I. LIFTING – A method for constructing consistent kinetic theories with electromagnetic interaction

II. HAMILTONIAN AND ACTION PRINCIPLE DERIVATIONS OF MAGNETOFLUID MODELS

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Both topics are about how to build Hamiltonian or Action Principle (HAP) models, plasma kinetic and/or fluid field theories, from scratch.

I. Kinetic theories $\longleftarrow$ From "orbit models"

II. Magnetofluid models $\longleftarrow$ Out of "nothing"

Part I

I. LIFTING

Definition: Given a set of ordinary differential equations for 'orbits' of some kind of charged entities in given 'electromagnetic' fields $\mathbf{E}, \mathbf{B}$, lifting is a prescription for building a consistent HAP kinetic theory.

$\longrightarrow$ pjm, PoP **20**, 012104 (2013)

Collaborators:

M. Vittot, L. de-Guillebon, ... J. Burby, A. Brizard, H. Qin

Orbits

Whence the orbits?

- Perturbation theory yield guiding center, gyrocenter, oscillation center, ... ODEs in given $\mathbf{E}, \mathbf{B}$ with small parameters.

- A priori modeling of matter with magnetization and polarization properties.

Orbit Theory Ingredients

Particle Hamiltonian/energy $\Rightarrow$ orbits:

$$\begin{aligned}\mathcal{E} &= \bar{\mathcal{K}}\,(\mathbf{p} - e\mathbf{A}/c, w, \mathbf{E}, \mathbf{B}, \nabla\mathbf{E}, \nabla\mathbf{B}, \dots) + e\phi\,,\\ &= \mathcal{K}\,(\mathbf{v}, w; \mathbf{E}, \mathbf{B}, \nabla\mathbf{E}, \nabla\mathbf{B}, \dots) + e\phi\,,\end{aligned}$$

Comments:

- Special from that gives rise to gauge invariant variational principles, e.g., Hamiltonian-Jacobi type of Pfirsch and pjm (1984, 1985, 1991), phase space action, etc.

- Any functional dependence on $\mathbf{E}, \mathbf{B}$ allowed.

- Can be written in terms of a canonical momentum $\mathbf{p}$ or kinetic momentum $m\mathbf{v}$.

- Can have 'internal' degrees of freedom, e.g., spin or angular momentum via w.

Orbit Theory as Matter Model

Examples:

- Linear materials:

  $$\mathcal{K}(\mathbf{v}, w, \mathbf{E}, \mathbf{B}) = h(\mathbf{v}, w, \mathbf{B}) + \mathcal{P}(\mathbf{v}, w, \mathbf{B}) \cdot \mathbf{E} + \frac{1}{2}\mathbf{E} \cdot \underline{\underline{k}}(\mathbf{v}, w, \mathbf{B}) \cdot \mathbf{E}\,,$$

  giving rise to $\mathbf{D} = \underline{\underline{\epsilon}} \cdot \mathbf{E}$ etc. as in elementary electromagnetism.

- Lorentzian dynamics $\mathcal{K} = m|v|^2/2 \rightarrow$ Maxwell-Vlasov theory.

- More interesting $\mathcal{K}$s are for guiding center or gyrokintic theories.

The Field Theory

Desiderata:

- Kinetic Theory/ transport equation

- Sources for Maxwell's equations

Provided by:

- Hamiltonian Functional.

- Noncanonical (field theory) Poisson bracket.

Sources

K-functional:

$$K[\mathbf{E},\mathbf{B},f] := \int d\mathbf{x}d\mathbf{v}dw\,\mathcal{K}\,f\,,$$

Polarization and Magnetization:

$$\mathbf{P}(\mathbf{x},t) = -\frac{\delta K}{\delta \mathbf{E}} \quad \text{and} \quad \mathbf{M}(\mathbf{x},t) = -\frac{\delta K}{\delta \mathbf{B}}$$

Sources:

$$\begin{aligned}
\rho(\mathbf{x},t) &= e\int d\mathbf{v}dw\, f - \nabla\cdot\frac{\delta K}{\delta \mathbf{E}} \\
\mathbf{J}(\mathbf{x},t) &= e\int d\mathbf{v}dw\,\frac{\partial \mathcal{K}}{\partial \mathbf{v}}\, f + \frac{\partial}{\partial t}\frac{\delta K}{\delta \mathbf{E}} + c\,\nabla\times\frac{\delta K}{\delta \mathbf{B}}
\end{aligned}$$

Generalization of Pfirsch & pjm (1984,1985,1991) in pjm (2013)

Constitutive Relations

General:

$$\mathbf{D}[\mathbf{E},\mathbf{B};f] = \mathbf{E} + 4\pi\mathbf{P}[\mathbf{E},\mathbf{B};f] \qquad \longleftarrow \text{operator}$$

Inverse:

$$\mathbf{E} = \mathbf{D}^{-1}[\mathbf{D},\mathbf{B};f] = \mathbf{E}[\mathbf{D},\mathbf{B};f]$$

Similarly,

$$\mathbf{H} = \mathbf{H}[\mathbf{B},\mathbf{E};f] = \mathbf{B} - 4\pi\mathbf{M}[\mathbf{B},\mathbf{E};f]$$

Inverse:

$$\mathbf{B} = \mathbf{B}[\mathbf{H},\mathbf{E};f] = \mathbf{H} + 4\pi\mathbf{M}[\mathbf{H},\mathbf{E};f]$$

Permitivity Operator:

$$\delta\mathbf{D} = \left(\underline{\underline{I}} - 4\pi\frac{\delta^2 K}{\delta\mathbf{E}\delta\mathbf{E}}\right)\cdot\delta\mathbf{E} =: \underline{\underline{\varepsilon}}\cdot\delta\mathbf{E} = \frac{\delta\mathbf{D}}{\delta\mathbf{E}}\cdot = \mathbf{D}_{\mathbf{E}}\cdot\delta\mathbf{E}$$

Hamiltonian and Bracket

Hamiltonian:

$$H[f,\mathbf{E},\mathbf{B}] \;=\; K - \int\! d\mathbf{x}\,\mathbf{E}\cdot\frac{\delta K}{\delta \mathbf{E}} + \frac{1}{8\pi}\int\! d\mathbf{x}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$$
$$\;=\; K + \int\! d\mathbf{x}\,\mathbf{E}\cdot\mathbf{P} + \frac{1}{8\pi}\int\! d\mathbf{x}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$$

Poisson Bracket:

$$\begin{aligned}
\{F,G\} \;=\;& \int\! d\mathbf{x}d\mathbf{v}dw\, f\left[F_f + \mathbf{E}_f^{\dagger}\cdot F_{\mathbf{E}}\,,\, G_f + \mathbf{E}_f^{\dagger}\cdot G_{\mathbf{E}}\right] \\
&+ \frac{4\pi e}{m}\int\! d\mathbf{x}d\mathbf{v}dw\, f\left(\left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot G_{\mathbf{E}}\right)\cdot\partial_{\mathbf{v}}\left(F_f + \mathbf{E}_f^{\dagger}\cdot F_{\mathbf{E}}\right)\right. \\
&\qquad - \left.\left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot F_{\mathbf{E}}\right)\cdot\partial_{\mathbf{v}}\left(G_f + \mathbf{E}_f^{\dagger}\cdot G_{\mathbf{E}}\right)\right) \\
&\qquad + 4\pi c\int\! d\mathbf{x}\Big(\left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot F_{\mathbf{E}}\right)\cdot\nabla\times\left(G_{\mathbf{B}} + \mathbf{E}_{\mathbf{B}}^{\dagger}\cdot G_{\mathbf{E}}\right) \\
&\qquad - \left(\mathbf{E}_{\mathbf{D}}^{\dagger}\cdot G_{\mathbf{E}}\right)\cdot\nabla\times\left(F_{\mathbf{B}} + \mathbf{E}_{\mathbf{B}}^{\dagger}\cdot F_{\mathbf{E}}\right)\Big)
\end{aligned}$$

Equations of Motion:

$$f_t = \{f,H\}\,, \qquad \mathbf{B}_t = \{\mathbf{B},H\}\,, \qquad \mathbf{E}_t = \{\mathbf{E},H\}$$

Part II

II. BUILDING MAGNETOFLUID ACTIONS

Goal: Describe a 4-step method for building physical matter actions (not necessarily fluid) out of nothing.

$\longrightarrow$ old papers plus several recent papers

Collaborators:

R. Acevedo, I. Keramidas Charidakos, M. Lingam, T. Andreussi, F. Pegoraro, R. White, A. Wurm, ...

Overview

- Classical Tools
- Building Actions
- Examples

Classical Tools

Action Principle

Hero of Alexandria (75 AD) $\longrightarrow$ Fermat (1600's) $\longrightarrow$

Hamilton's Principle (1800's)

The Procedure:

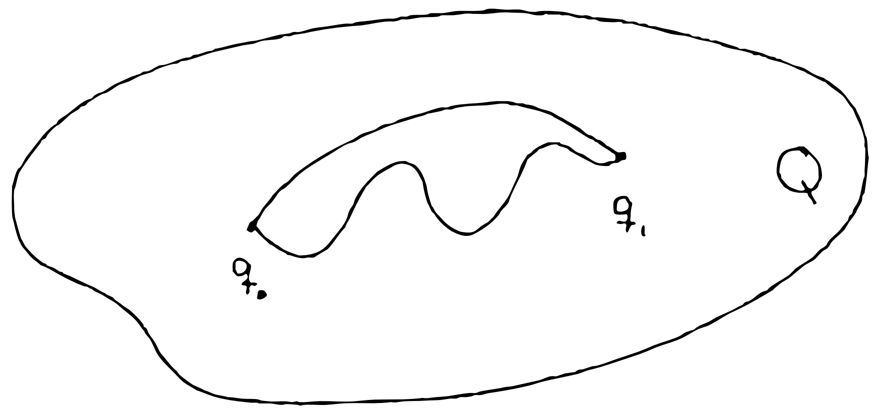
- Configuration Space: $q^i(t)\,, \qquad i = 1, 2, \ldots, N$ $\longleftarrow$ #DOF
- Kinetic & Potential: $L = T - V$ $\longleftarrow$ Kinetic Potential
- Action Functional:

$$S[q] = \int_{t_0}^{t_1} L(q, \dot{q}, t)\, dt\,, \qquad \delta q(t_0) = \delta q(t_1) = 0$$

Extremal path $\Longrightarrow$ Lagrange's equations

Variation Over Paths

$S[q_{\text{path}}] =$ number



Functional Derivative:

$$\frac{\delta S[q]}{\delta q^i} = 0 \qquad \Longrightarrow$$

Lagrange's Equations:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0\,.$$

Hamilton's Equations

Canonical Momentum: $p_i = \frac{\partial L}{\partial \dot{q}^i}$

Legendre Transform: $H(q,p) = p_i \dot{q}^i - L$

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,}$$

Phase Space Coordinates: $z = (q,p)$

$$\dot{z}^i = J_c^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad (J_c^{ij}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix},$$

symplectic 2-form = (cosymplectic form)$^{-1}$: $\omega^c_{ij} J_c^{jk} = \delta_i^k$,

Poisson Brackets

Noncanonical Coordinates:

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} = [z^i, H]\,, \qquad\qquad [A, B] = \frac{\partial A}{\partial z^i} J^{ij}(z) \frac{\partial B}{\partial z^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $[A, B] = -[B, A]\,,$

Jacobi identity $\longrightarrow$ $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$

G. Darboux: $det J \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus <u>Casimirs</u>

Eulerian Media: $J^{ij} = c_k^{ij} z^k$ $\longleftarrow$ Lie – Poisson Brackets

Hamiltonian Reduction

Bracket Reduction:

Reduced set of variables $(q,p) \to w(q,p)$

Bracket Closure:

$$[w,w] = c(w) \qquad\qquad f[q,p] = \widehat{f}[w(q,p)]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket

Hamiltonian Closure:

$$H(q,p) = \widehat{H}(w)$$

Eulerian variables are noncanonical variables

(pjm & John Greene 1980)

Plasma Parent Model

Relativistic N-Particle Action

Dynamical Variables: $q_i(t), \phi(x,t), A(x,t)$

$$
\begin{aligned}
S[q,\phi,A] \;&=\; -\sum_{i=1}^{N}\int dt\;\, mc^2\sqrt{1-\frac{\dot{q}_i^2}{c^2}} && \longleftarrow \text{ ptle kinetic energy}\\
\text{coupling} \longrightarrow \quad & -e\int dt\sum_{i=1}^{N}\int dx\,\left[\phi(x,t)+\frac{\dot{q}_i}{c}\cdot A(x,t)\right]\delta\left(x-q_i(t)\right)\\
\text{field 'energy'} \longrightarrow \quad & +\frac{1}{8\pi}\int dt\int dx\left[E^2(x,t)-B^2(x,t)\right].
\end{aligned}
$$

Variation:

$$\frac{\delta S}{\delta q^i(t)}=0 \quad\Longrightarrow\quad \text{Newton's 2nd \& Fields},$$

$$\frac{\delta S}{\delta\phi(x,t)}=0\,,\quad \frac{\delta S}{\delta A(x,t)}=0 \quad\Longrightarrow\quad \text{Maxwell \& Sources}$$

All done?

Building Actions

Senior Progeny

Computability and Intuition

Reductions $\Longrightarrow$

Vlasov-Maxwell, two-fluid theory, MHD, ...

Neglect clearly identifiable dissipation $\Longrightarrow$

Action principles and Hamiltonian structure

identified *ex post facto*

Simplifications: Reduced Fluid Models

Approximations:

asymptotic expansions, systematic ordering

Model Building:

Mutilations, put it what one this is important, closures etc.

Other Progeny:

Gyrokinetics, guiding-center kinetics, gyrofluids, ...

Hamiltonian? Action?

Building Action Principles

Step 1: Select Domain

For fluid a spatial domain; for kinetic theory a phase space

Step 2: Select Attributes – Eulerian Variables (Observables)

L to E, map e.g. MHD $\{v, \rho, s, B\}$. Builds in constraints!

Step 3: Eulerian Closure Principle

Terms of action must be 'Eulerianizable' $\Rightarrow$ EOMs are!

Step 4: Symmetries

Traditional. Rotation, etc. via Noerther $\rightarrow$ invariants

Examples

Fluid Action Kinematics

Lagrange (1788)

Lagrangian Variables:

Fluid occupies domain D e.g. (x, y, z) or (x, y)

Fluid particle position $q(a, t)$, $q_t : D \to D$ 1-1, invertible...

Particles label: a e.g. $q(a, 0) = a$.

Deformation: $\frac{\partial q^i}{\partial a^j} = q^i_{,j}$

Determinant: $\det(q^i_{,j}) \neq 0 \Rightarrow a(q, t)$

Identity: $q^i_{,k} a^k_{,j} = \delta^i_j$

Volume: $d^3q = \mathcal{J} d^3a$

Area: $(d^2q)_i = \mathcal{J} a^j_{,i} (d^2a)_j$

Line: $(dq)_i = q^i_{,j} (da)_j$

Eulerian Variables:

Observation point: r

Velocity field: $v(r,t) =?$ Probe sees $\dot{q}(a,t)$ for some a.

What is a ? $r = q(a,t) \quad \Rightarrow \quad a = q^{-1}(r,t)$

$$v(r,t) = \dot{q}(a,t)|_{a=q^{-1}(r,t)}$$

IDEAL MHD

Attributes:

Entropy (1-form):

$$s(r,t) = s_0|_{a=a(r,t)} \ ,$$

Mass (3-form):

$$\rho d^3x = \rho_0 d^3a \quad \Rightarrow \quad \rho(r,t) = \left.\frac{\rho_0}{\mathcal{J}}\right|_{a=a(r,t)} .$$

B-Flux (2-form):

$$B \cdot d^2x = B_0 \cdot d^2a \quad \Rightarrow \quad B^i(r,t) = \left.\frac{q^i_{,j} B^j_0}{\mathcal{J}}\right|_{a=a(r,t)} .$$

Kinetic Potential

Kinetic Energy:

$$K[q] = \frac{1}{2}\int_D d^3a\,\rho_0|\dot{q}|^2 = \frac{1}{2}\int_D d^3x\,\rho|v|^2$$

Potential Energy:

$$\begin{aligned} V[q] &= \int_D d^3a\,\rho_0\mathcal{V}(\rho_0/\mathcal{J}, s_0, |q^i_{,j}B^j_0|/\mathcal{J}) = \frac{1}{2}\int_D d^3x\,\rho\mathcal{V}(\rho, s, |B|) \\ &= \int_D d^3a\,\rho_0\mathcal{U}(\rho_0/\mathcal{J}, s_0) - \frac{1}{2}\frac{|q^i_{,j}B^j_0|^2}{\mathcal{J}^2} \end{aligned}$$

Action:

$$S[q] = \int dt\,(K - V)\,, \qquad \delta S = 0 \quad \Rightarrow \quad \text{Ideal MHD}$$

Alternative: Lagrangian variations induce constrained Eulerian variations $\Rightarrow$ Serrin, Newcomb, Euler-Poincaré, ...

Stability: δW, Lagrangian, Eulerian, dynamical accessible, Andreussi, Pegoraro, pjm. (2010 – 2014)

Braginskii MHD

$$\rho\,(v_t + v\cdot\nabla v) = -\nabla p + J\times B + \nabla\cdot\Pi$$

Gyroviscosity Tensor: $\Pi_{ij} = \frac{p}{B} N_{jsik}\frac{\partial v_s}{\partial x_k}$

Action:

$$S[q] = \int dt\,(K + G - V)\,,$$

Gyroscopic Term:

$$G[q] = \int_D d^3a\,\Pi^*\cdot\dot{q} = \int_D d^3x\,M^*\cdot v$$

where

$$\Pi^* = \nabla\times L^* = \frac{m}{2e}\mathcal{J}\,\hat{b}\times\nabla\left(\frac{p}{B}\right)$$

$\delta S[q] = 0 \quad \Rightarrow \quad$ Braginskii MHD

pjm, Lingam, Acevedo, Wurm 2014

Inertial MHD (Tassi)

Basic Idea: Can 'freeze-in' anything one likes! (2-form attribute)

Choose:

$$\mathbf{B}_e = \mathbf{B} + d_e^2 \nabla \times \mathbf{J},$$

Action:

$$S = \int dt \int d^3x \left(\rho \frac{v^2}{2} - \rho U(\rho, s) - \mathbf{B}_e \cdot \mathbf{B} \right).$$

Attributes:

$$\rho d^3x = \rho_0 d^3a, \qquad B_e^i = \frac{B_{e0}^j}{J} \frac{\partial q^i}{\partial a_j}$$

$\delta S[q] = 0 \quad \Rightarrow \quad$ IMHD

Hamiltonian Structure

Legendre Transformation:

$$p = \frac{\delta L}{\delta \dot{q}} \qquad\qquad L \to H$$

Poisson Bracket:

$$\{F, G\} = \int_D da \left(\frac{\delta F}{\delta q^i}\frac{\delta G}{\delta p_i} - \frac{\delta G}{\delta q^i}\frac{\delta F}{\delta p_i} \right)$$

EOM:

$$\dot{q} = \{q, H\} \qquad \dot{p} = \{p, H\}$$

Eulerian Reduction

$$F[q,p] = \widehat{F}[v,\rho,s,B]$$

Chain Rule $\Rightarrow$ yields noncanonical Poisson Bracket in terms of Eulerian variables (pjm & John Greene 1980)

It is an algorithmic process. Manipulations like calculus.

Example: 2D IMHD

$$\{F,G\} = -\int \frac{d^3x}{\rho}(\omega[F_\omega, G_\omega] + \psi_e([F_\omega, G_{\psi_e}] - [G_\omega, F_{\psi_e}]))$$

$$H = \int d^2x(d_e^2(\nabla^2\psi)^2 + |\nabla\psi|^2 + |\nabla\varphi|^2)$$

Produces 2D incompressible IMHD (Ottaviani-Porcelli model)!

Infinite Dimensional Hamiltonian Structure

Field Variables: $\psi(\mu, t)$ e.g. $\mu = x$, $\mu = (x, v)$, . . .

Poisson Bracket:

$$\{A, B\} = \int \frac{\delta A}{\delta \psi} \mathcal{J}(\psi) \frac{\delta A}{\delta \psi} \, d\mu$$

Lie-Poisson Bracket:

$$\{A, B\} = \left\langle \psi, \left[\frac{\delta A}{\delta \psi}, \frac{\delta A}{\delta \psi} \right] \right\rangle$$

Cosymplectic Operator:

$$\mathcal{J} \, \cdot \, \sim [\psi, \, \cdot \,]$$

Form for Eulerian theories: ideal fluids, Vlasov, Liouville eq, BBGKY, gyrokinetic theory, MHD, tokamak reduced fluid models, RMHD, H-M, 4-field model, ITG . . . .

Two-Fluid Action

Keramidas Charidakos, Lingam, pjm, R. White and A. Wurm

$$S[q_s, A, \phi] = \int dt\, \frac{1}{8\pi} \int d^3x\, \left[\left| -\frac{1}{c}\frac{\partial A(x,t)}{\partial t} - \nabla\phi(x,t) \right|^2 \right.$$
$$\left. - |\nabla \times A(x,t)|^2 \right] \tag{1}$$
$$+ \sum_s \int d^3a\, n_{s0}(a) \int d^3x\, \delta\left(x - q_s(a,t)\right)$$
$$\times \left[\frac{e_s}{c}\, \dot{q}_s \cdot A(x,t) - e_s\phi(x,t) \right] \tag{2}$$
$$+ \sum_s \int d^3a\, n_{s0}(a) \left[\frac{m_s}{2}\, |\dot{q}_s|^2 \right.$$
$$\left. - m_s U_s\left(m_s n_{s0}(a)/\mathcal{J}_s, s_{s0}\right) \right]. \tag{3}$$

Eulerian Observables:

$$\{n_\pm, v_\pm, A, \phi\}$$

Reduced Variables

New Lagrangian Variables:

$$Q(a,t) = \frac{1}{\rho_{m0}(a)} \left(m_i n_{i0}(a) q_i(a,t) + m_e n_{e0}(a) q_e(a,t)\right)$$

$$D(a,t) = e\left(n_{i0}(a) q_i(a,t) - n_{e0}(a) q_e(a,t)\right)$$

$$\rho_{m0}(a) = m_i n_{i0}(a) + m_e n_{e0}(a)$$

$$\rho_{q0}(a) = e\left(n_{i0}(a) - n_{e0}(a)\right) \, .$$

Consistent Expansion:

$$\frac{v_A}{c} << 1\,, \quad \frac{m_e}{m_i} << 1 \quad \Rightarrow \quad \text{quasineutrality}$$

Eulerian Closure:

$$\{n, s, s_e, V, J\}$$

Extended MHD

Ohm's Law:

$$E + \frac{V \times B}{c} = \frac{m_e}{e^2 n} \left(\frac{\partial J}{\partial t} + \nabla \cdot (VJ + JV) \right) - \frac{m_e}{e^2 n} (J \cdot \nabla) \left(\frac{J}{n} \right) + \frac{(J \times B)}{enc} - \frac{\nabla p_e}{en} \, .$$

Momentum:

$$nm \left(\frac{\partial V}{\partial t} + (V \cdot \nabla) V \right) \; = \; -\nabla p + \frac{J \times B}{c} - \frac{m_e}{e^2} (J \cdot \nabla) \left(\frac{J}{n} \right) \, .$$

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Consistent with an ordering of Lüst (1958)

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$$- \frac{m_e}{e^2} (J \cdot \nabla) \left(\frac{J}{n} \right) .$$

Consistent with an ordering of Lüst (1958)

Noether → Energy Conservation

Energy:

$$H = \int d^3x \left[\frac{|B|^2}{8\pi} + n\mathfrak{U}_i + n\mathfrak{U}_e + mn\frac{|V|^2}{2} + \frac{m_e}{ne^2}\frac{|J|^2}{2} \right]$$

Energy conservation requires

$$\frac{m_e}{e^2}(J \cdot \nabla)\left(\frac{J}{n}\right)$$

in momentum equation. Otherwise inconsistent.

Physical dissipation is real. Fake dissipation is troublesome, particularly for reconnection studies. Kimura and pjm (2014).

Summary

- Classical Tools
- Building Actions
- Examples