Stability Subject to Dynamical Accessibility

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Hamilton's Equations

Phase Space Coordinates: $(q, p) = (q^1, q^2, \ldots, q^N; p_1, p_2, \ldots, p_N)$,

Hamiltonian: $H(q, p)$ s.t. phase space $\mapsto \mathbb{R}$

Hamiltons Equations:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i}\,, \qquad \dot{q}^i = \frac{\partial H}{\partial p_i}\,,} \qquad i = 1, 2, \ldots, N$$

Equivalent Form $z = (q, p)$:

$$\dot{z}^\alpha = J_c^{\alpha\beta}\frac{\partial H}{\partial z^\beta} =: \{z^\alpha, H\}\,, \qquad (J_c^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}\,,$$

$$\alpha, \beta = 1, 2, \ldots, 2N$$

Geometry

Dynamics takes place in phase space, $\mathcal{Z}$ (needn't be T^*Q), a differential manifold endowed with a closed, nondegenerate 2-form ω. A patch has canonical coordinates $z = (q, p)$.

Hamiltonian dynamics $\Leftrightarrow$ flow on symplectic manifold: $i_X \omega = dH$

Poisson tensor (J_c) is bivector inverse of ω, defining the Poisson bracket

$$\{f, g\} = \langle df, J_c(dg)\rangle = \omega(X_f, X_g) = \frac{\partial f}{\partial z^\alpha} J_c^{\alpha\beta} \frac{\partial g}{\partial z^\beta}\,, \quad \alpha, \beta = 1, 2, \ldots 2N$$

Flows generated by Hamiltonian vector fields $Z_H = JdH$, H a 0-form, dH a 1-form. Poisson bracket = commutator of Hamiltonian vector fields etc.

Early refs.: Jost, Mackey, Souriau, Arnold, Abraham &Marsden

Noncanonical Hamiltonian Dynamics

Sophus Lie (1890)

Noncanonical Coordinates:

$$\dot{z}^a = J^{ab}\frac{\partial H}{\partial z^b} = \{z^a, H\}\,, \qquad \{f,g\} = \frac{\partial f}{\partial z^a} J^{ab}(z) \frac{\partial g}{\partial z^b}\,, \qquad a,b = 1,2,\ldots M$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{f,g\} = -\{g,f\}\,,$

Jacobi identity $\longrightarrow$ $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

G. Darboux: $detJ \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $detJ = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

$$J \to J_d = \begin{pmatrix} 0_N & I_N & 0 \\ -I_N & 0_N & 0 \\ 0 & 0 & 0_{M-2N} \end{pmatrix}.$$

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{M}$ is differentiable manifold with bracket $\{\,,\,\} : C^\infty(\mathcal{M}) \times C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ st $C^\infty(\mathcal{M})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, $Z_H = JdH$.

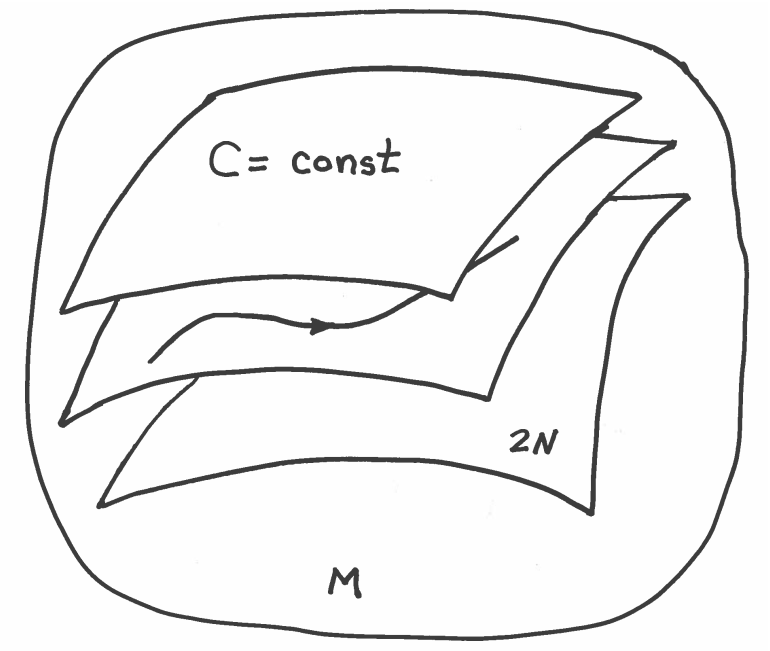
Because of degeneracy, $\exists$ functions C st $\{f, C\} = 0$ for all $f \in C^\infty(\mathcal{M})$. Called Casimir invariants (Lie's distinguished functions.)

Poisson Manifold $\mathcal{M}$ Cartoon

Degeneracy in $J \Rightarrow$ Casimirs:

$$\{f, C\} = 0 \quad \forall \ f : \mathcal{M} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Leaf vector fields, $Z_f = \{z, f\} = J df$ are tangent to leaves.

Lie-Poisson Brackets

Matter models in Eulerian variables:

$$J^{ab} = c_c^{ab} z^c$$

where c_c^{ab} are the structure constants for some Lie algebra.

Examples:

- 3-dimensional Bianchi algebras for free rigid body, Kida vortex, rattleback. (cf. Tokeida, Yoshida)

- Infinite-dimensional theories: Ideal fluid flow, **MHD**, shearflow, extended MHD, Vlasov-Maxwell, etc.

Lie-Poisson Geometry

Lie Algebra: $\mathfrak{g}$, a vector space with

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\,,$$

antisymmetric, bilinear, satisfies Jacobi identity

Pairing:

$$\langle\ \ ,\ \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$$

with $\mathfrak{g}^*$ vector space dual to $\mathfrak{g}$

Lie-Poisson Bracket:

$$\{f, g\} = \left\langle z, \left[\frac{\partial f}{\partial z}, \frac{\partial g}{\partial z}\right]\right\rangle\,, \qquad z \in \mathfrak{g}^*,\ \frac{\partial f}{\partial z} \in \mathfrak{g}$$

Magnetohydrodynamics

Equations of Motion:

$$\begin{aligned}
\text{Force} \qquad \rho\frac{\partial \boldsymbol{v}}{\partial t} &= -\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B} \\
\text{Density} \qquad \frac{\partial \rho}{\partial t} &= -\nabla\cdot(\rho\boldsymbol{v}) \\
\text{Entropy} \qquad \frac{\partial s}{\partial t} &= -\boldsymbol{v}\cdot\nabla s \\
\text{Magnetic Field} \qquad \frac{\partial \boldsymbol{B}}{\partial t} &= -\nabla\times\boldsymbol{E} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})
\end{aligned}$$

Energy:

$$H = \int_D d^3x \left(\rho|\boldsymbol{v}|^2/2 + \rho U(\rho, s) + |\boldsymbol{B}|^2/2\right)$$

Thermodynamics:

$$p = \rho^2\frac{\partial U}{\partial \rho} \qquad\qquad T = \frac{\partial U}{\partial s}$$

Noncanonical Lie-Poisson Bracket (pjm & Greene 1980):

$$
\begin{aligned}
\{F,G\} \;=\; & -\int_D d^3x \Bigg[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \\
+ \; & \rho \left(\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left(\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
+ \; & B \cdot \left[\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{B}} \right] \\
+ \; & + \boldsymbol{B} \cdot \left[\nabla \left(\frac{\delta F}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} - \nabla \left(\frac{\delta G}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta F}{\delta \boldsymbol{B}} \right] \Bigg] ,
\end{aligned}
$$

Dynamics:

$$
\frac{\partial \rho}{\partial t} = \{\rho, H\}\,, \quad \frac{\partial s}{\partial t} = \{s, H\}\,, \quad \frac{\partial \boldsymbol{v}}{\partial t} = \{\boldsymbol{v}, H\}\,, \text{ and } \quad \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}\,.
$$

Densities:

$$
\boldsymbol{M} := \rho \boldsymbol{v} \qquad\qquad \sigma := \rho s
$$

Hamiltonian Description of Idea MHD

Lagrangian fluid variable description is a continuum version of particle mechanics and, consequently, canonical, while Eulerian fluid variables are noncanonical variables. Lagrange to Euler is a reduction.

Eulerian Variable Description

Eulerian Variable Description

Observables $\{s(r,t), \rho(r,t), \boldsymbol{v}(r,t), \boldsymbol{B}(r,t)\}$ where $r \in D \subset \mathbb{R}^3$.

Eulerian Variable Description

Observables $\{s(r,t)\rho(r,t), \boldsymbol{v}(r,t), \boldsymbol{B}(r,t)\}$ constitue a Field Theory.

Lagrangian Variable Description

Assume a continuum of fluid particles or fluid elements and follow them around.

Lagrange *Mécanique Analytique* (1788) $\rightarrow$ Newcomb (1962) MHD

Lagrangian Variable Description

Lagrangian Variable Description

Dynamical canonical variables $\{q(a,t), \pi(a,t)\}$.

Stability in General Finite Systems

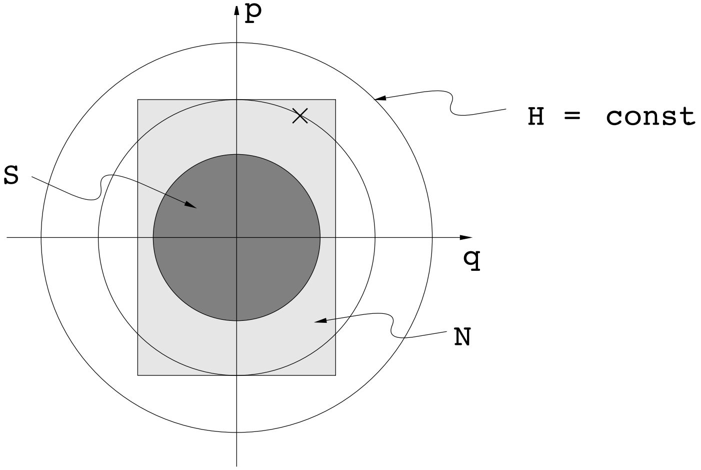
General system:

$$\dot{z}^i = V^i(z)\,, \qquad\qquad i = 1, 2, \ldots, M\,,$$

Equilibrium point, z_e, satisfies $V(z_e) = 0$.

Definition:

The equilibrium point z_e is said to be stable if, for any neighborhood N of z_e there exists a subneighborhood $S \subset N$ of z_e such that if $z(t = 0) \in S$ then $z(t) \in N$ for all time $t > 0$.



Stability in Canonical Hamiltonian Systems

Lagrange's Theorem:

For separable Hamiltonians, $H = p^2/2 + V(q)$*, an equilibrium point* $p_e = 0$ *and* q_e *a local minimum of* V *is stable.*

Converse?

Dirichlet's Theorem:

For general Hamiltonians, $H(q,p)$*, an equilibrium point* $\nabla H(q_e, p_e) = 0$ *is stable if* $\partial^2 H(z_e)/\partial z^i \partial z^j$ *is definite.*

Note, H could an energy minimum or a maximum, e.g. for a localized vortex in fluid mechanics.

Converse?

MHD δW Energy Principla

- Standard tool of MHD for stability. Bernstein et al.; Hain et al.

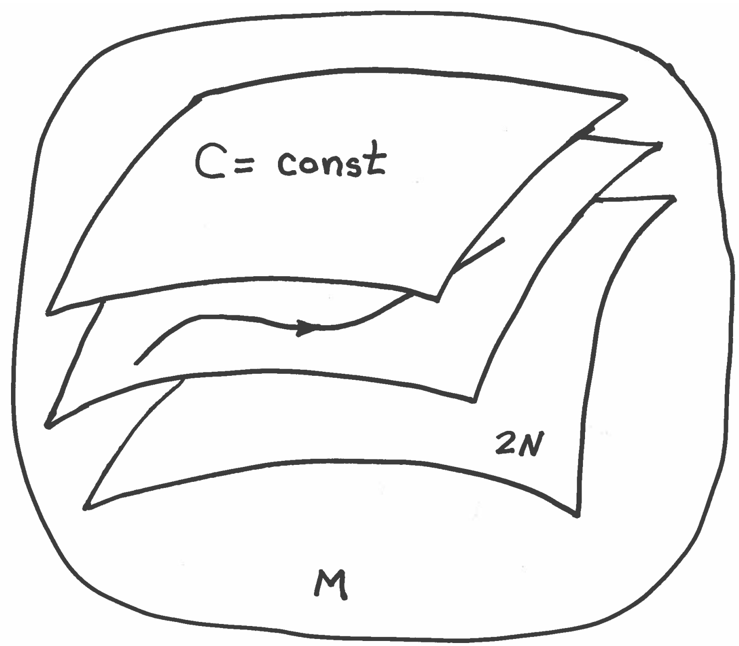
- Infinite-dimensional version of Lagrange's "necessary and sufficient" theorem.

- Flow in equilbrium considered by Frieman & Rotenberg. Sufficient condition is a version of Dirichlet's theorem.

Stability in Noncanonical Hamiltonian Systems

What do Casimirs do?

What does the cartoon geometry imply?



Stability in Noncanonical Hamiltonian Systems

I. Energy-Casimir Stability: In noncanonical variables w

$$\dot{w}^\alpha = \{w^\alpha, H\} = \{w^\alpha, H + C\} = J^{\alpha\beta}(w)\frac{\partial F}{\partial w^\beta} = 0$$

Equilibrium variational principle:

$$\delta F = \delta(H + C) = 0$$

Linear Hamiltonian:

$$\delta^2 F = \frac{\partial^2 F(w_e)}{\partial w^\alpha \partial w \beta}\,\delta w^\alpha \delta w^\beta = H_L \quad \text{definite} \Rightarrow \text{stability}$$

Kruskal-Oberman (1958), Gardner, Arnold, Marsden, pjm, etc.

II. Dynamically Accessible Stability: Constrained variations

$$\delta w^\alpha = J^{\alpha\beta}(w_e) g_\beta \quad \rightarrow \quad \delta^2 F|_{da} \quad \text{definite} \Rightarrow \text{stability}$$

pjm & Pfirsch (1989)

MHD Dynamically Stability

Generator:

$$G = \int x \left(\mathbf{g}_1 \cdot \mathbf{M} + g_2\sigma + g_3\rho + \mathbf{g}_4 \cdot \mathbf{B} \right)$$

where $\boldsymbol{M} = \rho\boldsymbol{v}$ and $\sigma = \rho s$. Then $\{\chi, G\} \Rightarrow$

$$\begin{aligned}
\delta\rho_{\text{da}} &= \nabla \cdot (\rho\mathbf{g}_1) \\
\delta\mathbf{M}_{\text{da}} &= \rho\nabla g_3 + (\nabla \times \mathbf{M}) \times \mathbf{g}_1 + \mathbf{M}\nabla \cdot \mathbf{g}_1 \\
&\quad + \nabla\left(\mathbf{M} \cdot \mathbf{g}_1\right) + \sigma\nabla g_2 + \mathbf{B} \times (\nabla \times \mathbf{g}_4) \\
\delta\sigma_{\text{da}} &= \nabla \cdot (\sigma\mathbf{g}_1) \\
\delta\mathbf{B}_{\text{da}} &= \nabla \times (\mathbf{B} \times \mathbf{g}_1)
\end{aligned}$$

Energy Functional:

$$\delta^2 H_{\text{da}}[\mathbf{g}] = \int d^3x\, \rho \left| \delta\mathbf{v}_{\text{da}} - \mathbf{g}_1 \cdot \nabla\mathbf{v} + \mathbf{v} \cdot \nabla\mathbf{g}_1 \right|^2 + \delta W_{\text{la}}[\mathbf{g}_1]$$

with

$$\delta\mathbf{v}_{\text{da}} = \nabla\left(g_3 + \mathbf{g}_1 \cdot \mathbf{v}\right) - \mathbf{g}_1 \times (\nabla \times \mathbf{v}) + \frac{\sigma}{\rho}\nabla g_2 + \frac{1}{\rho}\mathbf{B} \times (\nabla \times \mathbf{g}_4)$$

Hameiri: In certain relevant circumstances $\int d^3x\, \rho|...|^2$ stabelizes!