Magnetic Monopoles: Symmetry Gained is Symmetry Lost†

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Review Dirac's theory and some subsequent work.
Ellucidate a basic 'flaw' in the theory.

† with Jeffrey Heninger.

Maxwell's Equations

$$\frac{\partial \mathbf{B}}{\partial t} = -c\,\nabla \times \mathbf{E} \qquad \frac{\partial \mathbf{E}}{\partial t} = c\,\nabla \times \mathbf{B} - 4\pi \mathbf{J}_e$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 4\pi \rho_e$$

$$\mathbf{E}(\mathbf{x}, t),\ \mathbf{B}(\mathbf{x}, t),\ \rho_e(\mathbf{x}, t),\ \mathbf{J}_e(\mathbf{x}, t)$$

Coupling to Vlasov Matter

$$\frac{\partial f_s}{\partial t} = -\mathbf{v} \cdot \nabla f_s - \frac{e_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \cdot \frac{\partial f_s}{\partial \mathbf{v}}$$

$$\rho_e(\mathbf{x}, t) = \sum_s e_s \int f_s(\mathbf{x}, \mathbf{v}, t)\, d^3v\,, \qquad \mathbf{J}_e(\mathbf{x}, t) = \sum_s e_s \int \mathbf{v}\, f_s(\mathbf{x}, \mathbf{v}, t)\, d^3v$$

$f_s(\mathbf{x}, \mathbf{v}, t)$ is a phase space density for particles of species s with charge and mass, e_s, m_s.

An inclusive matter field theory that includes point particles and fluids as exact reductions.

Dirac's Electrodynamics

$$\frac{\partial \mathbf{B}}{\partial t} = -c\,\nabla \times \mathbf{E} - 4\pi \mathbf{J}_m \qquad \frac{\partial \mathbf{E}}{\partial t} = c\,\nabla \times \mathbf{B} - 4\pi \mathbf{J}_e$$

$$\nabla \cdot \mathbf{B} = 4\pi \rho_m \qquad \nabla \cdot \mathbf{E} = 4\pi \rho_e$$

$$\mathbf{E}(\mathbf{x}, t),\ \mathbf{B}(\mathbf{x}, t),\ \rho_e(\mathbf{x}, t),\ \mathbf{J}_e(\mathbf{x}, t),\ \rho_m(\mathbf{x}, t),\ \mathbf{J}_m(\mathbf{x}, t)$$

Coupling to Monopole Vlasov Matter

$$\frac{\partial f_s}{\partial t} = -\mathbf{v} \cdot \nabla f_s - \left(\frac{e_s}{m_s} \Big(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B} \Big) + \frac{g_s}{m_s} \Big(\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E} \Big) \right) \cdot \frac{\partial f_s}{\partial \mathbf{v}}$$

$$\rho_e(\mathbf{x}, t) = \sum_s e_s \int f_s(\mathbf{x}, \mathbf{v}, t)\, d^3v \,, \qquad \mathbf{J}_e(\mathbf{x}, t) = \sum_s e_s \int \mathbf{v}\, f_s(\mathbf{x}, \mathbf{v}, t)\, d^3v$$

$$\rho_m(\mathbf{x}, t) = \sum_s g_s \int f_s(\mathbf{x}, \mathbf{v}, t)\, d^3v \,, \qquad \mathbf{J}_m(\mathbf{x}, t) = \sum_s g_s \int \mathbf{v}\, f_s(\mathbf{x}, \mathbf{v}, t)\, d^3v$$

Now $f_s(\mathbf{x}, \mathbf{v}, t)$ is a phase space density for particles of species s with electric charge, magnetic charge, and mass, e_s, g_s, m_s, respectively. A species needn't carry either charge.

Some Hamiltonian Landmarks

Dirac (1931):

"... if we wish to put the equations of motion [of electromagnetism] in the Hamiltonian form, however, we have to introduce the electromagnetic potentials ..."

Born and Infeld (1934):

Obtained gauge-free Hamiltonian form for electromagnetism without introducing potentials for light in terms of 'noncanonical' Poisson bracket. Theory symmetric in terms of $\mathbf{E}$ and $\mathbf{B}$.

pjm (1981):

When coupling to Vlasov matter, the constraints $\nabla \cdot \mathbf{E} = 4\pi\rho_e$ and $\nabla \cdot \mathbf{B} = 0$ are not symmetric in the Hamiltonian theory and play different roles. Was seen via a grueling direct calculation of the Jacobi identity (pjm**9** 1982, pjm**165** 2013).

Noncanonical Hamiltonian Definition

A phase space $\mathcal{P}$ (manifold, function space, etc.) with binary bracket operation on functions (functionals) $F, G\colon \mathcal{P} \to \mathbb{R}$ in e.g. $C^\infty(\mathcal{P})$ s.t. $\{\cdot\,,\,\cdot\}\colon C^\infty(\mathcal{P})\times C^\infty(\mathcal{P}) \to C^\infty(\mathcal{P})$, that satisfies

- Bilinear: $\{F + \lambda G, H\} = \{F, H\} + \lambda\{G, H\}\,, \qquad \forall F, G, H$ and $\lambda \in \mathbb{R}$
- Antisymmetric: $\{F, G\} = -\{G, F\}\,, \qquad \forall F, G$
- Jacobi: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} \equiv 0\,, \qquad \forall F, G, H$
- Leibniz: $\{FG, H\} = F\{G, H\} + \{F, H\}G\,, \qquad \forall F, G, H\,.$

Above is a Lie algebra realization on functions. Take FG to be pointwise multiplication.

Equations of Motion: $\frac{\partial \Psi}{\partial t} = \{\Psi, \mathcal{H}\}$ for Ψ an observable and $\mathcal{H}$ a Hamiltonian.

Example: flows on Poisson manifolds, e.g. Weinstein 1983 ....

Maxwell-Vlasov Structure†

Hamiltonian:

$$\mathcal{H} = \sum_s \frac{m_s}{2}\int |\mathbf{v}|^2 f_s \, d^3x \, d^3v + \frac{1}{8\pi}\int (|\mathbf{E}|^2 + |\mathbf{B}|^2) \, d^3x \,,$$

Bracket:

$$\begin{aligned}\{F,G\} \;=\; & \sum_s \int \Big(\frac{1}{m_s} f_s \left(\nabla F_{f_s} \cdot \partial_{\mathbf{v}} G_{f_s} - \nabla G_{f_s} \cdot \partial_{\mathbf{v}} F_{f_s} \right) + \frac{e_s}{m_s^2 c} f_s \, \mathbf{B} \cdot \left(\partial_{\mathbf{v}} F_{f_s} \times \partial_{\mathbf{v}} G_{f_s} \right) \\ & + \frac{4\pi e_s}{m_s} f_s \left(G_{\mathbf{E}} \cdot \partial_{\mathbf{v}} F_{f_s} - F_{\mathbf{E}} \cdot \partial_{\mathbf{v}} G_{f_s} \right) \Big) d^3x \, d^3v + 4\pi c \int (F_{\mathbf{E}} \cdot \nabla \times G_{\mathbf{B}} - G_{\mathbf{E}} \cdot \nabla \times F_{\mathbf{B}}) \, d^3x \,,\end{aligned}$$

where $\partial_{\mathbf{v}} := \partial/\partial\mathbf{v}$, F_{f_s} means functional derivative of F with respect to f_s etc.

Equations of Motion:

$$\frac{\partial f_s}{\partial t} = \{f_s, \mathcal{H}\} \,, \qquad \frac{\partial \mathbf{E}}{\partial t} = \{\mathbf{E}, \mathcal{H}\} \,, \qquad \frac{\partial \mathbf{B}}{\partial t} = \{\mathbf{B}, \mathcal{H}\} \,.$$

† pjm**5,9** 1980,1982; Marsden and Weinstein 1982; Bialynicki-Birula et al. 1984

Maxwell-Vlasov Structure (cont)

Casimirs invariants:

$$
\begin{aligned}
\mathcal{C}_s^f[f_s] &= \int \mathcal{C}_s(f_s)\, d^3x d^3v \\
\mathcal{C}^E[\mathbf{E}, f_s] &= \int h^{\mathbf{E}}(x) \left(\nabla \cdot \mathbf{E} - 4\pi \sum_s e_s \int f_s\, d^3v \right) d^3x\,, \\
\mathcal{C}^B[\mathbf{B}] &= \int h^{\mathbf{B}}(x)\, \nabla \cdot \mathbf{B}\, d^3x\,,
\end{aligned}
$$

where $\mathcal{C}_s$, $h^{\mathbf{E}}$ and $h^{\mathbf{B}}$ are arbitrary functions of their arguments. These satisfy the degeneracy conditions

$$
\{F, C\} = 0 \quad \forall F\,.
$$

Lie's distinguished functions.

Jacobi Identity

The grueling direct calculation (pjm**9** 1982, pjm**165** 2013)

$$\{F,\{G,H\}\}+\{G,\{H,F\}\}+\{H,\{F,G\}\} \;=\; \sum_s \int f_s\,\nabla\cdot\mathbf{B}\,\left(\partial_{\mathbf{v}}F_{f_s}\times\partial_{\mathbf{v}}G_{f_s}\right)\cdot\partial_{\mathbf{v}}H_{f_s}\,d^3x\,d^3v \not\equiv 0\,.$$

Remedy: Lie algebra realization on functionals of closed, but not necessarily exact, 2-forms.

Asymmetry of constraints $\nabla\cdot\mathbf{B}=0$ & $\nabla\cdot\mathbf{E}=4\pi\rho_e$: 1st needed for Jacobi, 2nd not!

Attempts to remove this "taint": Was successful for MHD pjm**7** 1982. Closest for VM using Dirac constaint theory in Chandre et al. 2012, 2013 pjm**150**, pjm**158**.

Surely Dirac's theory should handle monopoles? Bull by the horns!

Monopole-Maxwell-Vlasov Structure†

Bracket:

$$
\begin{aligned}
\{F,G\} \;=\;& \sum_s \int \Big(\frac{1}{m_s} f_s \left(\nabla F_{f_s} \cdot \partial_{\mathbf{v}} G_{f_s} - \nabla G_{f_s} \cdot \partial_{\mathbf{v}} F_{f_s} \right) \\
+\;& \frac{e_s}{m_s^2 c} f_s \, \mathbf{B} \cdot \left(\partial_{\mathbf{v}} F_{f_s} \times \partial_{\mathbf{v}} G_{f_s} \right) - \frac{g_s}{m_s^2 c} f_s \, \mathbf{E} \cdot \left(\partial_{\mathbf{v}} F_{f_s} \times \partial_{\mathbf{v}} G_{f_s} \right) \\
+\;& \frac{4\pi e_s}{m_s} f_s \left(G_{\mathbf{E}} \cdot \partial_{\mathbf{v}} F_{f_s} - F_{\mathbf{E}} \cdot \partial_{\mathbf{v}} G_{f_s} \right) + \frac{4\pi g_s}{m_s} f_s \left(G_{\mathbf{B}} \cdot \partial_{\mathbf{v}} F_{f_s} - F_{\mathbf{B}} \cdot \partial_{\mathbf{v}} G_{f_s} \right) \Big) d^3x \, d^3v \\
+\;& 4\pi c \int \left(F_{\mathbf{E}} \cdot \nabla \times G_{\mathbf{B}} - G_{\mathbf{E}} \cdot \nabla \times F_{\mathbf{B}} \right) d^3x \, .
\end{aligned}
$$

Yields correct equations with same Hamiltonian.

† pjm**165** 2013

Monopole Jacobi Identity

Jacobi:

$$\{F,\{G,H\}\} + \text{cyc} \;=\; \sum_s \frac{1}{m_s^2}\int \partial_{\mathbf{v}} H_{f_s}\cdot\left(\partial_{\mathbf{v}} F_{f_s}\times \partial_{\mathbf{v}} G_{f_s}\right) \times f_s\,(e_s\nabla\cdot\mathbf{B} - g_s\nabla\cdot\mathbf{E})\,d^3x\,d^3v\,.$$

For Jacobi, above mush vanish for arbitrary F, G, H, and $f_s \Rightarrow$

$$e_s\nabla\cdot\mathbf{B} \equiv g_s\nabla\cdot\mathbf{E}\,,\qquad \forall s\,.$$

Monopole Jacobi Identity (cont)

Cases:

- All species have the same e_s/g_s. Duality transformation

$$\mathbf{E}' = \mathbf{E}\cos\xi + \mathbf{B}\sin\xi \quad , \quad \mathbf{B}' = -\mathbf{E}\sin\xi + \mathbf{B}\cos\xi\,,$$
$$e_s' = e_s\cos\xi + g_s\sin\xi \quad , \quad g_s' = -e_s\sin\xi + g_s\cos\xi\,,$$

with $\xi = \arctan(g_s/e_s)$, makes $g_s' = 0\ \forall s$.

> *"The only meaningful question is whether all particles have the same ratio of magnetic to electric charge"* — J. D. Jackson.

If so, by our calculation, Jacobi is satisfied, but the theory is equivalent to Maxwell's.

- Not all species have the same e_s/g_s. Then the Jacobi identity requires $\nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{B} = 0$, which implies there is no matter!

Conclusion:

Dirac's monopole theory with Vlasov matter is either trivial or not a Hamiltonian field theory.

Other Matter Models

Q: Could the problem be in the matter model?

A: No, because MV includes point particles and fluids.

Consider the reduction:

$$f_s = \sum_s \delta\Big(\mathbf{x} - \mathbf{X}_s(t)\Big)\delta\Big(\mathbf{v} - \mathbf{V}_s(t)\Big)$$

A simple chain rule calculation implies point particle Poisson bracket.

Special case of one electron and one monopole.

Hamiltonian:

$$\mathcal{H} = \frac{m_e}{2}\,|\mathbf{V}_e|^2 + \frac{m_m}{2}\,|\mathbf{V}_m|^2 + \frac{1}{8\pi}\int \Big(|\mathbf{E}|^2 + |\mathbf{B}|^2\Big)\, d^3x\,,$$

Other Matter Models (cont)

Bracket:

$$
\begin{aligned}
\{F, G\} \; = \; & \frac{1}{m_e}\left(\frac{\partial F}{\partial \mathbf{X}_e} \cdot \frac{\partial G}{\partial \mathbf{V}_e} - \frac{\partial G}{\partial \mathbf{X}_e} \cdot \frac{\partial F}{\partial \mathbf{V}_e}\right) + \frac{1}{m_m}\left(\frac{\partial F}{\partial \mathbf{X}_m} \cdot \frac{\partial G}{\partial \mathbf{V}_m} - \frac{\partial G}{\partial \mathbf{X}_m} \cdot \frac{\partial F}{\partial \mathbf{V}_m}\right) \\
& + \frac{e}{m_e^2 c} \mathbf{B}(\mathbf{X}_e) \cdot \left(\frac{\partial F}{\partial \mathbf{V}_e} \times \frac{\partial G}{\partial \mathbf{V}_e}\right) - \frac{g}{m_e^2 c} \mathbf{E}(\mathbf{X}_m) \cdot \left(\frac{\partial F}{\partial \mathbf{V}_m} \times \frac{\partial G}{\partial \mathbf{V}_m}\right) \\
& + \frac{4\pi e}{m_e}\left(\left.\frac{\delta G}{\delta \mathbf{E}}\right|_{\mathbf{X}_e} \cdot \frac{\partial F}{\partial \mathbf{V}_e} - \left.\frac{\delta F}{\delta \mathbf{E}}\right|_{\mathbf{X}_e} \cdot \frac{\partial G}{\partial \mathbf{V}_e}\right) + \frac{4\pi g}{m_m}\left(\left.\frac{\delta G}{\delta \mathbf{B}}\right|_{\mathbf{X}_m} \cdot \frac{\partial F}{\partial \mathbf{V}_m} - \left.\frac{\delta F}{\delta \mathbf{B}}\right|_{\mathbf{X}_m} \cdot \frac{\partial G}{\partial \mathbf{V}_m}\right), \\
& + 4\pi c \int \left(\frac{\delta F}{\delta \mathbf{E}} \cdot \nabla \times \frac{\delta G}{\delta \mathbf{B}} - \frac{\delta G}{\delta \mathbf{E}} \cdot \nabla \times \frac{\delta F}{\delta \mathbf{B}}\right) d^3x
\end{aligned}
$$

Other Matter Models (cont^2)

This Hamiltonian and Poisson bracket give the expected equations of motion:

$$
\begin{aligned}
\frac{\partial \mathbf{X}_e}{\partial t} &= \mathbf{V}_e \\
\frac{\partial \mathbf{V}_e}{\partial t} &= \frac{e}{m_e}\mathbf{E}(\mathbf{X}_e) + \frac{e}{m_e c}\mathbf{V}_e \times \mathbf{B}(\mathbf{X}_e)\,, \\
\frac{\partial \mathbf{X}_m}{\partial t} &= \mathbf{V}_m \\
\frac{\partial \mathbf{V}_m}{\partial t} &= \frac{g}{m_m}\mathbf{B}(\mathbf{X}_m) - \frac{g}{m_m c}\mathbf{V}_m \times \mathbf{E}(\mathbf{X}_m)\,, \\
\frac{\partial \mathbf{E}}{\partial t} &= c\nabla \times \mathbf{B} - 4\pi e\,\mathbf{V}_e\,\delta(x - \mathbf{X}_e)\,, \\
\frac{\partial \mathbf{B}}{\partial t} &= -c\nabla \times \mathbf{E} - 4\pi g\,\mathbf{V}_m\,\delta(x - \mathbf{X}_m)\,.
\end{aligned}
$$

Other Matter Models (cont^3)

Jacobi:

$$\{\{F,G\},H\} \;+\; \text{cyc} = \frac{12\pi eg}{c}\,\delta(\mathbf{X}_e - \mathbf{X}_m)$$
$$\times\left(\frac{1}{m_e^3}\frac{\partial F}{\partial \mathbf{V}_e}\left(\frac{\partial G}{\partial \mathbf{V}_e}\times\frac{\partial H}{\partial \mathbf{V}_e}\right) - \frac{1}{m_m^3}\frac{\partial F}{\partial \mathbf{V}_m}\cdot\left(\frac{\partial G}{\partial \mathbf{V}_m}\times\frac{\partial H}{\partial \mathbf{V}_m}\right)\right).$$

Jacobi identity is not satisfied globally; $\exists$ a singularity when positions coincide.

Classically, there is no reason why this coincidence can't happen. A stationary monopole produces a completely radial $\mathbf{B}$-field. An electron moving directly towards the monopole experiences a force $e\mathbf{V}_e \times \mathbf{B}/c = 0$.

$\therefore$ the electron passes through the monopole without experiencing any force at all!

Conclusion:

Interaction of an electron and a magnetic monopole is not Hamiltonian

Conclusions

• Maxwell's equations are special. The constraints $\nabla \cdot \mathbf{B} = 0$ and $\nabla \cdot \mathbf{E} = 4\pi\rho_e$ play different roles in Hamiltonian theory, which is violated if they are forced to look alike.

• Various geometrical, topological, operator algebras, etc. Dirac (1931,1948) quantization assumptions, Dirac string, defects, wave function conditions, etc. to dress up or bypass this basic flaw. (For GUTs, etc. too.)

• Various experimental attempts for finding monopoles have lead to zilch.

• Opinion: Scrap it! Maxwell's equations are special and maybe it is not a good idea to build new theories based on changing them.