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Transport & Mixing of Tracers: An Overview

Biased

* Kinematic Transport

* Realistic Transport & mixing

Reviews: Aref et al. 2017

Thiffeault 2012

See Bibliography

Three Pillars of Kinematic Transport

1) The Arena

Ocean, Atmosphere, Tokamak, Solar Wind, Accretion Disc, Blood Vessel

$\mathbb{R}^n$, $n \in \mathbb{Z}$, domain $D \subset \mathbb{R}^n$

domain of fluid $n = 2, 3$

differentiable manifold of any dim.

Phase space T^*Q w/ can. coords (q, p)

Symplectic manifold, Poisson manifold

General manifold w/ boundary

M any manifold w/ coords. $(z^1, z^2, \cdots z^n) = z$

Types: open vs. closed

pipe flow vs. Taylor-Couette

2) The Cargo

neutrally buoyant ptle, dye, entropy/mass, mass density, vorticity, magnetic flux, phase space density, ...

Differential Forms:

0 - form	entropy/mass	Attributes of Lagrangian fluid elements
1 - form		
2 - form	magnetic flux in MHD	
3 - form	fluid volume	
⋮		
n - form	phase space den of Vlasov for $n = 6$	

Any Tensor field on M:

$\mathring{T}(z) \longrightarrow T(z,t)$ meaning of transport

3) The Transporter

Often fluid vel. field in dim 2 or 3

Phase space dynamics, Vlasov

Vector field on M

in coords

$$V^i(z) \quad \text{or} \quad V^i(z,t) \qquad i=1,2,\dots n \longrightarrow i=1,2,\dots,n+1$$

$$\frac{dz^i}{dt} = V^i(z) \implies z_t = \phi_t \mathring{z} \iff z^i = z^i(\mathring{z},t)$$

Flow

$$\phi_t \circ \phi_{-t} = Id \implies \phi_{-t} \circ \phi_t \mathring{z} = \phi_{-t} z_t = \mathring{z}$$

Integral curves w/ V tangent

1- parameter Abelian group $\implies$ transport w/o mixing

Some vector fields:

2D incompressible Euler $\quad v = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$

NC Hamiltonian vector field

Poisson bracket $\{ , \}$ $\quad V^i = J^{ij} \frac{\partial H}{\partial z^j} = \{ z^i, H \}$

$\{ , \} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M)$, bilinear, antisym.

& Jacobi identity $\{ f, \{ g, h \} \} + \circlearrowleft \equiv 0$

Canonical Hamiltonian vector field

$$J_c = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$$

e.g. 2D Euler

Kinematic Transport is Lie Dragging!

Arena any manifold, M

Cargo any tensor field, T

Transporter any vector field, V

$$\boxed{\frac{\partial T}{\partial t} + \mathcal{L}_V T = 0}$$

Basic Equation

$\mathcal{L}_V$ is the Lie derivative along $V \in \mathfrak{X}(M)$

Yano 1957
etc.

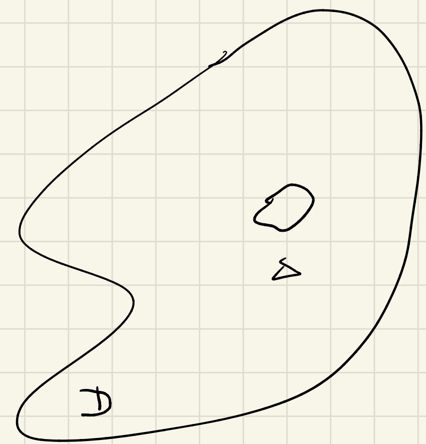
Why Lie Dragging?

Example 1 $\quad \rho$ a 3-form on 3D domain of fluid, D.

Consider arbitrary $\Delta \subset D$, a subvolume

$$M_\Delta(t) = \int_{\Delta(t)} \rho(x,t)\, d^3x$$

mass of fluid in Δ



Assume Δ moves w/ fluid velocity ξ

$$\frac{\partial \rho}{\partial t} + \mathcal{L}_v \rho = 0 \quad \Longrightarrow$$

$$\frac{dM_\Delta}{dt} = 0$$

Example 2 Kinematic Dynamo

$$\frac{\partial B}{\partial t} = -\nabla \times E = \nabla \times (v \times B) = -v \cdot \nabla B + B \cdot \nabla v - B\, \nabla \cdot v$$

$$= -\mathcal{L}_v B$$

B is a vector density

$\Longleftrightarrow$ 2-form

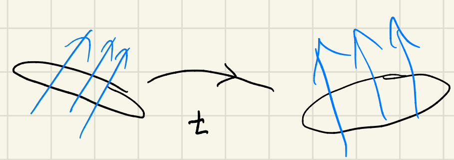
D 3 dim domain

S arbitrary 2dim surface

$B \cdot d^2x$ mag. flux through infinitesimal area

Alfven's Frozen-in Flux

$$\frac{d\Phi_S(t)}{dt} = \int_S B \cdot d^2x = 0$$



Example 3

Liouville's Equation on $6n$ dim phase space

$z_\alpha = (q_\alpha, p_\alpha)$ for ptle α

Phase space prob. density for $\alpha = 1, 2, \ldots n$ interacting ptles

$F(z_1, z_2, \ldots z_n)$

$$\frac{\partial F}{\partial t} + \mathcal{L}_V F = 0$$

V is a Ham. vector field

$$V\cdot = \{\cdot, h\}$$

2-Body interaction

$$h = \sum_\alpha \frac{|p_\alpha|^2}{2m_\alpha} + \frac{1}{2}\sum_{\alpha,\beta} \Phi_{\alpha\beta}(z_\alpha, z_\beta)$$

Self-Consistent Transport — Hamiltonian Mean Field Theories

$$H = \int h_1(z) f(z,t)\, d^n z + \frac{1}{2} \iint h_2(z,z') f(z,t) f(z',t)\, d^n z\, d^n z'$$
$$+ \cdots$$

$$\frac{\partial f}{\partial t} + \left\{ f, \frac{\delta H}{\delta f} \right\} = 0 \quad , \quad \frac{\delta H}{\delta f} = h_1 + \int h_2 f\, dz'$$

Variational derivative

$$\frac{\partial f}{\partial t} + \mathcal{L}_{JdH} f = 0$$

Depends on f globally

Lie dragging by a Ham. vector field depending on f !

Tennyson et al. 1994

PJM 2003

Why Lie dragging?

Assures important physical quantity conserved along <u>integral curves</u>.

$\Longrightarrow$

Kinematic Transport meets Dynamical Systems Theory

Yet $\quad \dot{\mathring{z}} = v(z) \quad$ —

$$\dot{\mathring{z}} = \phi_{-t} \circ \phi_t \, \dot{\mathring{z}} = \phi_{-t} z_t$$

$\Longrightarrow \quad \neg$ mixed

Dynamical Systems Theory – Phase Space Structures

* Periodic Orbits

  stable (elliptic), unstable (hyperbolic)

* Quasiperiodic Orbits

  e.g. attracting $\mathbb{T}^2$

* Invariant sets

  Barriers to transport; exact or sticky regions

  Cantor sets – strange attractors

* Regions of chaos, ergodicity, invariant measures

Tools – Fast Indicators

Lyapunov Exponent: calculation technique Benettin et al. 1980

experimental technique Wolf et al. 1985

FTLE Froeschle et al. 1997

FTLE as indicator of transport Haller 2000 →

others (celestial mech.): Lekien et al. 2007

Small alignment index (SALI) Skokos et al 2007

General alignment index (GALI)

mean exp. growth of nearby orbits (MEGNO) Cinotta 2000

Frequency Methods: Laskar et al. 1992

Finite Time Rotation Number: Szezech et al. 2013; Sander et al.

FTRN "dual" to FTLE – integrability vs. chaos

"Duality"

$\psi = A \sin[\pi f(x,t)] \times \sin(\pi y)$

FTRN exists where FTLE doesn't!

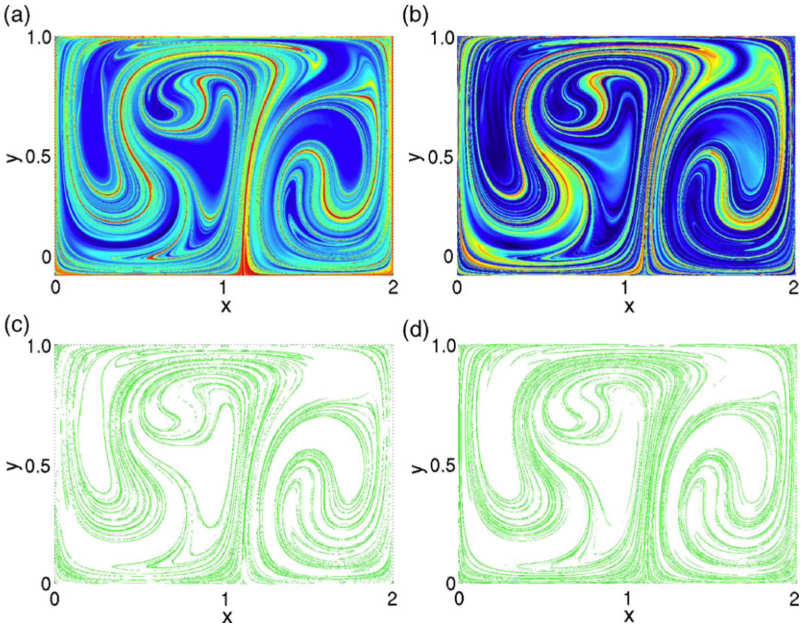


Fig. 1. (Color online.) (a) Time-4T Lyapunov exponent and (b) time-4T rotation number for the double gyre system, with period $T = 10$, amplitude $A = 0.1$ and forcing strength $\epsilon = 0.25$. (c) and (d) depict the Lagrangian coherent structures corresponding to ridges of (a) and (b), respectively.

Modelling w/ Symplectic Maps

Example: Standard Map — generic near elliptic periodic orbit

Charged ptle. in E-field $\qquad m\ddot{x} = eE(x,t)$

$$E = E_1 e^{ik_1 x - i\omega_1 t} + E_2 e^{ik_2 x - i\omega_2 t} + \cdots$$

$E_1 = E_2 = \cdots$ scaling $\Rightarrow$ $H = \frac{p^2}{2} + C\sin q \sum_{n\in\mathbb{Z}} \delta(t - \pi n)$

Standard Map:

$$\boxed{q_{n+1} = q_n + p_{n+1}\,; \qquad p_{n+1} = p_n - \frac{k}{2\pi}\sin(2\pi q_n)}$$

Invariant Circles are exact barriers to transport

KAM limit, Poincare-Birkhoff Thm, Island Overlap, Greene's Method, Renormalization & Scaling.

1991

del-Castillo-Negrete et al. 1992 Nontwist

Pumped rotating annulus on β-plane

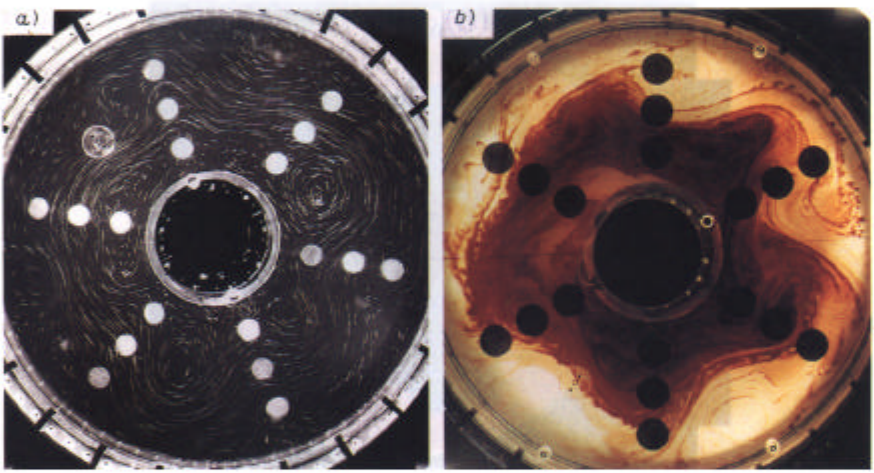
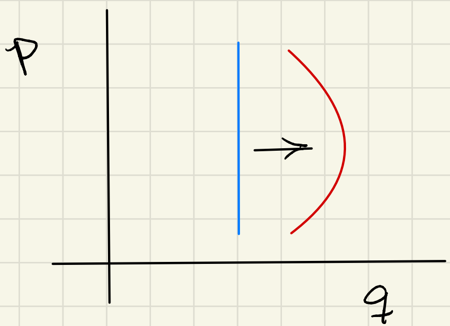


Fig. 11. – *a*) Streak photograph of an eastward jet generated in a slowly decelerating tank by pumping *only* through the middle ring of ports, which alternate as sources and sinks (acceleration rate = 0.013 rad/s^2, F = 137 cm^3/s, Ω = 25.1 rad/s, exposure time = 1/4 s). *b*) An eastward jet generated by pumping through three consecutive radial pairs of sources (at r = 35.1 cm) and sinks (at r = 27.0 cm) (Ω = 12.6 rad/s, F = 200 cm^3/s). Here the dye is injected on the inner side of the jet, filling the region of quasi-uniform q; there is only weak mixing across the center of the jet.

Zonal Flow $\Longrightarrow$ Nontwist $\Longleftrightarrow$ $\exists$ shearless Torus

del-Castillo-Negrete & PJM 1992, 1993

Moser twist condition – further up $\Rightarrow$ further over



Behavior not captured by the standard Map!

Standard Nontwist Map

$$\boxed{q_{n+1} = q_n + a(1 - p_{n+1}^2) \; ; \; p_{n+1} = p_n - b \sin q_n}$$

Precursors:
J. Howard
J. Weiss

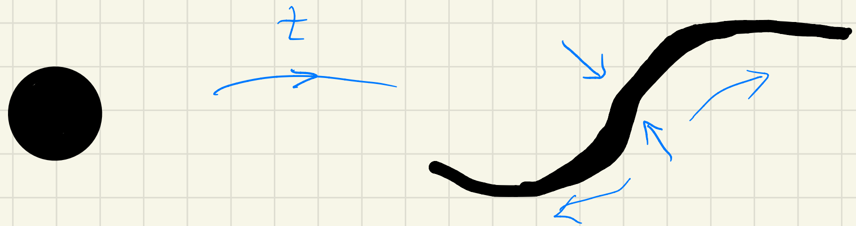
Large literature
Apte, Wurm, Fuchss
... Viana 2021
Javier Beron-Vera
today!

generic behavior of shearless tori
* nonstandard bifurcations
* nonstandard renormalization
* Broken Shearless Tori are sticky!

Realistic Particle Transport and Mixing

$$\overset{\circ}{z} = \phi_{-t} \circ \phi_t = \overset{\circ}{z}$$ Broken! How?

If $\exists$ stretching & contracting directions $\Rightarrow$



Generation of fine scales

$$\frac{\partial s}{\partial t} + v \cdot \nabla s = \nu \nabla^2 s$$

Someone's #

Gets activated on fine scales $\Rightarrow$ mixing

$\exists$ measures of mixing   J-L Thiffeault

Other Possibilities

* Collisional Kinetic theory

$$\frac{\partial f}{\partial t} + \left\{ \frac{\delta H}{\delta f}, f \right\} = \left.\frac{\partial f}{\partial t}\right)_c$$

Boltzmann

Landau

⋮

* Damping & Driving

$$\frac{\partial S}{\partial t} + v \cdot \nabla S = D + \mathcal{S}(S, x, t)$$

Source

Intentional vs. "Natural"

Natural: midlatitude ozone $\rightarrow$ ozone hole, impurities in tokamak, ...

Intentional: diagnostic dye, Barium, neutrally buoyant ptles (PIV) ...

Particle Entrainment

Weeks 1997

Ptle. (PIV, pollutant, midlatitude ozone, ...) in vel. field.

moves w/ fluid : $v_p = v_f$? Approx.

Sphere : radius, a

$m_f = \frac{4}{3}\pi a^3 \rho$

$$m \frac{dv_p}{dt} = 6\pi a \rho \nu (v_f - v_p)$$ Stokes drag — Simha et al. 2018, boundaries etc.

$$+ m_f \frac{dv_f}{dt}$$ Pressure, viscous stresses

$$+ \frac{1}{2} m_f \left(\frac{dv_f}{dt} - \frac{dv_p}{dt} \right)$$ added mass

$$+ 6a^2 \rho \sqrt{\pi \nu} \int_0^t \frac{\frac{d}{d\tau}(v_f - v_p)}{\sqrt{t-\tau}} d\tau$$ Basset history

$$+ (m - m_f) F$$ buoyancy, centripetal

That's All Folks!

BIBLIOGRAPHY FOR P. J. MORRISON ACP TALK 210601

Reviews with real mixing: [1–5]

Some Hamiltonian background: [3]

Lie derivatives: [4, 5]

Hamiltonian mean field theories: [6, 7]

Lyapunov exponents and fast indicators:

Famous papers on Lyapunov exponents: calculation [8] experiment [9]
Finite-time Lyapunov exponents (FTLE): [10]

Lagrangian Coherent Structures:

Based on FTLE: [11–14] and many newer papers by Haller
Based on Fast Indicators from celestial mechanics:
Small Alignment Index (SALI) and Generalized Alignment Index (GALI) [15]
Mean Exponential Growth of Nearby Orbits (MEGNO) [16]

Frequency Methods: [17]

Finite time rotation number (FTRN): [18, 19]

Swinney's Rotation Annulus: [20]

Zonal Flows have Shearless Tori/Standard Nontwist Map: [21–24]

Some later work on Shearless Tori/Standard Nontwist Map: [18, 19, 25–30]

Earlier nontwist maps: [31, 32]

Maxey – Riley equation discussion: [33]

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