Hamiltonian Description of Matter

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Goal: Discuss MSRI Fall 2018 Program, *Hamiltonian systems, from topology to applications through analysis*, via an excursion through the landscape of Hamiltonian dynamics.

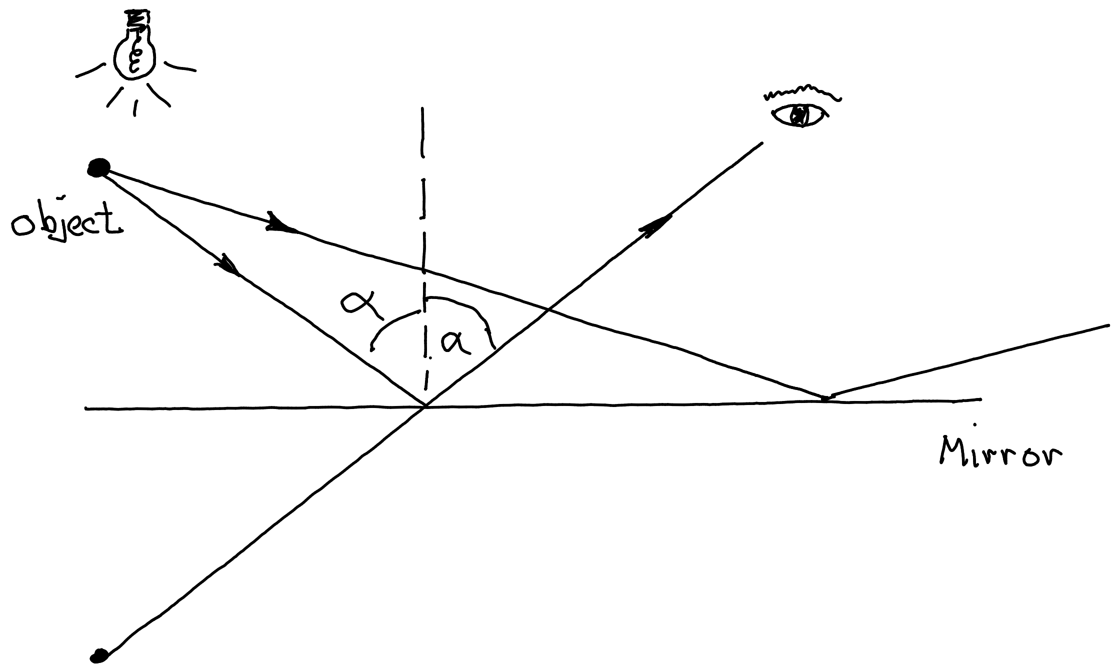
The Program

- Popular and Diverse

- Pure and Applied

Law of Reflection

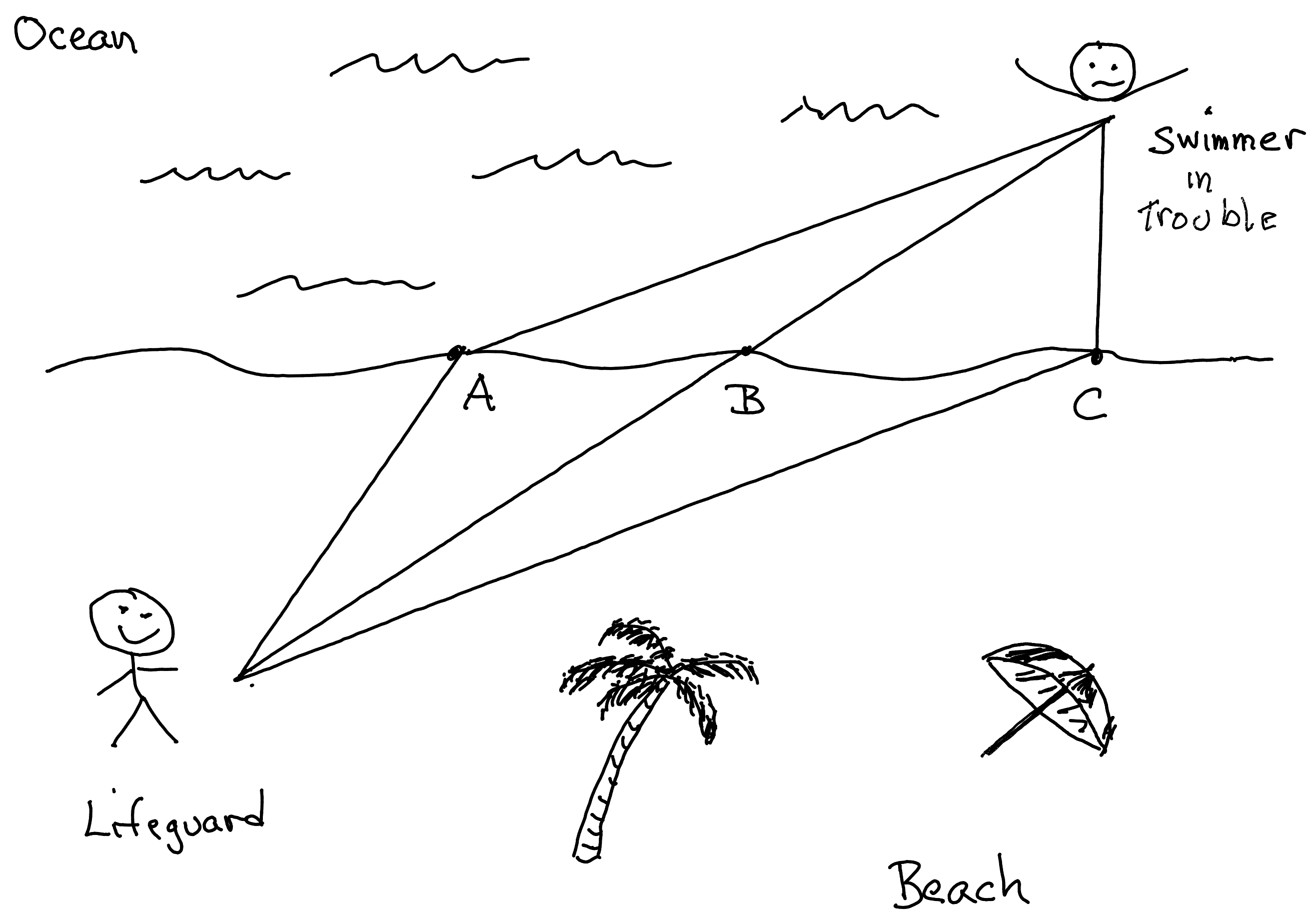
Hero of Alexandria, Greek geometer in his *Catoptrics*, c. 60 AD.



Light takes the shortest path if incident = reflected angle.

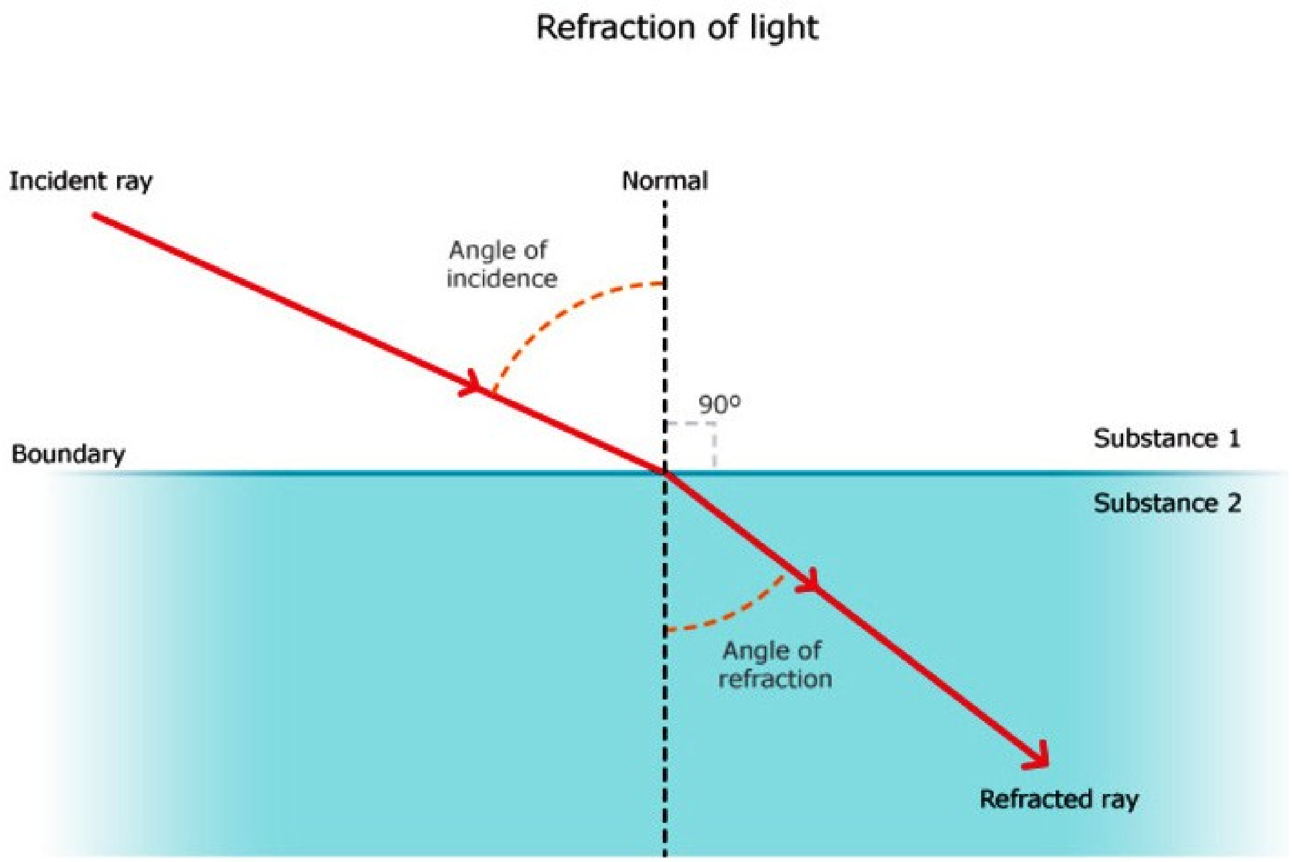
Lifeguard Problem

Goal: Minimize time to save the swimmer!



Snell and Fermat (17th Century)

Path of light minimizes time. Just like lifeguard!



Extremum Principles

Teleology: ancient and medieval philosophy ... explanation of phenomena by the purpose they serve rather than by postulated causes. Nature has goals.

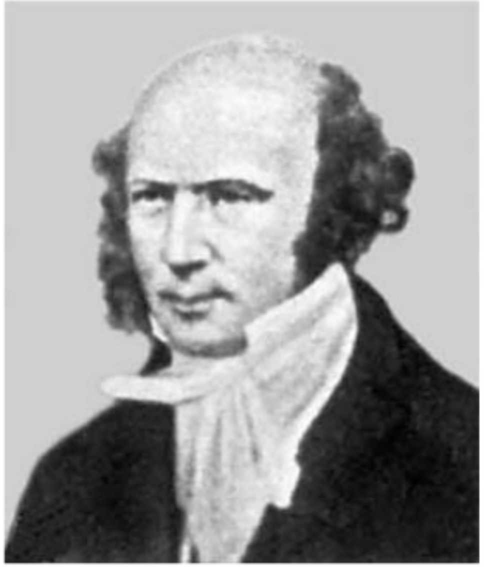
Aristotle: “nature does nothing in vain”

Things strive towards their natural place in the universe: Fire rises to reach the heavens, rocks fall to reach the center of the earth.

17th century Galileo and Descartes $\rightarrow$ Newton’s mathematizable **laws** of motion $\rightarrow$ Fermat, Lagrange, ... **Hamilton’s principle**.

Hamilton's Principle and Equations

William Rowan Hamilton, Irish mathematician
(August 4, 1805 - September 2, 1865)

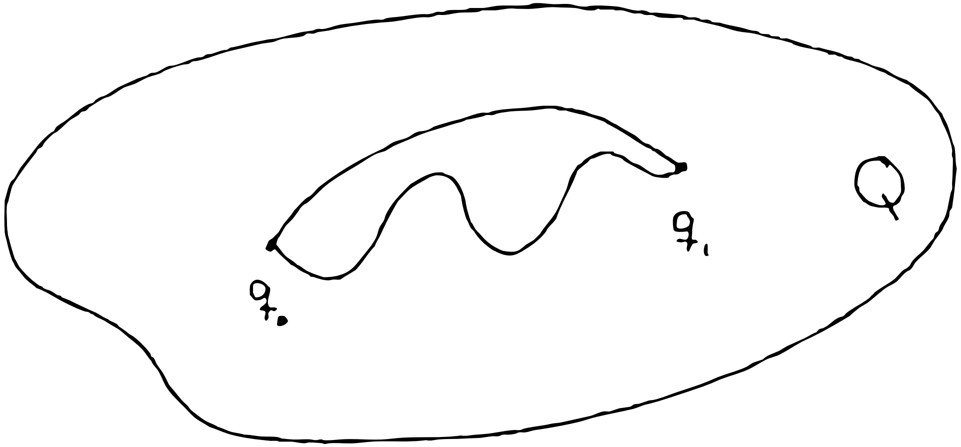


Hamilton in middle age

In 1820s and 1830s he studied both light and particles.

Minimum Paths

Hamilton's Action: $S[q_{\mathsf{path}}] =$ number



Action minimizing path is a solution to special equations,
a special **deterministic dynamical system,**
a Hamiltonian system.

Whence action? Make them up!

Extremization of Actions

Successful Physics Theories (without teleology; with or without being extremal, i.e., min/max. *(added 11/28/18)*)

Examples:

- Newton's Law of Gravitation
- Theory of Electricity and Magnetism, Maxwell's Equations.
- Fluid, Magnetofluid, and Plasma Dynamics
- Quantum Mechanics
- Einstein's Theory of Gravity, General Relativity
- Physics of Elementary Particles

Scales: From microscopic subatomic to the Cosmos

Why Hamiltonian?

- Beauty, $\neg$ Teleology, ...: Still a good reason!
- 20th Century framework for physics: Fluids, Plasmas, etc. too.
- Symmetries and Conservation Laws: energy-momentum ....
- Generality: do one problem $\Rightarrow$ do all.
- Approximation: perturbation theory, averaging, ... 1 function.
- Stability: built-in principle, Lagrange-Dirichlet, δW, ...
- Beacon: $\exists$ ∞-dim KAM theorem? Krein with Cont. Spec.?
- Numerical Methods: structure preserving algorithms: symplectic, conservative, Poisson integrators, ... e.g. GEMPIC.
- Statistical Mechanics: energy, measure ... e.g. absolute equil.

Deterministic Dynamics

Dynamical System: **Arena** plus **Event**

Arena: exact specification of point (position and velocity) determines exactly the future. Usual terminology: state space, phase space, manifold.

Examples: surface of sphere vs. inside a torus. Different kinds of ways to get to same place.

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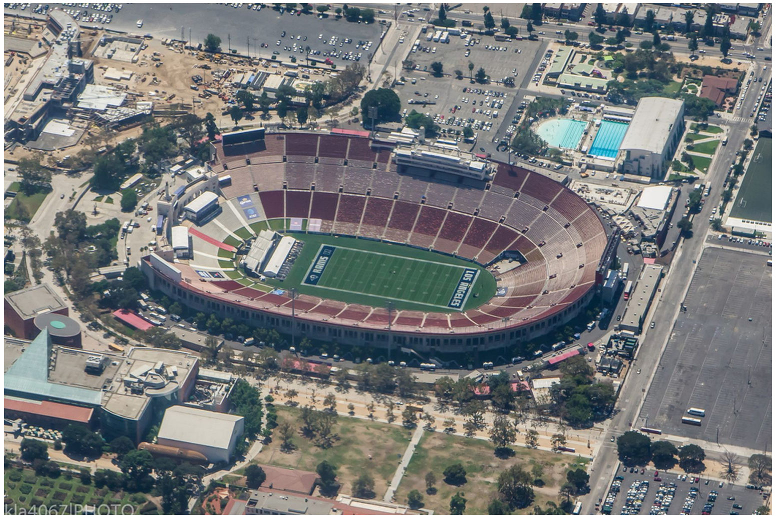

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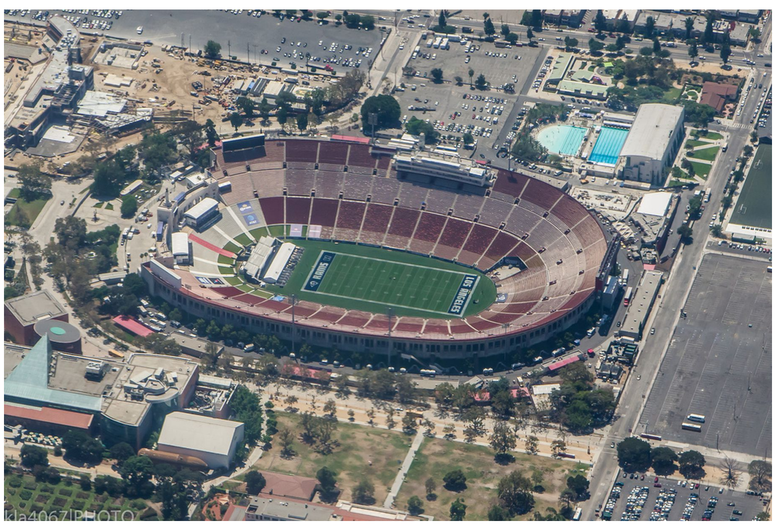

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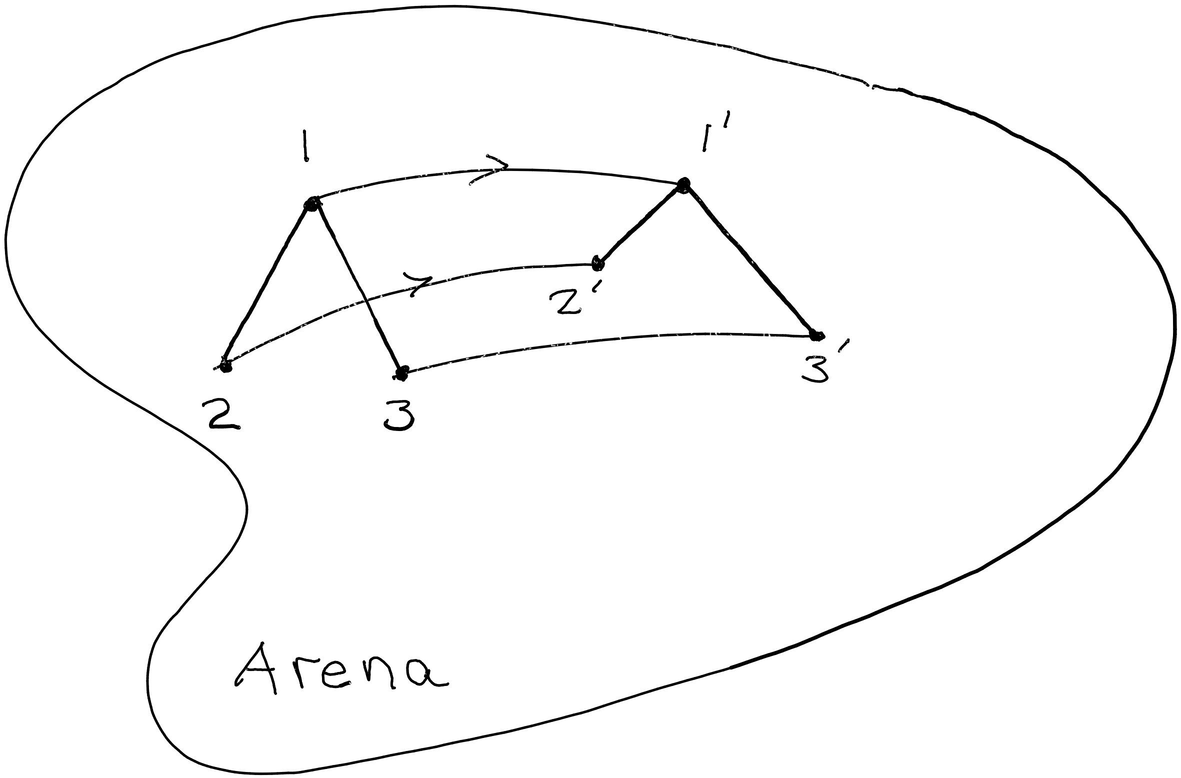
Event: Rules that determine how to go from one place to another as time progresses $\rightarrow$ trajectory or path in phase space. Paths of all points are "flows."

Hamiltonian systems have special arenas and rules, Hamiltonian vector fields on <u>symplectic manifolds</u>. Consequently some things are not possible.

Hamiltonian Flows

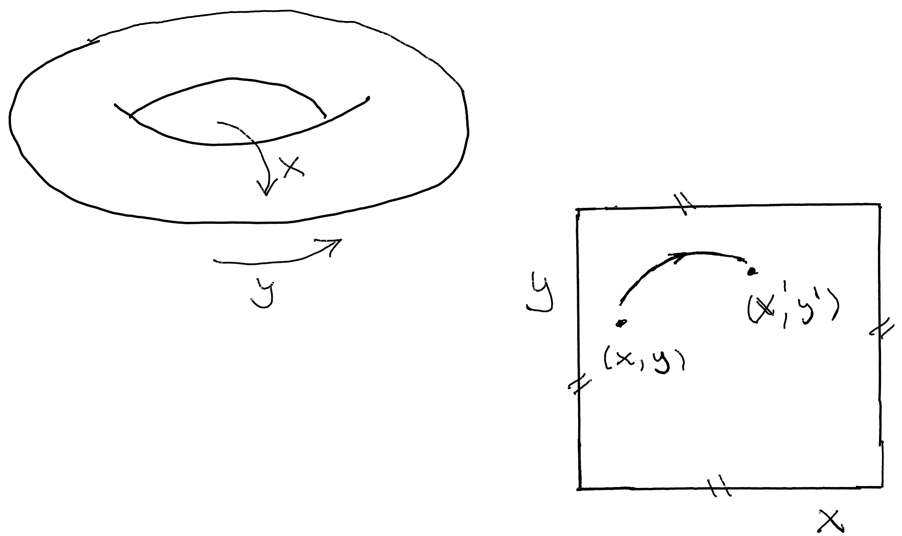
Nearby trajectories at 1, 2, 3 go to 1', 2', 3' in a time interval.



For plane arena **area** is preserved. Any arena, volume is preserved.

A Universal Symplectic Map of Dimension Two

Arena:

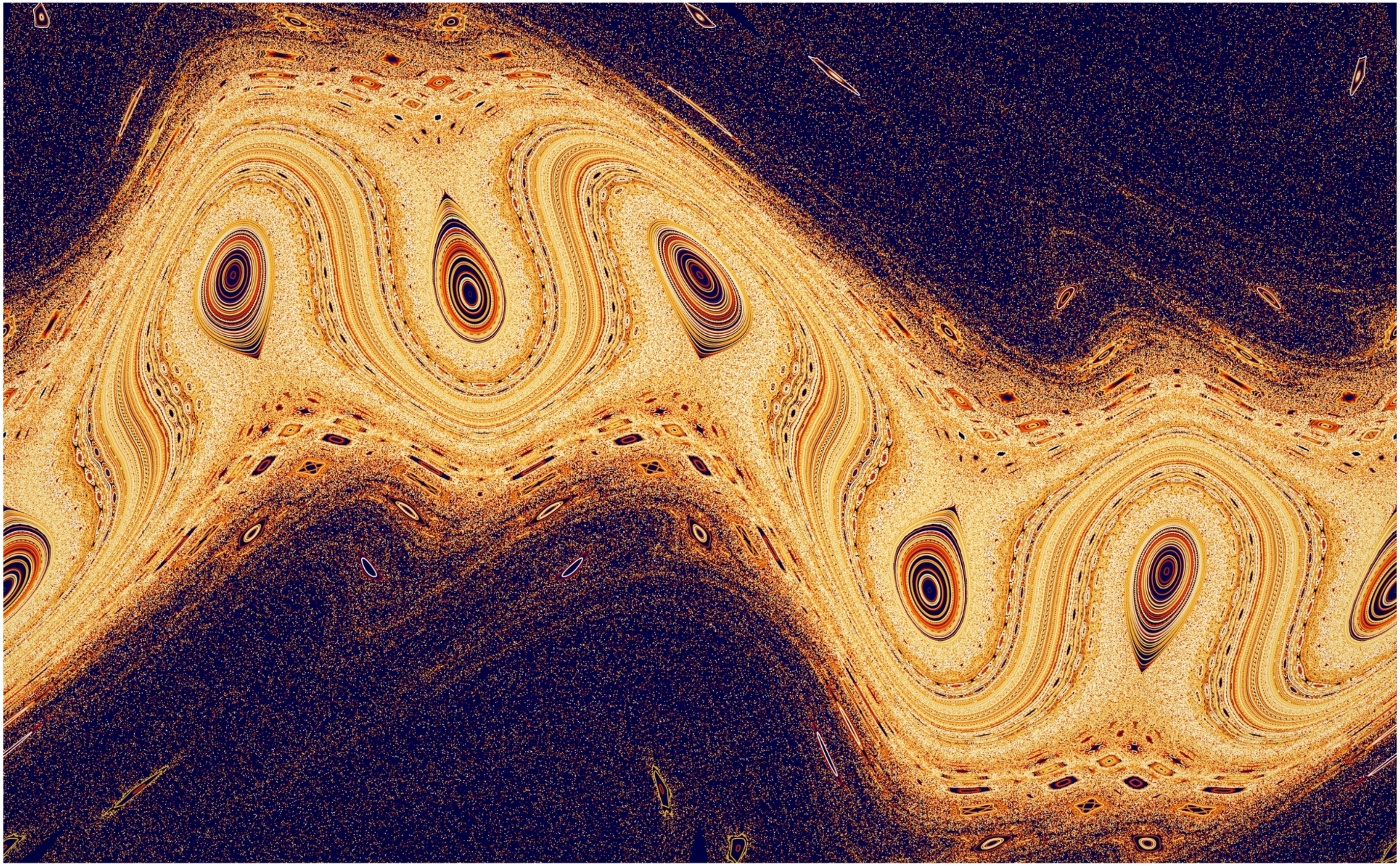


Standard Nontwist Map:

$$
\begin{aligned}
x' &= x + a(1 - y'^2) \\
y' &= y - b\sin(2\pi x)
\end{aligned}
$$

Time jumps in units. Parameters: a shear, b ripple.

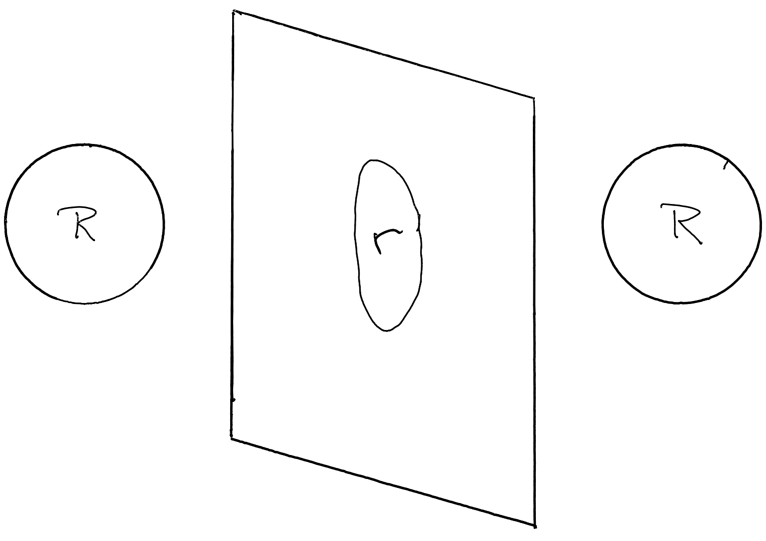
Nontwist Symplectic Maps

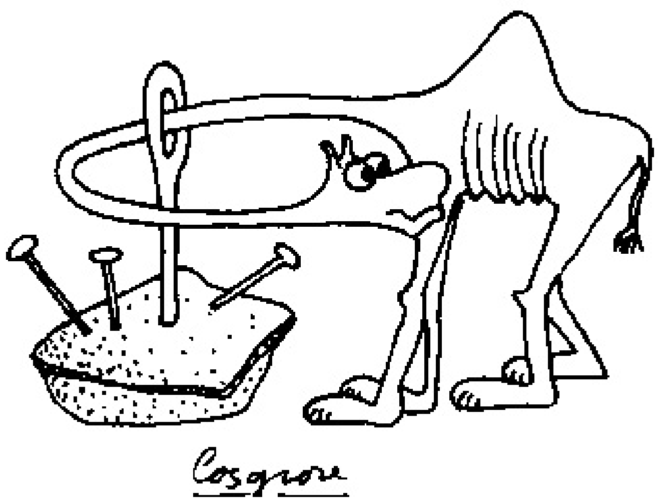


Nonsqueezing Theorem (symplectic camel)

What else? In higher dimension, volume is preserved, but ...

Cartoon:





Can a M-dimensional ball go through a (particular) 2-dimensional hole? Depends on R and r. Squeezing a camel through the eye of a needle. Constraint on Hamiltonian flow.

Applications of Hamiltonian Dynamics

Conventional Accelerators: CERN, Fermilab, ... Higgs.

General Relativity: LIGO, ... gravitational radiation.

Next Generation Accelerators: laser-plasma based, collective effects of many particles handing off enormous energy to few.

Celestial Mechanics: tracking astroids, satellite navigation, Applications to Space Missions, exploration, ...

Magnetic Fusion: ITER experiment in Marseille, France

Parker Solar Probe: Exploration of solar wind source near sun.

Stellar Dynamics: Understanding stars, motions in the cosmos.

Computation: Structure preservation.

Classical Dynamics of Matter

- N**-body problem (dust).** Swarm of interaction particles, like electrons or stars. Variables: positions and momenta.

- **Fluid description** with pressure and possibly electromagnetic or gravitational forces. Variables: velocity field, density, ...

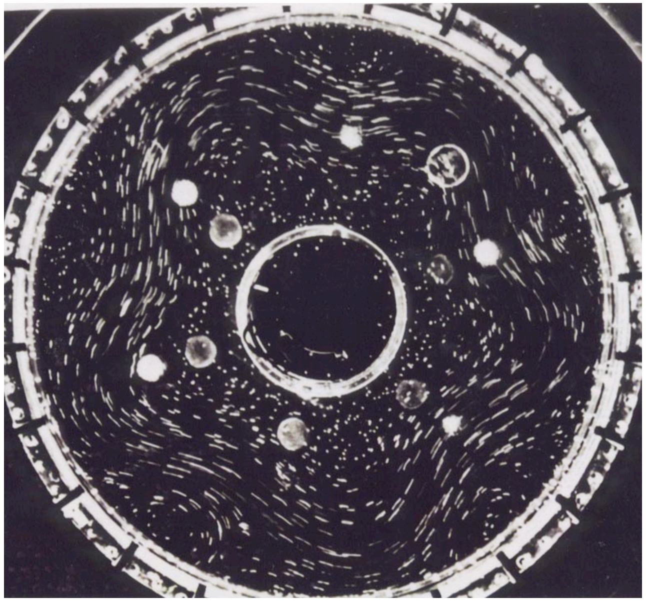
- **Kinetic description**, with gravitational or electromagnetic forces. Variables: phase space probability density, ...

Chaotic Advection - Particles

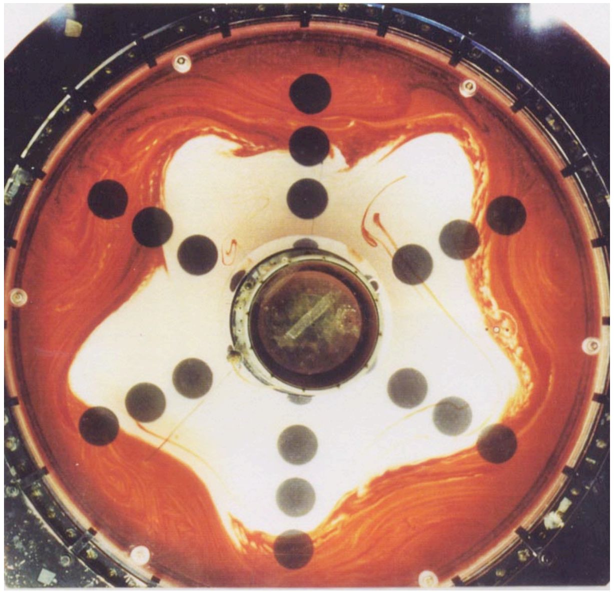
Swinney Lab, del-Castillo-Negrete, et al. Circa 1989.

Cyclonic (eastward) jet

particle streaks



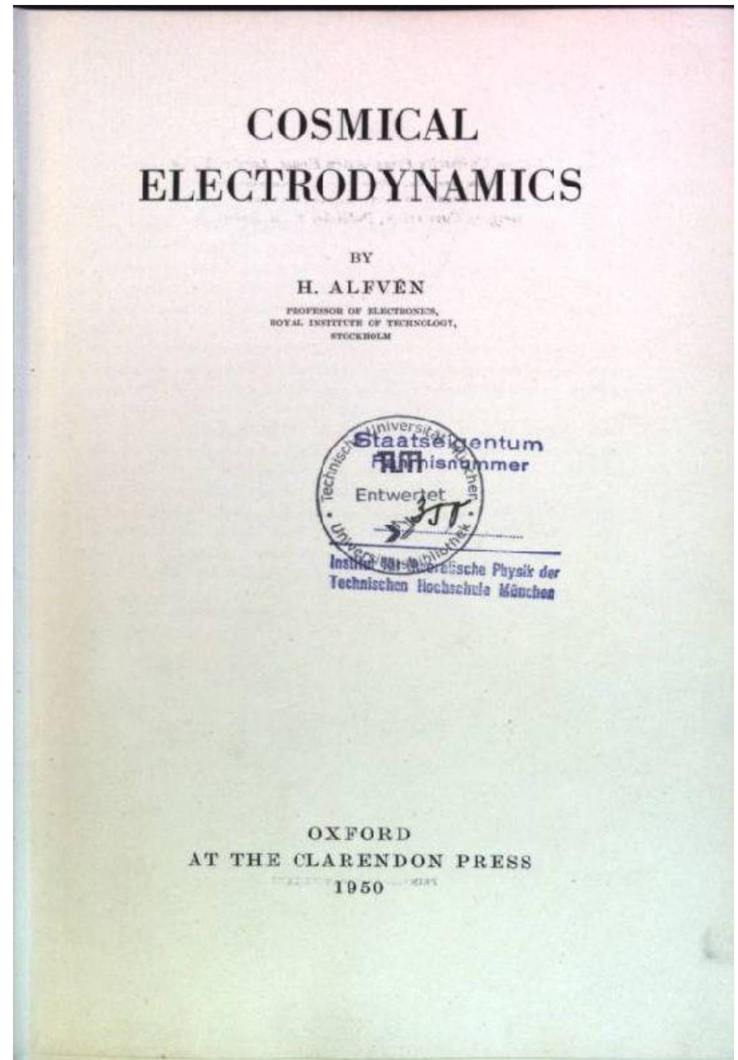
dye



Magnetohyrodynamics (MHD)

Fluid with pressure and magnetic forces.

Hannes Alfvén 1942



COSMICAL
ELECTRODYNAMICS

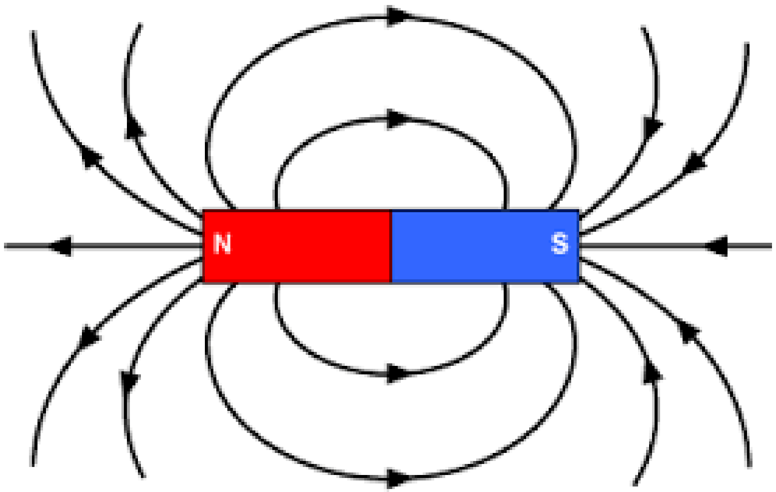
BY

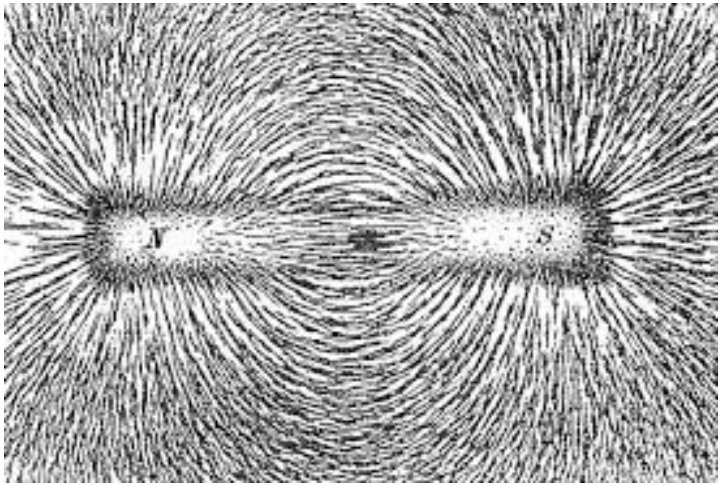
H. ALFVÉN

PROFESSOR OF ELECTRONICS,
ROYAL INSTITUTE OF TECHNOLOGY,
STOCKHOLM

OXFORD
AT THE CLARENDON PRESS
1950

Magnetic Field Lines





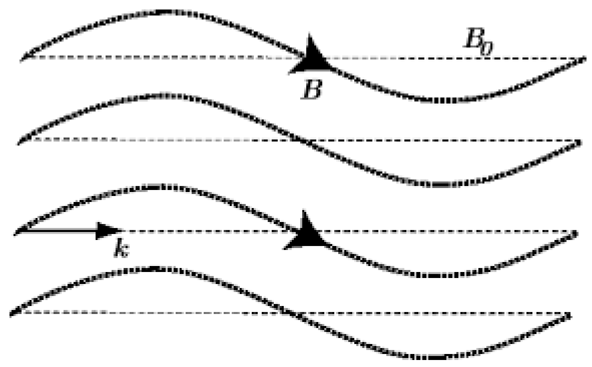
Iron filings like a bunch of little compasses! Trace out $\mathbf{B}$-lines

Currents, moving electrons, make $\mathbf{B}$-lines too. This happens in a magnetofluid such as MHD.

Frozen Flux

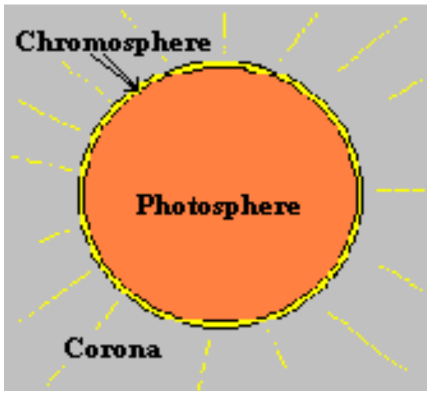
Alfv́en Waves (hydromagnetic waves):

$\boldsymbol{B}_0$ perturbed to $\boldsymbol{B}$
with $\boldsymbol{B}$ "Frozen-In"
i.e. $\boldsymbol{V} \sim \boldsymbol{B} - \boldsymbol{B}_0$.
Observed in Hg by
Lundquist in 1949



Magnetic forces of bits of fluid acting on other bits. Like the flipping of magnets.

Snapping of B-lines (reconnection) causes heating: Parker Solar Probe & solar wind origin? Corona 10^6 deg vs. Photosphere 5×10^3.



Fluid Descriptions

Two Kinds:

- Lagrangian Description
- Eulerian Description

Both exist for MHD and have Hamiltonian description.

Lagrangian Variable Description

Assumes a continuum of fluid particles and follows them around. Canonical variables $\{q(a,t), \pi(a,t)\}$, position and velocity.

Lagrangian Variable Description

A Field Theory!

Eulerian Variable Description

Observables $\{\rho(r,t), \boldsymbol{V}(r,t), \boldsymbol{B}(r,t)\}$ where $r \in D \subset \mathbb{R}^3$ with BC.

Eulerian Variable Description

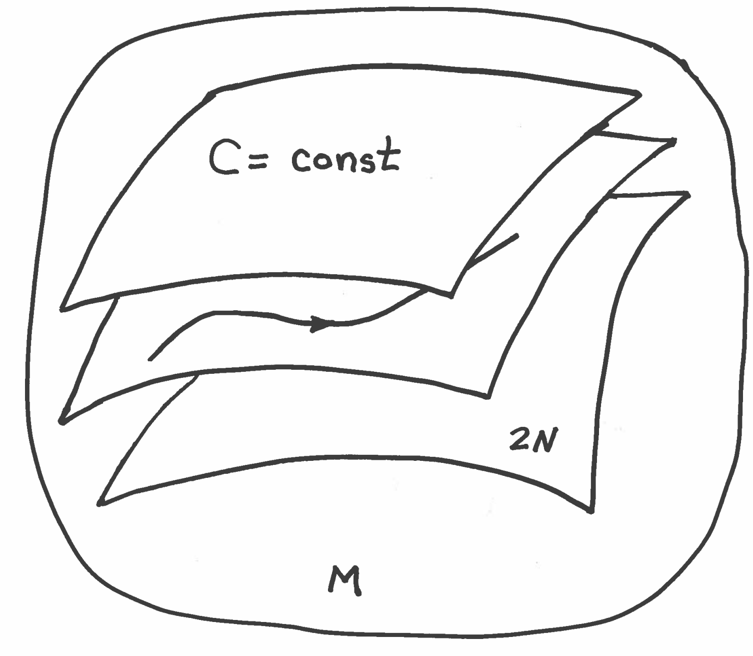
Also a Field Theory.

Eulerian Description Special Structure

Arena is a Poisson Manifold

Poisson Manifold $\mathcal{M}$ Cartoon

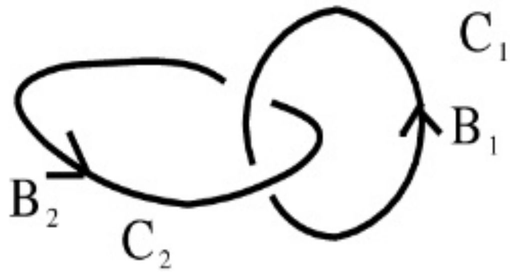
Foliation by symplectic leaves:



Casimir invariants restrict dynamics.

Helicities are MHD Casimir Invariants

Helicity is a “topological” quantity that measures, e.g., linking. where n is the linking number.



Two kinds: $\boldsymbol{B}$-linking and linking of $\boldsymbol{V}$ with $\boldsymbol{B}$, cross helicity.

<u>Relaxation Belief</u>: System minimizes magnetic energy at fixed helicity as a preferred state. Magnetic confinement, solar physics, ...

Back to Teleology?

No. The mechanisms for the relaxation are to be understood.