}(X,Y,X_\tau,Y_\tau,Y_{\tau\tau},X_{\tau\tau},\ldots)$$

$\sigma = \phi(\tau)$, $d\phi(\tau)/d\tau \neq 0$

Lie Algebra Realization:

$\mathbb{V}$ over $\mathbb{R}$ is set of parameterization invariant functionals with Poisson Bracket $\{\,,\,\}$

Bianchi identitiy:

$$\frac{\delta F}{\delta X(\sigma)}\, X_\sigma + \frac{\delta F}{\delta Y(\sigma)}\, Y_\sigma \equiv 0\,,$$

Noether & isoperimetric problems

Hamiltonian Form (cont)

Poisson Bracket:

$$\{F, G\} = \oint d\sigma \left[\frac{Y_\sigma \frac{\delta F}{\delta X} - X_\sigma \frac{\delta F}{\delta Y}}{X_\sigma^2 + Y_\sigma^2}\right] \frac{\partial}{\partial \sigma} \left[\frac{Y_\sigma \frac{\delta G}{\delta X} - X_\sigma \frac{\delta G}{\delta Y}}{X_\sigma^2 + Y_\sigma^2}\right]$$

Area/Casimir:

$$\Gamma = \frac{1}{2} \oint (X Y_\sigma - Y X_\sigma)\, d\sigma\,, \qquad \{\Gamma, F\} = 0 \;\forall\; F$$

Area Preservation:

$$\{\Gamma, F\} = \oint \frac{\partial}{\partial \sigma} \left[\frac{Y_\sigma \frac{\delta F}{\delta X} - X_\sigma \frac{\delta F}{\delta Y}}{X_\sigma^2 + Y_\sigma^2}\right] d\sigma = 0$$

Dynamics of closed curves with fixed areas for any H.

Contour Dynamics Clips – DSA

Built-in Invariants:

- Angular momentum:

$$L = \int_D (x^2 + y^2)\, d^2x$$

- Strain moment (2-fold symmetry):

$$K = \int_D xy\, d^2x$$

{1-Kellipse, 2-two.stationary, 3-two}

2D Euler Calculations

Four Types of Dynamics

$$
\begin{aligned}
\text{Hamiltonian}: \frac{\partial F}{\partial t} &= \{F, \mathcal{F}\} \quad (1) \\
\text{Hamiltonian Dirac}: \frac{\partial F}{\partial t} &= \{F, \mathcal{F}\}_D \quad (2) \\
\text{Simulated Annealing}: \frac{\partial F}{\partial t} &= \sigma\{F, \mathcal{F}\} + \alpha((F, \mathcal{F})) \quad (3) \\
\text{Dirac Simulated Annealing}: \frac{\partial F}{\partial t} &= \sigma\{F, \mathcal{F}\}_D + \alpha((F, \mathcal{F}))_D \quad (4)
\end{aligned}
$$

F an arbitrary observable, $\mathcal{F}$ generates time advancement. Equations (1) and (2) are ideal and conserve energy. In (3) and (4) parameters σ and α weight ideal and dissipative dynamics: $\sigma \in \{0, 1\}$ and $\alpha \in \{-1, 1\}$. $\mathcal{F}$, can have form

$$\mathcal{F} = H + \sum_i C_i + \lambda^i P_i \,,$$

Cs Casimirs and Ps dynamical invariants.

DSA is Dressed Advection

$$\frac{\partial \zeta}{\partial t} = -[\Psi, \zeta]\,,$$

$$\Psi = \psi + A^i c_i \quad \text{and} \quad A^i = -\frac{\int d\mathbf{x}\, c_j [\psi, \zeta]}{\int d\mathbf{x}\, \zeta [c_i, c_j]}\,.$$

with constraints

$$C_j = \int d\mathbf{x}\, c_j\, \zeta\,.$$

"Advection" of ζ by Ψ, with A^i just right to force constraints.

Easy to adapt existing vortex dynamics codes!!

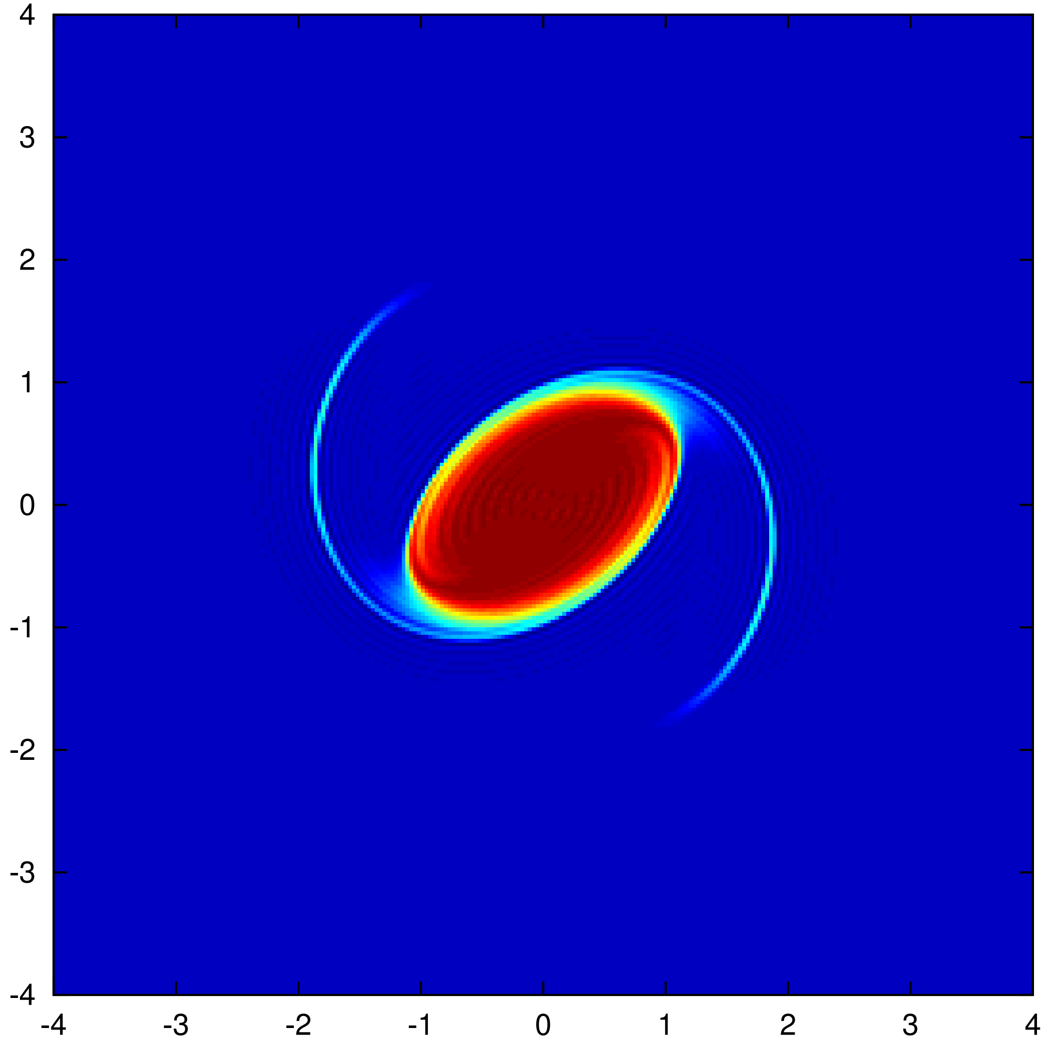
2D Euler Clip, 2-fold Symmetry – H

Initial Condition:

$$q = e^{-(r/r_0)^{10}}, \qquad r_0 = 1 + \epsilon \cos(2\theta), \qquad \epsilon = 0.4$$

{(fig3)els-1-m0}

Filamentation leading to ‘relaxed state’. How much? Which state?



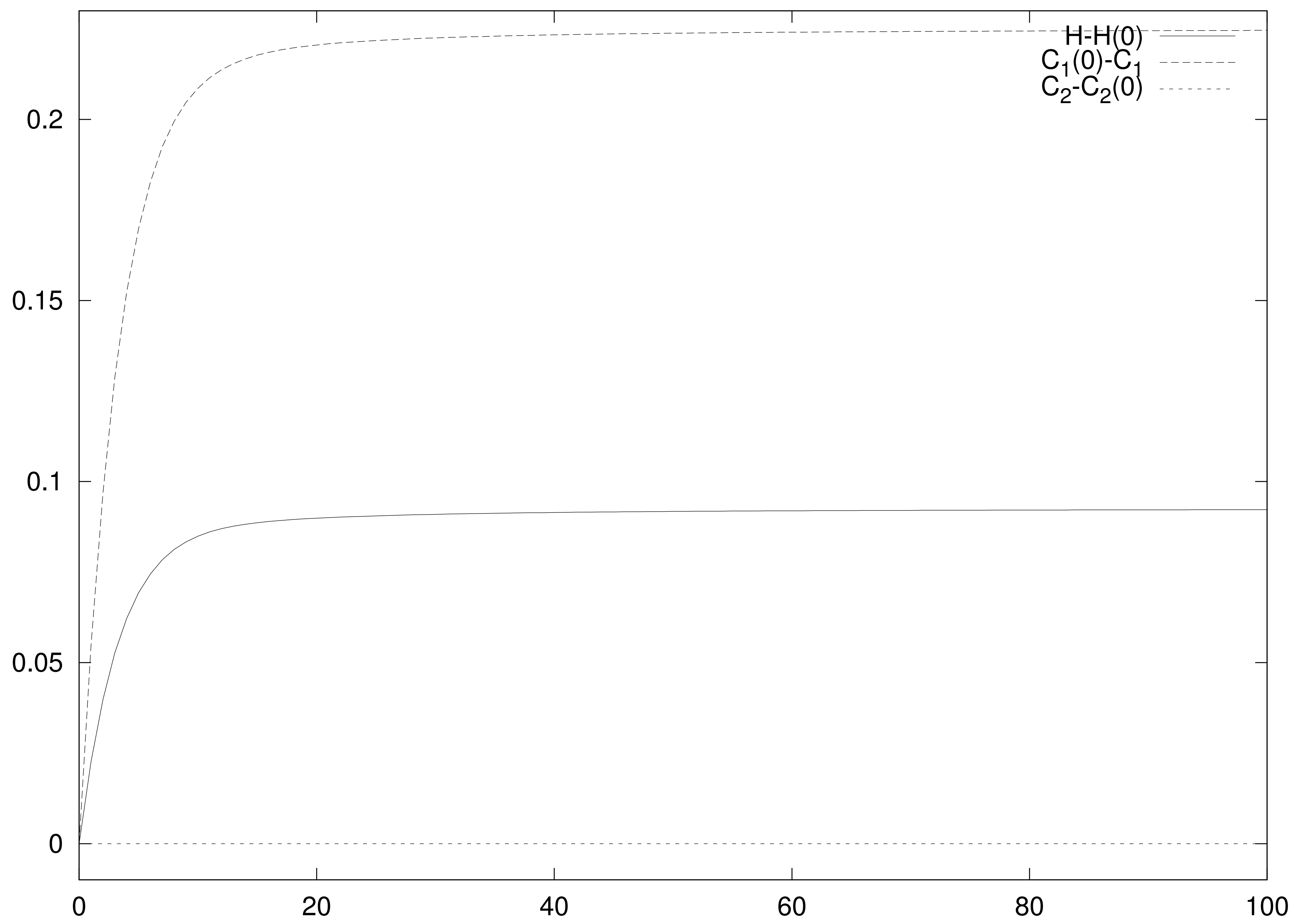
2D Euler Clip, 2-fold Symmetry – SA$_{\sigma=0}$

Initial Condition:

$$q = e^{-(r/r_0)^{10}}\,, \qquad r_0 = 1 + \epsilon \cos(2\theta)\,, \qquad \epsilon = 0.4$$

{(fig6)els-2-m0}

Constants vs. t; Kelvin's H-Maximization



2-fold Symmetry – HD vs. DSA$_{0,1}$

Initial Condition:

$$q = e^{-(r/r_0)^{10}}\,, \qquad r_0 = 1 + \epsilon\cos(2\theta)\,, \qquad \epsilon = 0.4$$

- Angular momentum:

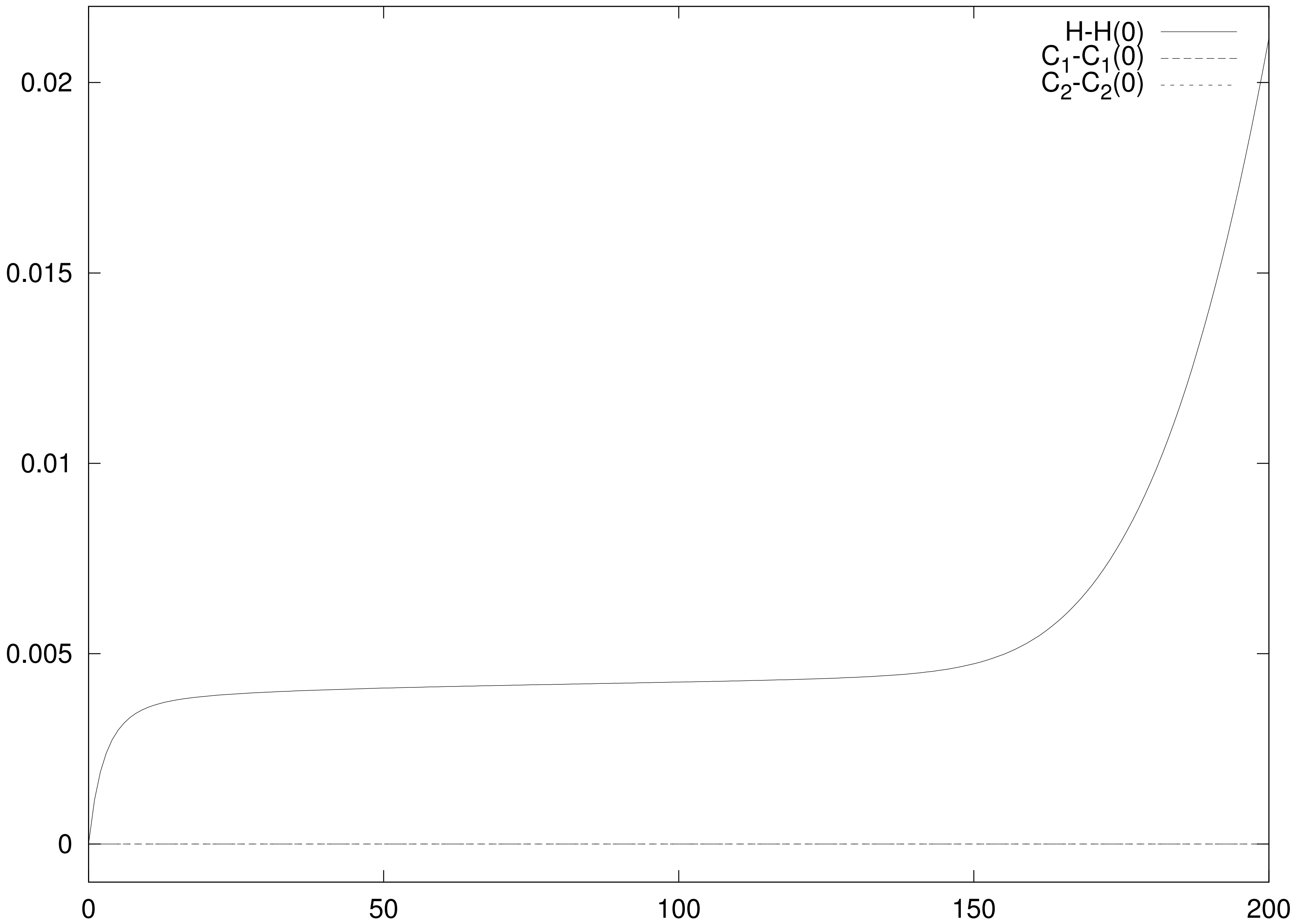
$$L = \int_D (x^2 + y^2)\, d^2x$$

- Strain moment (2-fold symmetry):

$$K = \int_D xy\, d^2x$$

{(fig8)els-3-m0, (fig10)els-4-m0,(fig12)els-4-m1}

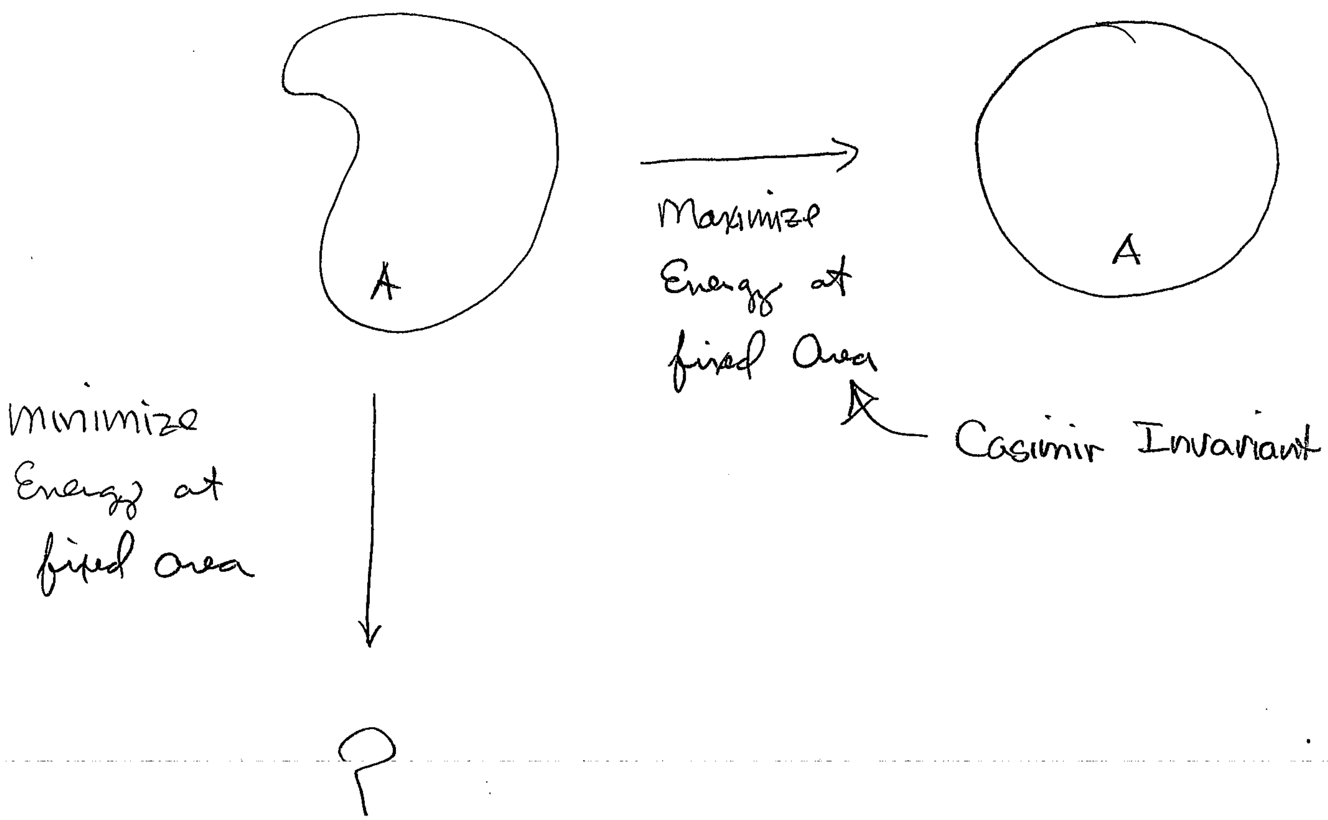
Constants vs. t for DSA_0



Kelvin's Sponge

Kelvin Sponge $\qquad H = \int_{Supp} \frac{v^2}{2}\, d\xi$

Example : Vortex Patch



Uniform positive vorticity inside circle. Net vorticity maintained. But, angular momentum not conserved? With Dirac, angular momentum conserved. Then what?

2-fold Symmetry – Minimizing SA vs. DSA$_0$

Initial Condition:

$$q = e^{-(r/r_0)^{10}}, \qquad r_0 = 1 + \epsilon \cos(2\theta), \qquad \epsilon = 0.4$$

- Angular momentum:

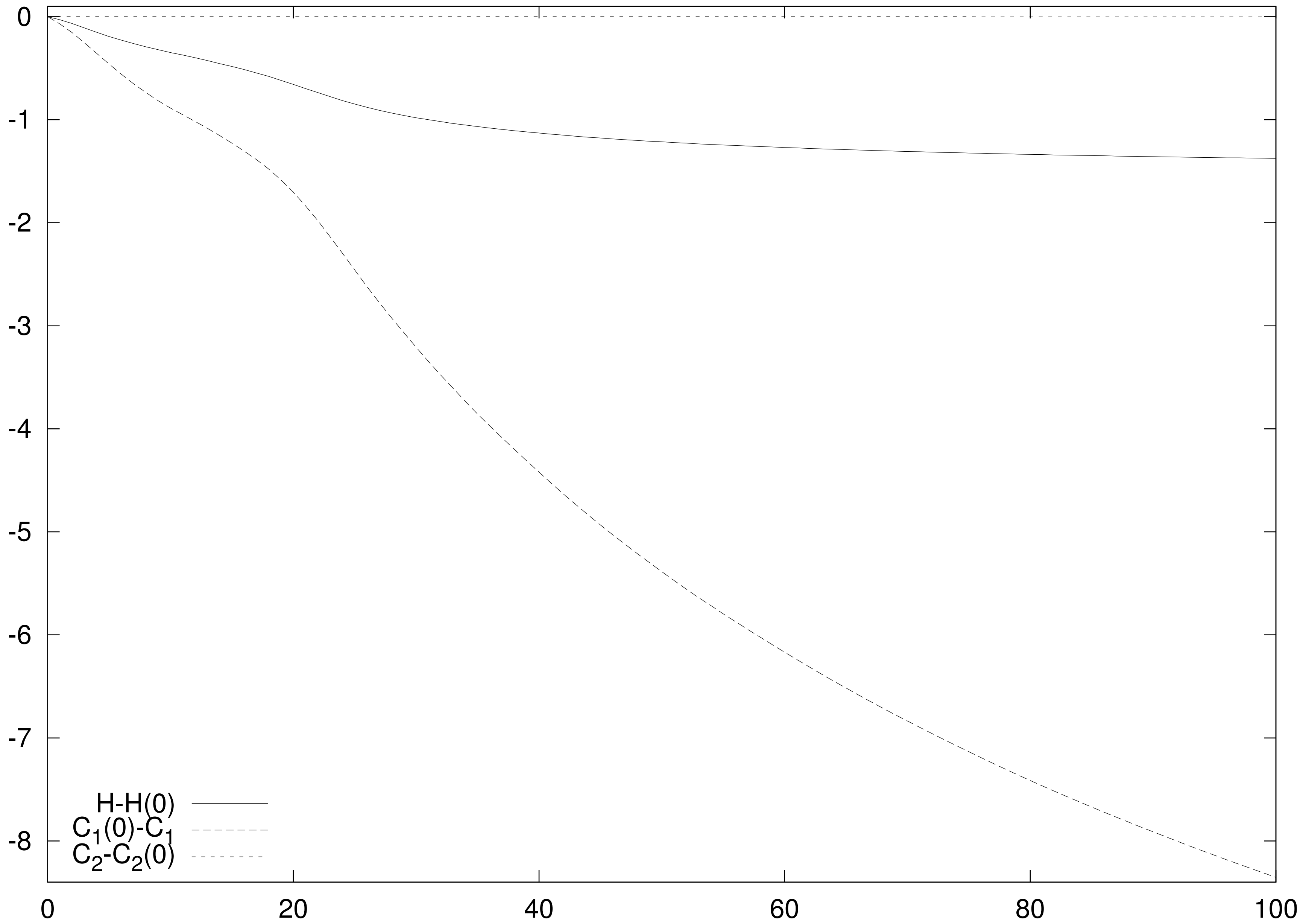
$$L = \int_D (x^2 + y^2)\, d^2x$$

- Strain moment (2-fold symmetry):

$$K = \int_D xy\, d^2x$$

{(fig14)els-2-p0,(fig16)els-4-p0}

Constants vs. t for SA_0



3-fold Symmetry and Dipole DSA

skipping details

{(fig21)tri-db2, (fig27)dip-4-m0}

Underview

1. Dissipative Structures
   (a) Rayleigh, Cahn-Hilliard
   (b) Hamilton Preliminaries
   (c) Hamiltonian Based Dissipative Structures
      i. Metriplectic Dynamics
      ii. Double Bracket Dynamics
2. Computations
   (a) XXXX Contour Dynamics
   (b) 2D Euler Vortex States