Noncanonical Poisson, Dirac, and Nambu Brackets for Classical Fluid and Plasma Dynamics and Quantum Mechanics

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Memories

Yoichiro Nambu a friend to University of Texas at Austin Physics Department.

Visits to Austin:

- 1970. An international Symposium: The Past Decade in Particle Theory
- 1991. Workshop in Honor of E.C.G. Sudarshan's Contributions to Theoretical Physics
- 2006. Sudarshan's 75th Birthday Celebration

1970 Workshop Press Release

UT NEWS

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RELEASE ON RECEIPT

AUSTIN, Texas (Spl.)--An international symposium, "The Past Decade in Particle Theory," will be held at The University of Texas at Austin Tuesday through Friday (April 14-17).

Dr. Harry Ransom, Chancellor of the UT System, will open the meeting at 8:45 a.m. Tuesday in the Texas Union Auditorium.

Sponsored by the UT Physics Department's Center for Particle Theory, the meeting is expected to attract more than 100 internationally known scientists, including several Nobel laureates.

1970 Workshop Press Release (cont)

The Particle Theory Center, founded only 18 months ago at UT Austin, has already achieved a national reputation for its work.

"The Center has become one of the most active areas in research in particle physics in the United States," says Dr. Yoichiro Nambu of the University of Chicago, one of the Tuesday speakers. "It is known both nationally and internationally through the importance of the work of Drs. Ne'eman and Sudarshan, as well as the recent contributions of its other members to almost every important aspect of particle physics," he adds.

1970 Workshop (cont^2)

SYMMETRY BREAKDOWN AND SMALL MASS BOSONS*

Yoichiro Nambu

THE ENRICO FERMI INSTITUTE AND DEPARTMENT OF PHYSICS
THE UNIVERSITY OF CHICAGO

Abstract: In a talk presented at the symposium "The Past Decade In Particle Theory", Prof. Y. Nambu reviews the meaning and mechanism for spontaneous breakdown of a symmetry. After reviewing the history of the Goldstone boson, he describes the relationship of the Goldstone theories to PCAC, Current Algebra, and soft pion limits. These problems are unified in the approach of utilizing the non-linear relizations of the symmetries. A discussion of Prof. Nambu's talk follows this paper.

1991 Workshop

Upcoming Events

Workshop in Honor of
E. C. G. Sudarshan's 60th Birthday
and his Contributions to Theoretical Physics

September 15, 16, & 17, 1991

PLENARY SESSION SPEAKERS:

Modern Optics
E. Wolf, W. Lamb, & J. Kimble

Classical and Quantum Mechanics:
I. Prigogine, C. Teitelboim & J. Klauder

Particle Phenomenology:
Y. Nambu, S. Glashow & F. Gilman

Symmetries
L. O'Raifeartaigh, S. Weinberg & P. Ramond

Contributed Sessions:
Classical and Quantum Mechanics and Particle Phenomenology in Parallel
Quantum Optics and Symmetry in Parallel

1991 Workshop (cont)

Physics Letters A 158 (1991) 453–457
North-Holland

PHYSICS LETTERS A

Quantum mechanics as a generalization of Nambu dynamics to the Weyl–Wigner formalism

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2006 Reception

Yoichiro Nambu

2006 Reception (cont)

Iwo Bialynicki-Birula

E. C. G. Sudarshan

Yoichiro Nambu

Overview

I. Noncanonical Hamiltonian Dynamics

II. Lie-Algebra generalization of Nambu Dynamics

III. Generalized Nambu Brackets & Weyl-Wigner Quantum Mechanics

IV. Incompressible Fluid: Lagrange's volume preserving diffeomorphisms, geodesics, and Dirac brackets

I. Noncanonical Hamiltonian Dynamics

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980)....

Noncanonical Coordinates:

$$\dot{w}^i = J^{ij}\frac{\partial H}{\partial w^j} = \{w^j, H\}\,, \qquad\qquad \{A,B\} = \frac{\partial A}{\partial w^i} J^{ij}(w) \frac{\partial B}{\partial w^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{A,B\} = -\{B,A\}\,,$

Jacobi identity $\longrightarrow$ $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$

G. Darboux: $det J \neq 0 \Longrightarrow J \rightarrow J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

Matter models in Eulerian variables: $J^{ij} = c_k^{ij} w^k$ $\leftarrow$ Lie – Poisson Brackets

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\,,\,\}: C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

i) bilinear,
ii) antisymmetric,
iii) Jacobi, and
iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

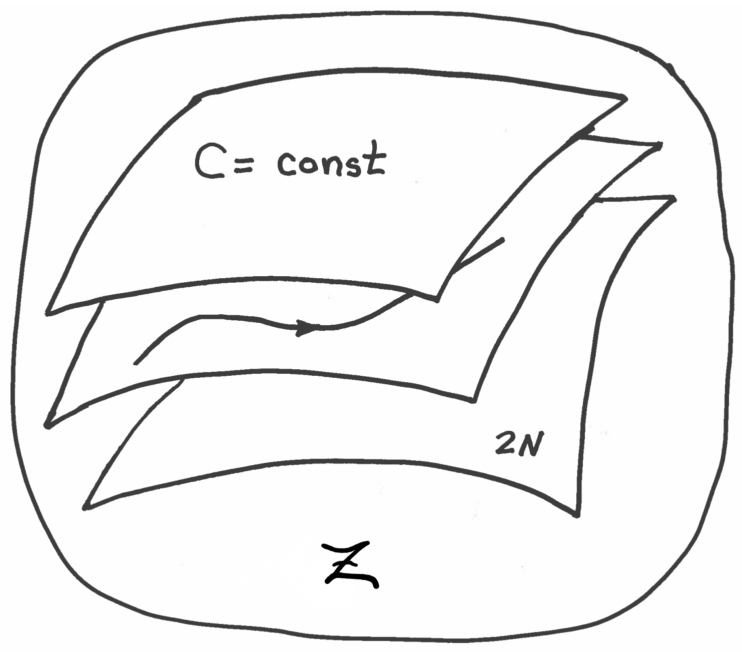
Because of degeneracy, $\exists$ functions C st $\{A,C\} = 0$ for all $A \in C^\infty(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{A, C\} = 0 \quad \forall\ A : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Lie Poisson Flows

$\mathfrak{g}$ Lie algebra; basis $\{E_1, E_2, \ldots, E_n\}$; structure constants c_{ij}^k, i.e., $[E_i, E_j] = c_{ij}^k E_k$;

Dual $\mathfrak{g}^*$; dual basis $\{E_*^1, E_*^2, \ldots, E_*^n\}$; $\left\langle E_*^i, E_j \right\rangle = \delta_j^i$; standard pairing $\langle \cdot\,, \cdot \rangle \colon \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$.

Smooth $A\colon \mathfrak{g}^* \to \mathbb{R}$ has derivative $DA(\mu) \in \mathfrak{g}$ at $\mu \in \mathfrak{g}^*$ for any $\delta\mu \in \mathfrak{g}^*$,

$$\langle \delta\mu, DA(\mu) \rangle = \frac{d}{ds} A(\mu + s\delta\mu) \Big|_{s=0} \qquad \Rightarrow \qquad DA(\mu) = \frac{\partial A}{\partial \mu_i}(\mu)\, E_i.$$

Lie-Poisson bracket on $\mathfrak{g}^*$, for all $A, B\colon \mathfrak{g}^* \to \mathbb{R}$,

$$\{A, B\}_{LP} := \langle \mu, [DA, DB] \rangle = \mu_k c_{ij}^k \frac{\partial A}{\partial \mu_i} \frac{\partial B}{\partial \mu_j}.$$

Dynamics with Hamiltonian $H\colon \mathfrak{g}^* \to \mathbb{R}$

$$\dot{\mu}_i = \{\mu_i, H\}_{LP} = \mu_k c_{ij}^k \frac{\partial H}{\partial \mu_j} \qquad \Leftrightarrow \qquad \dot{\mu} = -\,\mathrm{ad}_{DH}^* \, \mu$$

WLOG duality up $(w) \leftrightarrow (\mu)$ down, Hamiltonian bivector (Poisson tensor): $J^{ij} = c_k^{ij} w^k$.

II. Lie-Algebra generalization of Nambu Dynamics

Lie algebraic generalization of Nambu dynamics

$\mathfrak{g}$ semisimple $\Rightarrow$ Cartan-Killing metric:

$$g_{ij} = -c_{il}^{k}\, c_{jk}^{l}\,,$$

where metric is used to raise and lower indices.

Recall LP bracket:

$$\{A, B\}_{LP} = w^k\, c_k^{ij} \frac{\partial A}{\partial w^i} \frac{\partial B}{\partial w^j}$$

Fully antisymmetric structure constants

$$c^{ijk} = g^{im}\, g^{jn}\, c_{mn}^{k}.$$

Triple bracket:

$$[A, B, C] = c^{ijk} \frac{\partial A}{\partial w^i} \frac{\partial B}{\partial w^j} \frac{\partial C}{\partial w^k}.$$

Generalization of Nambu dynamics (cont)

Quadratic Casimir:

$$S = \frac{1}{2} g_{ij} w^i w^j \qquad \{S, A\}_{LP} = 0\,, \text{ i.e.} \quad \forall\ A$$

Main Theorem:

$$[A, B, S] = \{A, B\}_{LP}$$

Noncanonical Hamiltonian Dynamics:

$$\frac{dF}{dt} = [F, H, S]\,.$$

Nambu's example had $\mathfrak{g} = \mathfrak{so}(3)$, i.e., $c^{ijk} = \varepsilon_{ijk}$

$$[A, B, C] = \nabla A \cdot (\nabla B \times \nabla C)$$

Nambu chose S to be the rotational kinetic energy and H to be the Casimir, the square of the total angular momentum. Naturally $S \leftrightarrow H$.

Lie algebraic generalization of Nambu dynamics (cont)^2

$\mathfrak{g}$ $\neg$ semisimple? Still have quantity (flip indices up-down)

$$g_{ij} = -c_{il}^{k}\, c_{jk}^{l}\,,$$

but now degenerate!

Still have fully antisymmetric structure constants:

$$c_{ijk} = g_{im}\, c_{jk}^{m}\,.$$

Degenerate triple bracket:

$$[A,B,C]_* = c_{ijk}\frac{\partial A}{\partial \mu_i}\frac{\partial B}{\partial \mu_j}\frac{\partial C}{\partial \mu_k}\,.$$

Lie-Poisson Bracket?

$$\{A,B\}_* = [A,B,S]_* \quad \text{where if} \quad S = \Sigma^{mn}\mu_m\mu_n/2\,, \quad \{A,B\}_* = (g_{im}\, c_{j\ell}^{m}\, \Sigma^{\ell k})\, \mu_k \frac{\partial A}{\partial \mu_i}\frac{\partial B}{\partial \mu_j}\,.$$

Any S is conserved, but condition on Σ for Jacobi! Inverse difficulty for field theories.

III. Generalized Nambu Brackets

&

Weyl-Wigner Quantum Mechanics

Triple bracket formulation of quantum mechanics

Lie Algebra Basis Operators:

$$\widehat{E}(\mathbf{r}',\mathbf{p}') = \int d\Gamma \; e^{i(\mathbf{r}'\cdot\mathbf{p}-\mathbf{p}'\cdot\mathbf{r})/\hbar} \, e^{i(\mathbf{r}\cdot\widehat{\mathbf{p}}-\mathbf{p}\cdot\widehat{\mathbf{r}})/\hbar}\,, \qquad \text{where} \quad d\Gamma := d^n r \, d^n p/(2\pi\hbar)^n\,.$$

Here $\widehat{}$ indicates operator and $\{\widehat{E}(\mathbf{r},\mathbf{p})\}$ is basis spanning all QM operators

Wigner function is projection of density operator $\widehat{\rho}$ onto basis

$$W(\mathbf{r},\mathbf{p}) = Tr\{\widehat{\rho}\,\widehat{E}(\mathbf{r},\mathbf{p})\}, \qquad \widehat{\rho} = \int d\Gamma \; W(\mathbf{r},\mathbf{p})\widehat{E}(\mathbf{r},\mathbf{p})\,.$$

Wigner functions are coordinates of Lie algebra spanned $\{\widehat{E}(\mathbf{r},\mathbf{p})\}$.

For pure state:

$$W(\mathbf{r},\mathbf{p}) = Tr\{|\Psi\rangle\langle\Psi|\widehat{E}(\mathbf{r},\mathbf{p})\} = \langle\Psi|\widehat{E}(\mathbf{r},\mathbf{p})|\Psi\rangle = \int d^n s \; e^{-i\mathbf{s}\cdot\mathbf{p}/\hbar}\,\psi(\mathbf{r}+\mathbf{s}/2)\,\psi^*(\mathbf{r}-\mathbf{s}/2),$$

Wigner's original formula.

Triple bracket formulation of quantum mechanics (cont)

Commutator Lie Algebra:

$$(i\hbar)^{-1}[\widehat{E}(z_1), \widehat{E}(z_2)] = \int d\Gamma_3\, C(z_1, z_2, z_3)\widehat{E}(z_3)\,, \quad \text{where} \qquad z := (\mathbf{r}, \mathbf{p})$$

Lie algebra realization on phase-space functions:

$$[A, B]_M(z) = \int d\Gamma_1 d\Gamma_2\, C(z, z_1, z_2)\, A(z_1)\, B(z_2).$$

M is for Moyal

$$[A, B]_M(\mathbf{r}, \mathbf{p}) = \frac{2}{\hbar} A(\mathbf{r}, \mathbf{p}) \sin \frac{\hbar}{2}(\overleftarrow{\partial}_{\mathbf{r}} \cdot \overrightarrow{\partial}_{\mathbf{p}} - \overleftarrow{\partial}_{\mathbf{p}} \cdot \overrightarrow{\partial}_{\mathbf{r}})\, B(\mathbf{r}, \mathbf{p}),$$

Lie-Poisson-Moyal:

$$\begin{aligned} \{\mathcal{A}, \mathcal{B}\}_{LPM} &= \int d\Gamma_1\, d\Gamma_2\, d\Gamma_3\, W(z_1)\, C(z_1, z_2, z_3) \frac{\delta \mathcal{A}}{\delta W(z_2)} \frac{\delta \mathcal{B}}{\delta W(z_3)} \\ &= \int d\Gamma\, W \left[\frac{\delta \mathcal{A}}{\delta W}, \frac{\delta \mathcal{B}}{\delta W}\right]_M \end{aligned}$$

Triple bracket formulation of quantum mechanics (cont)^2

Hamiltonian:

$$\mathcal{H}[W] = \int d\Gamma\, W(\mathbf{r},\mathbf{p})\, H(\mathbf{r},\mathbf{p})\,, \quad \text{where} \quad H(\mathbf{r},\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{r}).$$

Casimir:

$$\mathcal{S} = \frac{1}{2}\int d\Gamma\, W^2(z)\,, \quad \text{where} \quad \{\mathcal{A},\mathcal{S}\}_{LPM} = 0 \quad \forall\, \mathcal{A}$$

Triple Bracket:

$$[\mathcal{A},\mathcal{B},\mathcal{C}] = \int d\Gamma_1\, d\Gamma_2\, d\Gamma_3\, C(z_1,z_2,z_3) \frac{\delta\mathcal{A}}{\delta W(z_1)} \frac{\delta\mathcal{B}}{\delta W(z_2)} \frac{\delta\mathcal{C}}{\delta W(z_3)}\,,$$

where $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ arbitrary functionals of W.

Dynamics:

$$\frac{d\mathcal{F}}{dt} = [\mathcal{F},\mathcal{H},\mathcal{S}] \quad \Rightarrow \quad \frac{\partial W}{\partial t} = [W,\mathcal{H},\mathcal{S}] = \{W,H\}_{LPM} = -\,[W,H]_M\,.$$

Triple bracket formulation of quantum mechanics (cont)^3

Classical limit, $\hbar \to 0$:

- Moyal bracket, $[\cdot,\cdot]_M \quad \to \quad$ canonical Poisson bracket, $[\cdot,\cdot]_c$
- Lie-Poisson-Moyal bracket, $\{W,H\}_{LPM} \quad \to \quad$ LP bracket for Vlasov eq., $\{W,H\}_{LP}$

  PJM (1980)
- Triple Bracket, $[\mathcal{A},\mathcal{B},\mathcal{C}] \quad \to \quad [\mathcal{A},\mathcal{B},\mathcal{C}]_c = \int d\Gamma \, \frac{\delta \mathcal{A}}{\delta W(z_1)} \left[\frac{\delta \mathcal{B}}{\delta W(z_2)}, \frac{\delta \mathcal{C}}{\delta W(z_3)} \right]_c$

$[\mathcal{A},\mathcal{B},\mathcal{C}]$ **is a one parameter family (deformation) that includes** $[\mathcal{A},\mathcal{B},\mathcal{C}]_c$ above

Appearance in J. Meiss and PJM (1984).

Class of QM Mean Field Theories Like Vlasov

Simply insert the new Hamiltonian:

$$\mathcal{H}[W] = \int d\Gamma\, W(\mathbf{r},\mathbf{p})\, H_1(\mathbf{r},\mathbf{p}) + \int d\Gamma \int d\Gamma'\, W(\mathbf{r},\mathbf{p}) W(\mathbf{r}',\mathbf{p}')\, H_2(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}')$$

where e.g.

$$H_1(\mathbf{r},\mathbf{p}) = \frac{|\mathbf{p}|^2}{2m} + V(\mathbf{r})$$

and $H_2(\mathbf{r},\mathbf{p};\mathbf{r}',\mathbf{p}')$ is an interaction kernel, e.g., $\Rightarrow$ Poisson's equation if Wigner-Poisson.

IV. Incompressible Fluid:

Lagrange's volume preserving diffeomorphisms,

geodesics, & and Dirac brackets

Brief Summary

- Lagrange (1788):
  - Lagrangian description of the ideal fluid: $q(a,t)$ diffeomorphism? $a = q_0 \mapsto q \ \forall t$
  - Lagrangian (Hamilton's action principle): $\rho_0 |\dot{q}|^2/2$ KE density of free particle
  - Lagrange multiplier for incompressibility constraint: $\mathcal{J} = \det(\partial q/\partial a) = 1$ holonomic

- First year physics course: **free particle with holonomic constraints** $\rightarrow$ **geodesic flow**
  - Arnold: geodesic flow on the group of volume preserving diffeomorphisms. Curvature for 2D Euler on domain $\mathbb{T}^2$.

Brief Summary 2

Three dichotomies:

- Lagrangian vs. Eulerian descriptions of the ideal fluid
- Lagrange Multiplier vs. Dirac constraint theory
- Lagrangian vs. Hamiltonian descriptions

Results:

- Explicit expressions for the dynamics in terms of constraints and original variables.
- Lagrangian and Eulerian conservation laws are not identical, various methods compared
- Christoffel symbol & Riemann curvature in terms of original Euclidean coordinates

A Few References

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- PJM, T. Andreussi, and F. Pegoraro, *Lagrangian and Dirac Constraints for the Ideal Incompressible Fluid and Magnetohydrodynamics*, J. Plasmas Phys. **86**, 835860301 (2020).