# LIFTING – A method for constructing consistent kinetic theories with electromagnetic interaction


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CIRM, Kinetic Equations
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**Definition**: Given a set of ordinary differential equations for 'orbits' of some kind of charged entities in given 'electromagnetic' fields $\mathbf{E}, \mathbf{B}$, lifting is a prescription for building a consistent Hamiltonian kinetic theory.

pjm, "A General Theory for Gauge-Free Lifting," Phys. Plasmas **20**, 012104 (2013)

New:

pjm, M. Vittot, and L. de Guillebon, "Lifting Particle Coordinate Changes of Magnetic Moment Type to Vlasov-Maxwell Hamiltonian Dynamics," Phys. Plasmas **20**, 032109 (2013).

J. Burby, A. Brizard, pjm, and H. Qin, "Hamiltonian Formulation of the Gyrokinetic Vlasov-Maxwell Equations," arXiv:1411.1790 [physics.plasm-ph] (2014).

Old:

D. Pfirsch and pjm, Phys. Fluids B (1985, 1991) on Guiding-Center Theories

# Orbits

Whence the orbits?

- Perturbation theory yield guiding center, gyrocenter, oscillation center, .... ODEs in given $\mathbf{E}, \mathbf{B}$ with small parameters.

- A priori modeling of matter with magnetization and polarization properties.

# Orbit Theory Ingredients

Particle Hamiltonian/energy $\Rightarrow$ orbits:

$$\begin{aligned}\mathcal{E} &= \bar{\mathcal{K}}\,(\mathbf{p} - e\mathbf{A}/c, w, \mathbf{E}, \mathbf{B}, \nabla\mathbf{E}, \nabla\mathbf{B}, \ldots) + e\phi\,, \\ &= \mathcal{K}\,(\mathbf{v}, w; \mathbf{E}, \mathbf{B}, \nabla\mathbf{E}, \nabla\mathbf{B}, \ldots) + e\phi(\mathbf{x})\,,\end{aligned}$$

Poisson Bracket:

$$[\,,\,] : C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z}) \qquad \text{where } z = (\mathbf{x}, \mathbf{v}, w) \in \mathcal{Z}$$

Comments:

- Special from that gives rise to gauge invariant variational principles, e.g., Hamiltonian-Jacobi type of Pfirsch and pjm (1984, 1985, 1991), phase space action, etc.
- Any functional dependence on $\mathbf{E}, \mathbf{B}$ allowed.
- Can be written in terms of a canonical momentum $\mathbf{p}$ or kinetic momentum $m\mathbf{v}$.
- Can have ‘internal’ degrees of freedom, e.g., spin or angular momentum via $w$.

# I. ODEs

## Orbits From Actions

# Action Principle

Hero of Alexandria (75 AD) $\longrightarrow$ Fermat (1600's) $\longrightarrow$

Hamilton's Principle (1800's)

The Procedure:

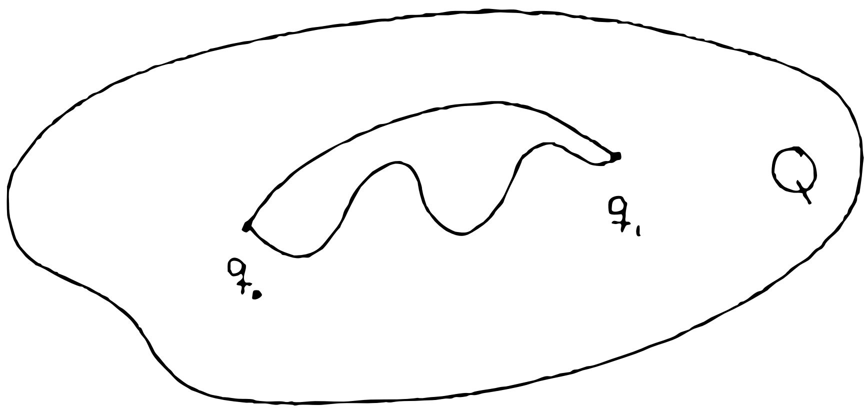
- Configuration Space: $q^i(t)\,, \qquad i = 1, 2, \ldots, N$ $\longleftarrow$ #DOF
- Kinetic & Potential: $L = T - V$ $\longleftarrow$ Kinetic Potential
- Action Functional:

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\,, \qquad \delta q(t_0) = \delta q(t_1) = 0$$

Extremal path $\Longrightarrow$ Lagrange's equations

# Variation Over Paths

$S[q_{\mathrm{path}}] =$ number



Functional Derivative:

$$\frac{\delta S[q]}{\delta q^i} = 0 \qquad \Longrightarrow$$

Lagrange's Equations:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0\,.$$

# Hamilton's Equations

Canonical Momentum: $p_i = \frac{\partial L}{\partial \dot{q}^i}$

Legendre Transform: $H(q,p) = p_i \dot{q}^i - L$

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,}$$

Phase Space Coordinates: $z = (q,p)$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

symplectic 2-form = (cosymplectic form)$^{-1}$: $\omega^c_{ij} J_c^{jk} = \delta_i^k,$

# Phase-Space Action

Gives Hamilton's equations directly

$$S[q,p] = \int_{t_0}^{t_1} dt\, \left(p_i \dot{q}^i - H(q,p)\right)$$

Defined on paths $\gamma$ in phase space $\mathcal{P}$ (e.g. $T^*Q$) parameterized by time, $t$, i.e., $z_\gamma(t) = (q_\gamma(t), p_\gamma(t))$. Then $S : \mathcal{P} \to \mathbb{R}$. Domain of $S$ any smooth path $\gamma \in \mathcal{P}$.

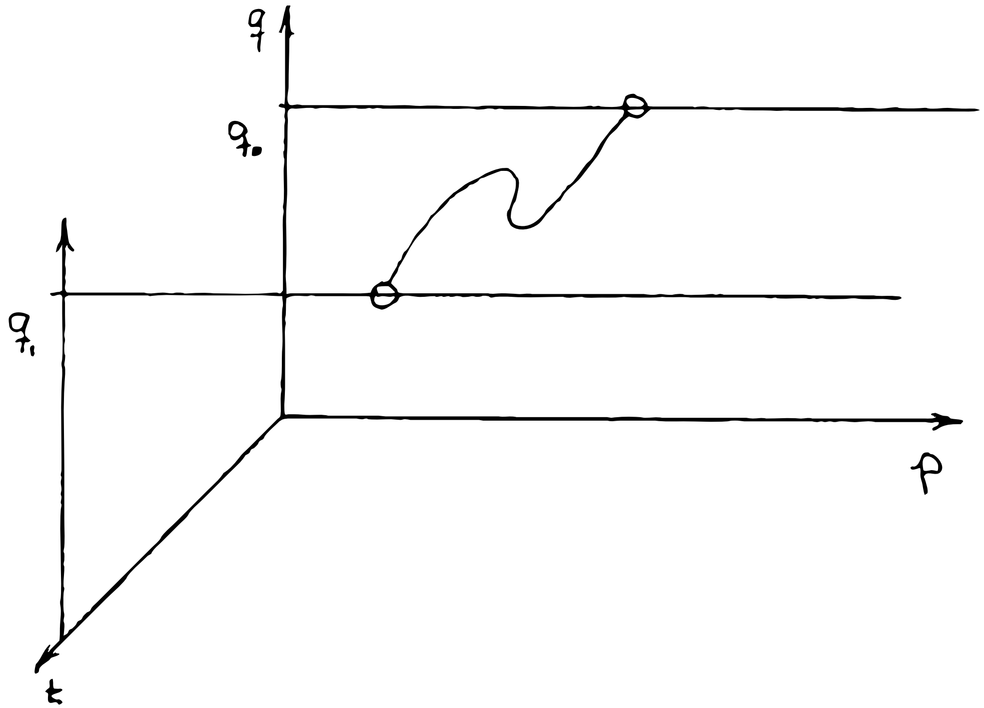
Law of nature, set Fréchet or functional derivative,to zero. Varying $S$ by perturbing path, $\delta z_\gamma(t)$ , gives

$$\delta S[z_\gamma; \delta z_\gamma] = \int_{t_0}^{t_1} dt \left[\delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i}\right) - \delta q^i \left(\dot{p}_i + \frac{\partial H}{\partial q^i}\right) + \frac{d}{dt}\left(p_i \delta q^i\right)\right].$$

Under the assumption $\delta q(t_0) = \delta q(t_1) \equiv 0$ , with no restriction on $\delta p$, boundary term vanishes.

Admissible paths in $\mathcal{P}$ have 'clothesline' boundary conditions.

# Phase-Space Action Continued




$$\delta S \equiv 0 \quad \Rightarrow \quad \dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \quad i = 1, 2, \ldots, N\,,$$

Thus, extremal paths satisfy Hamilton's equations.

# Alternatives

Rewrite action $S$ as follows:

$$S[z] = \int_{t_0}^{t_1} dt \left( \frac{1}{2} \omega^c_{\alpha\beta} z^\alpha \dot{z}^\beta - H(z) \right) =: \int_\gamma (d\theta - H dt)$$

where $d\theta$ is a differential one-form.

Particle motion in given electromagnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$

$$S[\mathbf{r}, \mathbf{p}] = \int_{t_0}^{t_1} dt \left[ \mathbf{p} \cdot \dot{\mathbf{r}} - \frac{1}{2m} \left| \mathbf{p} - \frac{e}{c} \mathbf{A}(\mathbf{r}, t) \right|^2 - e\phi(\mathbf{r}, t) \right] .$$

The Lorentz force law arises from $S$. **Generalize this to** $\rightarrow$

$$S[\mathbf{r}, \mathbf{p}] = \int_{t_0}^{t_1} dt \left[ \mathbf{p} \cdot \dot{\mathbf{r}} - \bar{\mathcal{K}} \left( \mathbf{p} - e\mathbf{A}/c, w, \mathbf{E}, \mathbf{B}, \nabla\mathbf{E}, \nabla\mathbf{B}, \dots \right) - e\phi \right] .$$

Orbit dynamics that describes matter (plasma) arises from $S$.

# Generalized Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980)....

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad [A, B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $[A, B] = -[B, A]\,,$

Jacobi identity $\longrightarrow$ $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus <u>Casimirs</u>

Matter models in Eulerian variables: $J^{ij} = c_k^{ij} z^k$ $\longleftarrow$ Lie – Poisson Brackets

Finite dimensions to infinite dimensions!

# Flow on Poisson Manifold

**Definition.** A Poisson manifold $\mathcal{P}$ is differentiable manifold with bracket $[\,,\,] : C^\infty(\mathcal{P}) \times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$ st $C^\infty(\mathcal{P})$ with $[\,,\,]$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $JdH$.

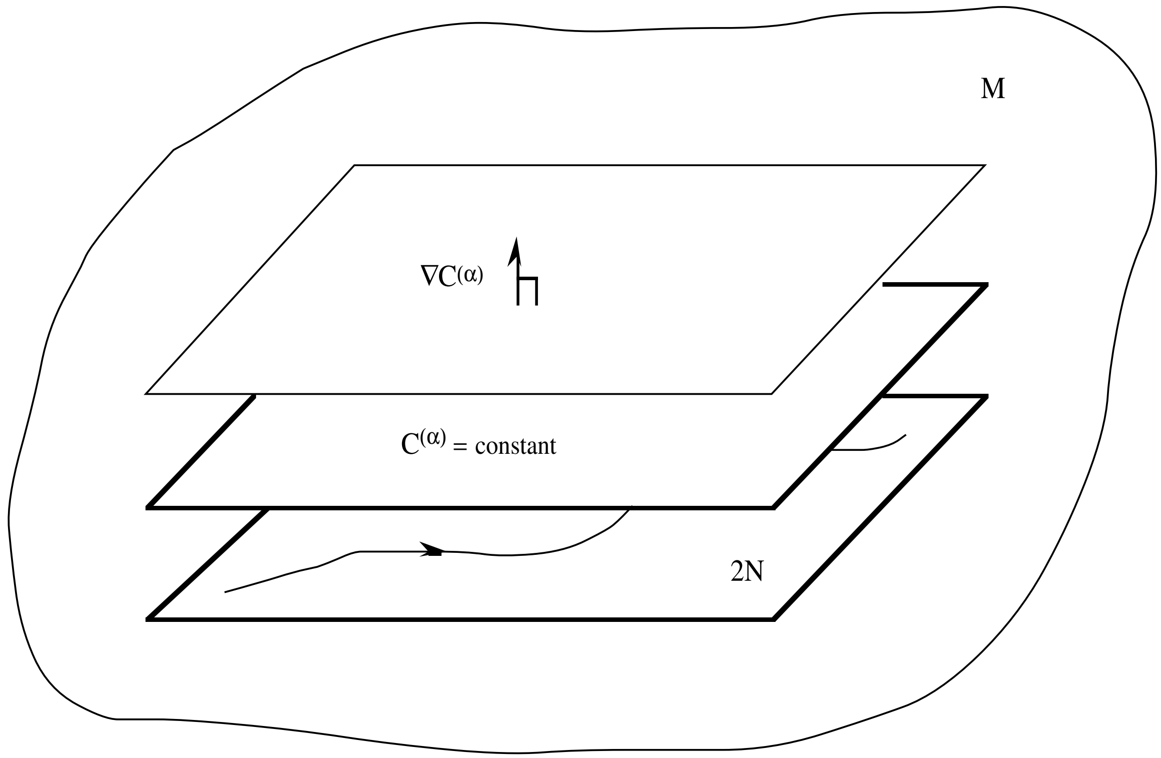
Because of degeneracy, $\exists$ functions $C$ st $[f, C] = 0$ for all $f \in C^\infty(\mathcal{P})$. Called Casimir invariants (Lie's distinguished functions.)

# Poisson Manifold $\mathcal{P}$ Cartoon

Degeneracy in $J$ $\Rightarrow$ Casimirs:

$$[f, C] = 0 \quad \forall\ f : \mathcal{P} \rightarrow \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:




Leaves are symplectic rearrangements in infinite dimensions.

# II. PDEs

## Lifting Orbits:

Hamiltonian Matter & Field Theory

# The Field Theory

Desiderata:

- Kinetic Theory/ transport equation

- Sources for Maxwell's equations

Provided by:

- Hamiltonian Functional.

- Noncanonical (field theory) Poisson bracket.

# Example: Vlasov-Poisson Hamiltonian Structure

Noncanonical Poisson Bracket (pjm 1980):

$$\{F, G\} = \int f \left[ \frac{\delta F}{\delta f}, \frac{\delta G}{\delta f} \right] dx dv$$

$F$ and $G$ are functionals. VP $\Longleftrightarrow$

$$\frac{\partial f}{\partial t} = \{f, H\} = \left[ f, \mathcal{E} = \frac{\delta H}{\delta f} \right].$$

where $\mathcal{E} = mv^2/2 + e\phi$ and

$$[f, \mathcal{E}] = \frac{1}{m} \left( \frac{\partial f}{\partial x} \frac{\partial \mathcal{E}}{\partial v} - \frac{\partial \mathcal{E}}{\partial x} \frac{\partial f}{\partial v} \right)$$

Casimir Invariants;

$$C[f] = \int \mathcal{C}(f) \, dx dv$$

Organizes: VP, Euler, QG, Defect Dyn, Benny-Dirac, ....

# Orbit Theory as Matter Model

Examples:

- Linear materials:

  $$\mathcal{K}(\mathbf{v}, w, \mathbf{E}, \mathbf{B}) = h(\mathbf{v}, w, \mathbf{B}) + \mathcal{P}(\mathbf{v}, w, \mathbf{B}) \cdot \mathbf{E} + \frac{1}{2}\mathbf{E} \cdot \underline{\underline{k}}(\mathbf{v}, w, \mathbf{B}) \cdot \mathbf{E}\,,$$

  giving rise to $\mathbf{D} = \underline{\underline{\epsilon}} \cdot \mathbf{E}$, which is $\underline{\underline{\epsilon}}$ constant, in elementary electromagnetism.

- Lorentzian dynamics $\mathcal{K} = m|v|^2/2 \rightarrow$ Maxwell-Vlasov theory.

- More interesting $\mathcal{K}$s are for guiding center or gyrokintic theories.

# Sources

$K$-functional:

$$K[\mathbf{E},\mathbf{B},f] := \int d\mathbf{x}d\mathbf{v}dw\,\mathcal{K}\,f\,,$$

Polarization and Magnetization:

$$\mathbf{P}(\mathbf{x},t) = -\frac{\delta K}{\delta \mathbf{E}} \quad \text{and} \quad \mathbf{M}(\mathbf{x},t) = -\frac{\delta K}{\delta \mathbf{B}}$$

Sources:

$$\begin{aligned}
\rho(\mathbf{x},t) &= e\int d\mathbf{v}dw\,f - \nabla\cdot\frac{\delta K}{\delta \mathbf{E}} \\
\mathbf{J}(\mathbf{x},t) &= e\int d\mathbf{v}dw\,\frac{\partial \mathcal{K}}{\partial \mathbf{v}}\,f + \frac{\partial}{\partial t}\frac{\delta K}{\delta \mathbf{E}} + c\,\nabla\times\frac{\delta K}{\delta \mathbf{B}}
\end{aligned}$$

Generalization of Pfirsch & pjm (1984,1985,1991) in pjm (2013)

# Constitutive Relations

General:

$$\mathbf{D}[\mathbf{E},\mathbf{B};f] = \mathbf{E} + 4\pi\mathbf{P}[\mathbf{E},\mathbf{B};f] \qquad \longleftarrow \text{operator}$$

Inverse:

$$\mathbf{E} = \mathbf{D}^{-1}[\mathbf{D},\mathbf{B};f] = \mathbf{E}[\mathbf{D},\mathbf{B};f]$$

Similarly,

$$\mathbf{H} = \mathbf{H}[\mathbf{B},\mathbf{E};f] = \mathbf{B} - 4\pi\mathbf{M}[\mathbf{B},\mathbf{E};f]$$

Inverse:

$$\mathbf{B} = \mathbf{B}[\mathbf{H},\mathbf{E};f] = \mathbf{H} + 4\pi\mathbf{M}[\mathbf{H},\mathbf{E};f]$$

Permitivity Operator:

$$\delta\mathbf{D} = \left(\underline{\underline{I}} - 4\pi\frac{\delta^2 K}{\delta\mathbf{E}\delta\mathbf{E}}\right)\cdot\delta\mathbf{E} =: \underline{\underline{\varepsilon}}\cdot\delta\mathbf{E} = \frac{\delta\mathbf{D}}{\delta\mathbf{E}}\cdot = \mathbf{D}_{\mathbf{E}}\cdot\delta\mathbf{E}$$

# Hamiltonian and Bracket

Hamiltonian:

$$H[f,\mathbf{E},\mathbf{B}] = K - \int\! d\mathbf{x}\,\mathbf{E}\cdot\frac{\delta K}{\delta \mathbf{E}} + \frac{1}{8\pi}\int\! d\mathbf{x}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$$
$$= K + \int\! d\mathbf{x}\,\mathbf{E}\cdot\mathbf{P} + \frac{1}{8\pi}\int\! d\mathbf{x}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$$

Poisson Bracket:

$$\begin{aligned}\{F,G\} =& \int\! d\mathbf{x}d\mathbf{v}dw\, f\left[F_f + \mathbf{E}_f^{\dagger}\cdot F_{\mathbf{E}}\,,\, G_f + \mathbf{E}_f^{\dagger}\cdot G_{\mathbf{E}}\right] \\ &+ \frac{4\pi e}{m}\int\! d\mathbf{x}d\mathbf{v}dw\, f\left(\left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot G_{\mathbf{E}}\right)\cdot\partial_{\mathbf{v}}\left(F_f + \mathbf{E}_f^{\dagger}\cdot F_{\mathbf{E}}\right)\right. \\ &\left. - \left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot F_{\mathbf{E}}\right)\cdot\partial_{\mathbf{v}}\left(G_f + \mathbf{E}_f^{\dagger}\cdot G_{\mathbf{E}}\right)\right) \\ &+ 4\pi c\int\! d\mathbf{x}\Big(\left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot F_{\mathbf{E}}\right)\cdot\nabla\times\left(G_{\mathbf{B}} + \mathbf{E}_{\mathbf{B}}^{\dagger}\cdot G_{\mathbf{E}}\right) \\ &- \left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot G_{\mathbf{E}}\right)\cdot\nabla\times\left(F_{\mathbf{B}} + \mathbf{E}_{\mathbf{B}}^{\dagger}\cdot F_{\mathbf{E}}\right)\Big)\end{aligned}$$

Equations of Motion:

$$f_t = \{f,H\}\,, \qquad \mathbf{B}_t = \{\mathbf{B},H\}\,, \qquad \mathbf{E}_t = \{\mathbf{E},H\}$$

Significant generalization of MV bracket:

pjm (1980,1982); Marsden &Weinstein (1982)

# Conclusion/ Summary

1. Described lifting in general terms for matter description

2. Origin of Orbits via Action Principle

3. Described noncanononcal Poisson brackets, finite $\rightarrow$ infinite

4. Consistent Theory with Polarization and Magnetization formulas