HAMILTONIAN AND ACTION PRINCIPLE DERIVATIONS OF REDUCED MAGNETOFLUID MODELS FOR PLASMA DYNAMICS – CONSEQUENCES

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Goal: Describe a prescription for building action functionals for plasma models. Apply it to the derivation of reduced fluid models

"Hamiltonian systems .... are the basis of physics." M. Gutzwiller

Many collaborators and students over the past 35 years
(AIP Conf. Proc. (1982), Rev Mod Phys (1998), PoP (2005), ...)

Overview

- Classical Tools
- Building Actions
- Examples

Classical Tools

Action Principle

Hero of Alexandria (75 AD) $\longrightarrow$ Fermat (1600's) $\longrightarrow$

Hamilton's Principle (1800's)

The Procedure:

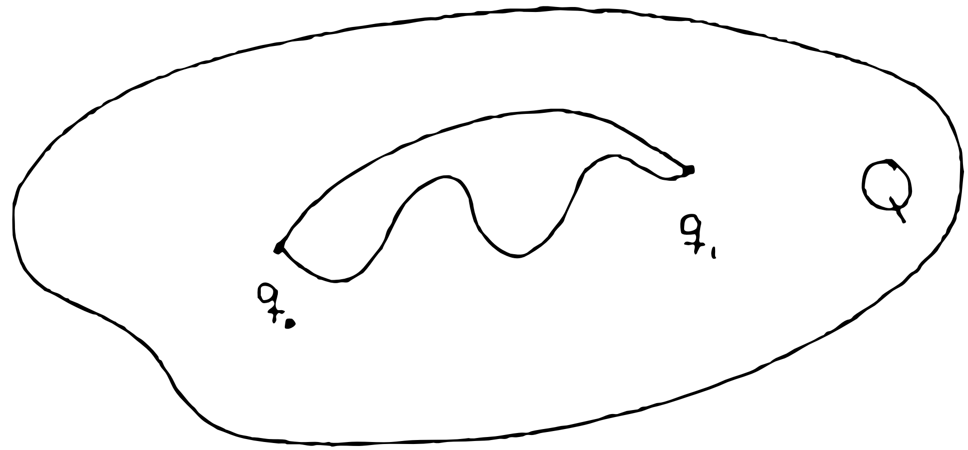
- Configuration Space: $q^i(t)\,, \qquad i = 1, 2, \ldots, N$ $\longleftarrow$ #DOF
- Kinetic & Potential: $L = T - V$ $\longleftarrow$ Kinetic Potential
- Action Functional:

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\,, \qquad \delta q(t_0) = \delta q(t_1) = 0$$

Extremal path $\Longrightarrow$ Lagrange's equations

Variation Over Paths

$S[q_{\text{path}}] = $ number



Functional Derivative:

$$\frac{\delta S[q]}{\delta q^i} = 0 \qquad \Longrightarrow$$

Lagrange's Equations:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0\,.$$

Hamilton's Equations

Canonical Momentum: $p_i = \frac{\partial L}{\partial \dot{q}^i}$

Legendre Transform: $H(q,p) = p_i \dot{q}^i - L$

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,}$$

Phase Space Coordinates: $z = (q,p)$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

symplectic 2-form = (cosymplectic form)$^{-1}$: $\omega^c_{ij} J_c^{jk} = \delta_i^k$,

Poisson Brackets

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad [A,B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $[A,B] = -[B,A]\,,$

Jacobi identity $\longrightarrow$ $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$

G. Darboux: $det J \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

Eulerian Media: $J^{ij} = c_k^{ij} z^k$ $\longleftarrow$ Lie – Poisson Brackets

Hamiltonian Reduction

Bracket Reduction:

Reduced set of variables $(q,p) \to w(q,p)$

Bracket Closure:

$$[w,w] = c(w) \qquad\qquad f[q,p] = \widehat{f}[w(q,p)]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket

Hamiltonian Closure:

$$H(q,p) = \widehat{H}(w)$$

Eulerian variables are noncanonical variables

(pjm & John Greene 1980)

Plasma Parent Model

Relativistic N-Particle Action

Dynamical Variables: $q_i(t), \phi(x,t), A(x,t)$

$$S[q,\phi,A] \;=\; -\sum_{i=1}^{N}\int dt\;\, mc^2\sqrt{1-\frac{\dot q_i^2}{c^2}}$$ $\longleftarrow$ ptle kinetic energy

coupling $\longrightarrow$ $$-e\int dt\sum_{i=1}^{N}\int dx\,\left[\phi(x,t)+\frac{\dot q_i}{c}\cdot A(x,t)\right]\delta\left(x-q_i(t)\right)$$

field 'energy' $\longrightarrow$ $$+\frac{1}{8\pi}\int dt\int dx\left[E^2(x,t)-B^2(x,t)\right].$$

Variation:

$$\frac{\delta S}{\delta q^i(t)}=0 \quad\Longrightarrow$$ Newton's 2nd & Fields,

$$\frac{\delta S}{\delta\phi(x,t)}=0\,,\qquad \frac{\delta S}{\delta A(x,t)}=0 \quad\Longrightarrow$$ Maxwell & Sources

All done?

Building Actions

Senior Progeny

Computability and Intuition

Reductions $\Longrightarrow$

Vlasov-Maxwell, two-fluid theory, MHD, ...

Neglect clearly identifiable dissipation $\Longrightarrow$

Action principles and Hamiltonian structure

identified *ex post facto*

Simplifications: Reduced Fluid Models

Approximations:

asymptotic expansions, systematic ordering

Model Building:

Mutilations, put it what one this is important, closures etc.

Other Progeny:

Gyrokinetics, guiding-center kinetics, gyrofluids, ...

Hamiltonian? Action?

Building Action Principles

Step 1: Select Domain

For fluid a spatial domain; for kinetic theory a phase space

Step 2: Select Attributes – Eulerian Variables (Observables)

L to E, map e.g. MHD $\{v, \rho, s, B\}$. Builds in constraints!

Step 3: Eulerian Closure Principle

Terms of action must be 'Eulerianizable' $\Rightarrow$ EOMs are!

Step 4: Symmetries

Traditional. Rotation, etc. via Noerther $\rightarrow$ invariants

Examples

Fluid Action Kinematics

Lagrange (1788)

Lagrangian Variables:

Fluid occupies domain D e.g. (x, y, z) or (x, y)

Fluid particle position $q(a, t)$, $q_t : D \to D$ 1-1, invertible...

Particles label: a e.g. $q(a, 0) = a$.

Deformation: $\frac{\partial q^i}{\partial a^j} = q^i_{,j}$

Determinant: $= \det(q^i_{,j}) \neq 0 \Rightarrow a(q, t)$

Identity: $q^i_{,k} a^k_{,j} = \delta^i_j$

Volume: $d^3 q = \mathcal{J} d^3 a$

Area: $(d^2 q)_i = \mathcal{J} a^j_{,i} (d^2 a)_j$

Line: $(dq)_i = q^i_{,j} (da)_j$

Eulerian Variables:

Observation point: r

Velocity field: $v(r,t) = ?$ Probe sees $\dot{q}(a,t)$ for some a.

What is a ? $r = q(a,t) \quad \Rightarrow \quad a = q^{-1}(r,t)$

$$v(r,t) = \dot{q}(a,t)|_{a=q^{-1}(r,t)}$$

IDEAL MHD

Attributes:

Entropy (1-form):

$$s(r,t) = s_0|_{a=a(r,t)} \; ,$$

Mass (3-form):

$$\rho d^3x = \rho_0 d^3a \quad \Rightarrow \quad \rho(r,t) = \left.\frac{\rho_0}{\mathcal{J}}\right|_{a=a(r,t)} \; .$$

B-Flux (2-form):

$$B \cdot d^2x = B_0 \cdot d^2a \quad \Rightarrow \quad B^i(r,t) = \left.\frac{q^i_{,j} B^j_0}{\mathcal{J}}\right|_{a=a(r,t)} \; .$$

Kinetic Potential

Kinetic Energy:

$$K[q] = \frac{1}{2}\int_D d^3a\,\rho_0|\dot{q}|^2 = \frac{1}{2}\int_D d^3x\,\rho|v|^2$$

Potential Energy:

$$\begin{aligned} V[q] &= \int_D d^3a\,\rho_0\mathcal{V}(\rho_0/\mathcal{J}, s_0, |q^i_{,j}B^j_0|/\mathcal{J}) = \frac{1}{2}\int_D d^3x\,\rho\mathcal{V}(\rho, s, |B|) \\ &= \int_D d^3a\,\rho_0\mathcal{U}(\rho_0/\mathcal{J}, s_0) - \frac{1}{2}\frac{|q^i_{,j}B^j_0|^2}{\mathcal{J}^2} \end{aligned}$$

Action:

$$S[q] = \int dt\,(K - V)\,, \qquad \delta S = 0 \quad \Rightarrow \quad \text{Ideal MHD}$$

Alternative: Lagrangian variations induce constrained Eulerian variations $\Rightarrow$ Serrin, Newcomb, Euler-Poincaré, ...

Stability: δW, Lagrangian, Eulerian, dynamical accessible, Andreussi, Pegoraro, pjm. (2010 – 2014)

Braginskii MHD

$$\rho\,(v_t + v\cdot\nabla v) = -\nabla p + J\times B + \nabla\cdot\Pi$$

Gyroviscosity Tensor: $\Pi_{ij} = \frac{p}{B}N_{jsik}\frac{\partial v_s}{\partial x_k}$

Action:

$$S[q] = \int dt\,(K + G - V)\,,$$

Gyroscopic Term:

$$G[q] = \int_D d^3a\,\Pi^*\cdot\dot{q} = \int_D d^3x\,M^*\cdot v$$

where

$$\Pi^* = \nabla\times L^* = \frac{m}{2e}\mathcal{J}\,\hat{b}\times\nabla\left(\frac{p}{B}\right)$$

$\delta S[q] = 0 \quad \Rightarrow \quad$ Braginskii MHD

pjm, Lingam, Acevedo, Wurm 2014

Inertial MHD (Tassi)

Basic Idea: Can 'freeze-in' anything one likes! (2-form attribute)

Choose:

$$\mathbf{B}_e = \mathbf{B} + d_e^2 \nabla \times \mathbf{J},$$

Action:

$$S = \int dt \int d^3x \left(\rho \frac{v^2}{2} - \rho U(\rho, s) - \mathbf{B}_e \cdot \mathbf{B} \right).$$

Attributes:

$$\rho d^3x = \rho_0 d^3a, \qquad B_e^i = \frac{B_{e0}^j}{J} \frac{\partial q^i}{\partial a_j}$$

$\delta S[q] = 0 \quad \Rightarrow \quad$ IMHD

Hamiltonian Structure

Legendre Transformation:

$$p = \frac{\delta L}{\delta \dot{q}} \qquad\qquad L \to H$$

Poisson Bracket:

$$\{F, G\} = \int_D da \left(\frac{\delta F}{\delta q^i} \frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i} \frac{\delta F}{\delta p_i} \right)$$

EOM:

$$\dot{q} = \{q, H\} \qquad\qquad \dot{p} = \{p, H\}$$

Eulerian Reduction

$$F[q,p] = \widehat{F}[v,\rho,s,B]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket in terms of Eulerian variables (pjm & John Greene 1980)

It is an algorithmic process. Manipulations like calculus.

Example: 2D IMHD

$$\{F,G\} = -\int \frac{d^3x}{\rho}(\omega[F_\omega, G_\omega] + \psi_e([F_\omega, G_{\psi_e}] - [G_\omega, F_{\psi_e}]))$$

$$H = \int d^2x(d_e^2(\nabla^2\psi)^2 + |\nabla\psi|^2 + |\nabla\varphi|^2)$$

Produces 2D incompressible IMHD (Ottaviani-Porcelli model)!

Infinite Dimensional Hamiltonian Structure

Field Variables: $\psi(\mu, t)$ e.g. $\mu = x$, $\mu = (x, v)$, . . .

Poisson Bracket:

$$\{A, B\} = \int \frac{\delta A}{\delta \psi} \mathcal{J}(\psi) \frac{\delta A}{\delta \psi} \, d\mu$$

Lie-Poisson Bracket:

$$\{A, B\} = \left\langle \psi, \left[\frac{\delta A}{\delta \psi}, \frac{\delta A}{\delta \psi} \right] \right\rangle$$

Cosymplectic Operator:

$$\mathcal{J} \,\cdot\, \sim [\psi, \,\cdot\,]$$

Form for Eulerian theories: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

Two-Fluid Action

Keramidas Charidakos, Lingam, pjm, R. White and A. Wurm

$$S[q_s, A, \phi] = \int dt \, \frac{1}{8\pi} \int d^3x \, \left[\left| -\frac{1}{c}\frac{\partial A(x,t)}{\partial t} - \nabla\phi(x,t) \right|^2 - |\nabla \times A(x,t)|^2 \right] \tag{1}$$

$$+ \sum_s \int d^3a \, n_{s0}(a) \int d^3x \, \delta\left(x - q_s(a,t)\right) \times \left[\frac{e_s}{c}\, \dot{q}_s \cdot A(x,t) - e_s\phi(x,t) \right] \tag{2}$$

$$+ \sum_s \int d^3a \, n_{s0}(a) \left[\frac{m_s}{2} \, |\dot{q}_s|^2 - m_s U_s \left(m_s n_{s0}(a)/\mathcal{J}_s, s_{s0} \right) \right]. \tag{3}$$

Eulerian Observables:

$$\{n_\pm, v_\pm, A, \phi\}$$

Reduced Variables

New Lagrangian Variables:

$$
\begin{aligned}
Q(a,t) &= \frac{1}{\rho_{m0}(a)}\left(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)\right) \\
D(a,t) &= e\left(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)\right) \\
\rho_{m0}(a) &= m_i n_{i0}(a) + m_e n_{e0}(a) \\
\rho_{q0}(a) &= e\left(n_{i0}(a) - n_{e0}(a)\right) .
\end{aligned}
$$

Consistent Expansion:

$$
\frac{v_A}{c} << 1\,, \quad \frac{m_e}{m_i} << 1 \quad \Rightarrow \quad \text{quasineutrality}
$$

Eulerian Closure:

$$
\{n, s, s_e, V, J\}
$$

Extended MHD

Ohm's Law:

$$E + \frac{V \times B}{c} = \frac{m_e}{e^2 n}\left(\frac{\partial J}{\partial t} + \nabla \cdot (VJ + JV)\right) - \frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en}\,.$$

Momentum:

$$nm\left(\frac{\partial V}{\partial t} + (V \cdot \nabla)V\right) = -\nabla p + \frac{J \times B}{c} - \frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)\,.$$

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Consistent with an ordering of Lüst (1958)

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$$-\frac{m_e}{e^2 n}(J \cdot \nabla)\left(\frac{J}{n}\right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en} .$$

Momentum:

$$nm\left(\frac{\partial V}{\partial t} + (V \cdot \nabla)V\right) \;=\; -\nabla p + \frac{J \times B}{c}$$
$$-\frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right) .$$

Consistent with an ordering of Lüst (1958)

Noether → Energy Conservation

Energy:

$$H = \int d^3x \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e}{ne^2}\frac{|J|^2}{2} \right]$$

Energy conservation requires

$$\frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)$$

in momentum equation. Otherwise inconsistent.

Physical dissipation is real. Fake dissipation is troublesome, particularly for reconnection studies. Kimura and pjm (2014).

Summary

- Classical Tools
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