Integral Transform for a Class of Mean-Field Theories: Action-Angle Variables for the Continuous Spectrum

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Finite Hamiltonian Systems $\rightarrow$ Hamiltonian Field Theories

$\exists$ large amount of finite degree-of-freedom (DOF) Hamiltonian systems lore:

1	DOF	Integrable
2	DOF	Nonintegrable, broken tori, chaos
3	DOF	Tori are not barriers, "diffusion" around tori
N	DOF	Linear & nonlinear bifurcation theory, e.g. Hamiltonian – Hopf & Krein – Moser
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∞	DOF	All of above plus?

What doesn't carry over?

$\rightarrow$ **Continuous Spectrum**

Hamiltonian Field Theories $\rightarrow$ Finite Hamiltonian Systems
• Vlasov Equation (1980) $\rightarrow$ Poisson Geometry

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\,,\,\}\colon C^\infty(\mathcal{Z}) \times C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

st $C^\infty(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

i) bilinear, ii) antisymmetric, iii) Jacobi, and iv) Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

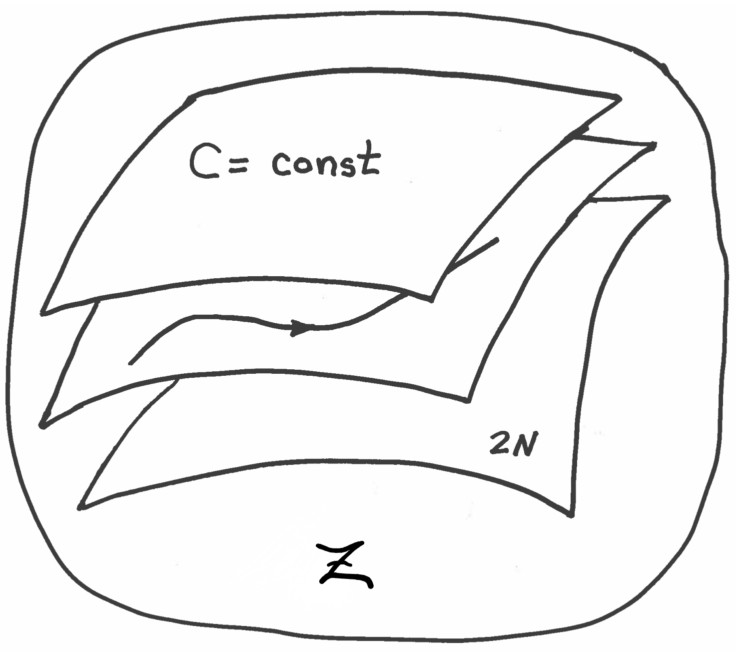
Because of degeneracy, $\exists$ functions C st $\{F, C\} = 0$ for all $F \in C^\infty(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold (phase space) $\mathcal{Z}$ Cartoon

Degeneracy in J $\Rightarrow$ Casimirs:

$$\{F, C\} = 0 \quad \forall\ F : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Lie-Poisson Brackets

Lie-Poisson brackets are special kind of noncanonical Poisson bracket that are associated with any Lie algebra, say $\mathfrak{g}$.

Natural phase space $\mathfrak{g}^*$. For $f, g \in C^\infty(\mathfrak{g}^*)$ and $z \in \mathfrak{g}^*$.

Lie-Poisson bracket has the form

$$\begin{aligned} \{f, g\} &= \langle z, [\nabla f, \nabla g] \rangle \\ &= \frac{\partial f}{\partial z^i} \, c^{ij}{}_k \, z_k \, \frac{\partial g}{\partial z^j}\,, \qquad\qquad i, j, k = 1, 2, \ldots, \dim \mathfrak{g} \end{aligned}$$

Pairing $<\ ,\ >: \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$, z^i coordinates for $\mathfrak{g}^*$, and $c^{ij}{}_k$ structure constants of $\mathfrak{g}$.

Vlasov Lie-Poisson Bracket:

$$\{F, G\} = \left\langle f, \left[\frac{\delta F}{\delta f}, \frac{\delta F}{\delta f}\right]\right\rangle = \int_{\mathcal{Z}} d^6 z \, f \left[\frac{\delta F}{\delta f}, \frac{\delta F}{\delta f}\right]$$

General Class of Mean-Field Hamiltonian Theories

Density:

$$\zeta(q,p,t) \qquad \text{s.t.} \qquad \zeta\colon \mathcal{Z}\times\mathbb{R}\to\mathbb{R}$$

Phase Space:

$$z := (q,p) \in \mathcal{Z} = \Pi \times \mathbb{R}$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} + [\zeta, \mathcal{E}] = 0$$

Particle Poisson Bracket:

$$[f,g] = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p}$$

Particle Energy:

$$\mathcal{E}[\zeta] = \quad ? \qquad \Rightarrow \qquad \text{nonlinear}$$

Lie-Poisson Hamiltonian Structure

Hamiltonian (energy):

$$H[\zeta] = H_1 + H_2 + \cdots = \int_{\mathcal{Z}} d^2z\, h_1(z)\, \zeta(z) + \frac{1}{2} \int_{\mathcal{Z}} d^2z \int_{\mathcal{Z}} d^2z'\, \zeta(z) h_2(z, z')\, \zeta(z') + \ldots$$

Lie-Poisson Bracket:

$$\{F, G\} = \int_{\mathcal{Z}} d^2z\ \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta} \right]$$

← arbitrary inner algebra

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = -\left[\zeta, \frac{\delta H}{\delta \zeta} \right] = -[\zeta, \mathcal{E}]$$

Casimir Invariants:

$$C[\zeta] = \int_{\mathcal{Z}} d^2z\ \mathcal{C}(\zeta)$$

Lie-Poisson Mean-Field Examples

Vlasov Poisson: $z = (x, p = mv)\,, \qquad \zeta \to f(x,p,t) =$ phase space density

$$
\begin{aligned}
H[f] &= \int_{\Pi\times\mathbb{R}} dxdp\,\frac{p^2}{2m}\,f(x,p) + \frac{1}{8\pi}\int_{\mathbb{R}} dx\,E^2 \\
&= \int_{\Pi\times\mathbb{R}} dxdp\,\frac{p^2}{2m}\,f(x,p) + c\int_{\Pi\times\mathbb{R}} dxdp\int_{\Pi\times\mathbb{R}} dx'dp'\,f(x,p)|x-x'|f(x',p') \\
\mathcal{E} &= \frac{\delta H}{\delta f} = \frac{p^2}{2m} + e\phi[f](x)
\end{aligned}
\tag{1}
$$

2D Euler: $z = (x, y)\,, \qquad \zeta \to \omega(x,p,t) =$ scalar vorticity

$$
\begin{aligned}
H[\omega] &= \int_{\Pi^2} dxdy\,\frac{v^2}{2} = \int_{\Pi^2} dxdy\,\frac{|\nabla\psi|^2}{2} \\
&= c\int_{\Pi^2} dxdy\int_{\Pi^2} dx'dy'\,\omega(x,y)\,\ln[(x-x')^2 + (y-y')^2]\,\omega(x',y') \\
\mathcal{E} &= \frac{\delta H}{\delta\omega} = \psi[\omega](x,y)
\end{aligned}
$$

Other: Jeans equation, quasigeostrophy, Hasegawa-Mima, ...

Other Lie-Poisson Mean-Field Examples

- Quantum Mechanics:

Use Wigner-Weyl representation with f replaced by Wigner function and inner algebra by the Moyal bracket (Birula & pjm 1981)

$$f(\mathbf{r},\mathbf{v},t) \longrightarrow W(\mathbf{r},\mathbf{p},t)\,, \qquad [f,g] \longrightarrow [f,g]_M = \frac{2}{\hbar} f(\mathbf{r},\mathbf{p}) \sin\frac{\hbar}{2}(\overleftarrow{\partial}_\mathbf{r}\cdot\overrightarrow{\partial}_\mathbf{p} - \overleftarrow{\partial}_\mathbf{p}\cdot\overrightarrow{\partial}_\mathbf{r})\, g(\mathbf{r},\mathbf{p})$$

$$H[W] = \int d\Gamma\; W(\mathbf{r},\mathbf{p}) \left(\frac{\mathbf{p}^2}{2m} + V(\mathbf{r})\right)\,, \qquad \text{where } d\Gamma \equiv d^n r\, d^n p/(2\pi\hbar)^n$$

- Schrödinger-Poisson:

$$H[W] = \int d\Gamma\, W(\mathbf{r},\mathbf{p}) \frac{\mathbf{p}^2}{2m} + \frac{1}{2}\int d\Gamma' \int d\Gamma\, W(\mathbf{r}',\mathbf{p}') W(\mathbf{r},\mathbf{p}) V(\mathbf{r},\mathbf{r}').$$

Other Lie-Poisson Mean-Field Examples Cont.

- Lie-Poisson $\rightarrow$ Hamilton-Jacobi Formulation:

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left|\frac{\partial S}{\partial \mathbf{q}}\right|^2 + e\phi[S](\mathbf{q},t) = H_0$$

where and H_0 is an arbitrary reference Hamiltonian, ϕ is given by Poisson's equation with $f(\mathbf{z},t)$ defined by

$$\Phi(\mathbf{q},\mathbf{P},t) = f\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right)\left|\left|\frac{\partial^2 S}{\partial \mathbf{q}\partial \mathbf{P}}\right|\right| \qquad \leftarrow \text{Van Vleck determinant}$$

Mean field (self-consistent) Hamiltonian-Jacobi theory of Vlasov in terms of mixed variable generating function $S(q,P,t)$ with Van Vleck determinant (Pfirsch, pjm 1985,2012).
Hamilton-Jacobi $\leftrightarrow$ Wigner formulation for QM? H-J hierarchy?

- Leaf Formulation:

Replace S by Lie generator on a symplectic leaf. (Ye et al. 1991.)

Linear Normal Form with Continuous Spectrum

Stable Normal Form:

$$H = \sum_i^N \frac{\sigma_i|\omega_i|}{2}\left(p_i^2 + q_i^2\right) = i\sum_i^N \omega_i\, Q_i\, P_i = \sum_i^N \sigma_i|\omega_i|J_i \;\rightarrow\; \int du\, \sigma(u)|\omega(u)|\, J(u)$$

Stable when ∃ a canonical transformation to Action-Angle variables. Note important signature: $\sigma_i \in \{-1, 1\}$. Negative energy modes and Krein-Moser.

Two Complications: Noncanonical & ∞-dimensional $\rightarrow$ Continuous Spectrum

Noncanonical: $\dot{z} = \mathcal{J}(z)\partial H/\partial z = \mathcal{J}(z)\partial(H+C)/\partial z$; $\delta(H+C) = 0 \Rightarrow z_e$. $z = z_e + \hat{z}$

$$\dot{\hat{z}} = \mathcal{J}(z_e)\partial H_L/\partial\hat{z} \qquad \text{where} \qquad H_L = \hat{z}^T\cdot D^2F(z_e)\cdot\hat{z}/2$$

$\rightarrow$ easy matrix calculation to reduce to Casimir leaf

Infinite Dimensions: Integral transform/Coordinate Change $\rightarrow$

$$g = \widehat{G}[f]$$

$\rightarrow$ general class of transforms for Lie – Poisson brackets with CS

Example: Linear Vlasov-Poisson

Equilibrium & Linearization: $\delta(H + C) = 0 \Rightarrow f_e(v)$

$$f = f_e(v) + \hat{f}(x, v, t)$$

Linearized EOM:

$$\frac{\partial \hat{f}}{\partial t} + v\frac{\partial \hat{f}}{\partial x} - \frac{e}{m}\frac{\partial \hat{\phi}[x, t; \hat{f}]}{\partial x}\frac{\partial f_e}{\partial v} = 0$$

$$\hat{\phi}_{xx} = -4\pi e \int_{\mathbb{R}} dv\, \hat{f}(x, v, t)$$

Linearized Energy (Kruskal and Oberman):

$$H_L = -\frac{m}{2}\int_{\Pi\times\mathbb{R}} dv dx\, \frac{v\, \hat{f}^2}{f_e'} + \frac{1}{8\pi}\int_{\Pi} dx\, \hat{\phi}_x^2$$

Vlasov Lie-Poisson Bracket:

$$\{F, G\}_L = \int_{\Pi\times\mathbb{R}} dv dx\, f_e(v)\left[\frac{\delta F}{\delta \hat{f}}, \frac{\delta G}{\delta \hat{f}}\right]$$

Example: Linear Vlasov-Poisson Canonization

Fourier Series: $\hat{f} = \sum_{k\in\mathbb{Z}} f_k(v,t)e^{ikx}$ and $\hat{\phi} = \sum_{k\in\mathbb{Z}} \phi_k(t)e^{ikx}$

Linearized EOM:

$$\begin{aligned} \frac{\partial f_k}{\partial t} &+ ikvf_k - ik\phi_k \frac{e}{m}\frac{\partial f_e}{\partial v} = 0 \\ k^2\phi_k &= 4\pi e \int_{\mathbb{R}} f_k(v,t)\, dv \end{aligned} \qquad \text{(LVP)}$$

Canonical Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m}\int_{\mathbb{R}} dv\;\, f_e' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}}\right) = \sum_{k=1}^{\infty}\int_{\mathbb{R}} dv \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k}\right)$$

where $q_k(v,t) = \frac{m}{ikf_e'} f_k(v,t)$ and $p_k(v,t) = f_{-k}(v,t)$

Linear KO Hamiltonian:

$$H_L = -\frac{m}{2}\sum_k \int_{\mathbb{R}} dv\, \frac{v}{f_e'}\, |f_k|^2 + \frac{1}{8\pi}\sum_k k^2|\phi_k|^2 = \sum_{k,k'}\int_{\mathbb{R}} dv \int_{\mathbb{R}} dv'\, f_k(v)\, \mathcal{H}_{k,k'}(v|v')\, f_{k'}(v')$$

Good Equilibria and Initial Conditions

Definition (VP1). A function $f_e(v)$ is a good equilibrium if $f_e'(v)$ satisfies

(i) $f_e' \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, q st $1 < q < \infty$, and α st $0 < \alpha < 1$,

(ii) $\exists\, v* > 0$ st $|f_e'(v)| < A|v|^{-\mu}$ $\forall\, |v| > v*$, where $A > 0$ and $\mu > 0$, and

(iii) $f_e'/v < 0$ $\forall\, v \in \mathbb{R}$ or f_e is Penrose stable. Assume $f_e'(0) = 0$.

Definition (VP2). A function, $\mathring{f}_k(v)$, is a good initial condition if it satisfies

(i) $\mathring{f}_k(v), v\mathring{f}_k(v) \in L^p(\mathbb{R})$,

(ii) $\int_{\mathbb{R}} \mathring{f}_k(v)\, dv < \infty$.

Good equilibria imply only continuous spectrum, while good initial conditions are physically reasonable and make theorems work. Not optimal.

Hilbert Transform Review

Hilbert transform:

$$H[g](x) := \frac{1}{\pi} \int_{\mathbb{R}} dt \, \frac{g(t)}{t - x}$$

$\exists$ theorems about Hilbert transforms in $L^p(\mathbb{R})$ and $C^{0,\alpha}(\mathbb{R})$. Plemelj, M. Riesz, Zygmund, and Titchmarsh $\cdots$ Can be extracted from Calderón-Zygmund theory.

Theorem (H1).

(ii) $H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, for $1 < p < \infty$, is a bounded linear operator:

$$\|H[g]\|_p \leq A_p \, \|g\|_p \, ,$$

A_p depends only on p,

(ii) H has an inverse on $L^p(\mathbb{R})$, given by

$$H[H[g]] = -g \, ,$$

(iii) $H\colon L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}) \to L^p(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$.

Hilbert Transform Review Continued

Theorem (H2). If $g_1 \in L^p(\mathbb{R})$ and $g_2 \in L^q(\mathbb{R})$ with $\frac{1}{p} + \frac{1}{q} < 1$, then

$$H\Big[g_1 H[g_2] + g_2 H[g_1]\Big] = H[g_1]H[g_2] - g_1 g_2 \,.$$

The proof, based on the Hardy-Poincaré-Bertrand theorem, is due to Tricomi.

Lemma (H3). If $vg \in L^p(\mathbb{R})$, then

$$H[vg](u) = u\, H[g](u) + \frac{1}{\pi}\int_{\mathbb{R}} g\, dv \,.$$

prf. $\frac{v}{v-u} = \frac{u+v-u}{v-u} = \frac{u}{v-u} + 1$

G-Transform

Definition (G1). The G-transform is defined by

$$\begin{aligned} f(v) &= G[g](v) \\ &:= \epsilon_R(v)\, g(v) + \epsilon_I(v)\, H[g](v)\,, \end{aligned}$$

where

$$\epsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_e(v)}{\partial v}\,, \qquad \epsilon_R(v) = 1 + H[\epsilon_I](v)\,.$$

Remarks.

- We suppress the dependence of $\epsilon_{I,R}$ on k throughout. Note, $\omega_p^2 := 4\pi n_0 e^2/m$ is the plasma frequency corresponding to an equilibrium of number density n_0.
- $\epsilon = \epsilon_R + i\epsilon_I$ (complex extended) is the plasma dispersion relation s.t. vanishing $\Rightarrow$ discrete normal eigenmodes. When $\epsilon \neq 0$ $\exists$ only continuous spectrum; i.e. no dispersion relation.
- $\epsilon_I \propto f_e' \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R}) \Rightarrow \epsilon_R - 1 \in L^q(\mathbb{R}) \cap C^{0,\alpha}(\mathbb{R})$, and since $\lim_{|v|\to\infty} \epsilon_I = 0$, $\lim_{|v|\to\infty} \epsilon_R = 1$, both $\epsilon_R\,, \epsilon_I \in L_\infty(\mathbb{R})$.

G-Transform Properties

Theorem (G2). $G\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator:

$$\|G[g]\|_p \leq B_p \, \|g\|_p \,,$$

where B_p depends only on p.

Theorem (G3). If f_e is a good equilibrium, then $G[g]$ has an inverse,

$$\widehat{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \widehat{G}[f](u) \\ &:= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} \, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} \, H[f](u) \,. \end{aligned}$$

where $|\epsilon|^2 := \epsilon_R^2 + \epsilon_I^2$.

(G3) Proof

That $\widehat{G}$ is the inverse follows directly upon inserting $G[g]$ of (G1) into $g = \widehat{G}[G[g]]$, and using (H2) and $\epsilon_R(v) = 1 + H[\epsilon_I]$.

$$
\begin{aligned}
g(u) &= \widehat{G}[f](u) = \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H[f](u) \\
&= \frac{\epsilon_R(u)}{|\epsilon(u)|^2}\, [\epsilon_R(u)\, g(u) + \epsilon_I(u)\, H[g](u)] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2}\, H\left[\epsilon_R(u')\, g(u') + \epsilon_I(u')\, H[g](u')\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H\left[H[\epsilon_I]\, g + \epsilon_I\, H[g]\right](u) \\
&= \frac{\epsilon_R^2(u)}{|\epsilon(u)|^2}\, g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} \left[H[\epsilon_I](u) H[g](u) - g(u)\, \epsilon_I(u)\right] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g] - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[\epsilon_I] H[g] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u)\, [1 + H[\epsilon_I](u)] \\
&= g(u) + \frac{\epsilon_R(u)\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} H[g](u) \epsilon_R(u) = g(u)
\end{aligned}
$$

G - Transform Properties Continued

Lemma (G4). If ϵ_I and ϵ_R are as above, then

(i) for $vf \in L^p(\mathbb{R})$,

$$\widehat{G}[vf](u) = u\,\widehat{G}[f](u) - \frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{\mathbb{R}} f\,dv\,,$$

(ii) $\widehat{G}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$

(iii) and if $f(u,t)$ and $g(v,t)$ are strongly differentiable in t; i.e. the mapping $t \mapsto f(t) = f(t,\cdot) \in L^p(\mathbb{R})$ is differentiable, (the usual difference quotient converges in the L^p sense), then

a) $\widehat{G}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial \widehat{G}[f]}{\partial t} = \frac{\partial g}{\partial t}$,

b) $G\left[\frac{\partial g}{\partial t}\right] = \frac{\partial G[g]}{\partial t} = \frac{\partial f}{\partial t}$.

prf. (i) goes through like (H3), (ii) follows from $\epsilon_R = 1 + H[\epsilon_I]$, and (iii) follows because G is bounded and linear.

G-Morphism?

$$G[f] = \epsilon_R(v)\, f(v) + \epsilon_I(v)\, H[f](v)$$

$$G^{-1}[f](u) = \frac{\epsilon_R(u)}{\epsilon_R^2 + \epsilon_I^2}\, f(u) - \frac{\epsilon_I(u)}{\epsilon_R^2 + \epsilon_I^2}\, H[f](u)$$

Compare with $z \in \mathbb{C}$: complex numbers $i^2 = -1$, while $H \circ H = -I$.

$$z = x + iy$$

$$z^{-1} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Algebraic structure?

Diagonalize via Mixed Variable Generating Function

Generating Functional:

$$\begin{aligned}\mathcal{F}[q,P] &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, G[P_k](v)\, dv \\ &= \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} \epsilon_R(v)\, q_k(v)\, P_k(v)\, dv + \frac{1}{\pi} \fint_{\mathbb{R}} \int_{\mathbb{R}} \frac{\epsilon_I(v)}{u-v} q_k(v)\, P_k(u)\, dv du \right)\end{aligned}$$

Canonical Coordinate change $(q,p) \longleftrightarrow (Q,P)$:

$$p_k(v) = \frac{\delta \mathcal{F}[q,P]}{\delta q_k(v)} = G[P_k](v) \qquad Q_k(u) = \frac{\delta \mathcal{F}[q,P]}{\delta P_k(u)} = G^{\dagger}[q_k](u)$$

Hamiltonian in new coords:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} i\, \omega_k(u)\, Q_k(u)\, P_k(u)\, du$$

where $\omega_k(u) = ku$.

Signature and Action-Angle variables

Elementary Coord. Change:

$$(Q_k, P_k) \quad \longleftrightarrow \quad (\theta_k, J_k)$$

Hamiltonian in new coords.:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \sigma_k(u)\, \omega_k(u) J_k(u,t)\, du\,,$$

where $\omega_k(u) := |ku|$ and $\sigma_k(u) :=$ sign$(ku\epsilon_I)$.

Poisson bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(\frac{\delta F}{\delta \theta_k} \frac{\delta G}{\delta J_k} - \frac{\delta G}{\delta \theta_k} \frac{\delta F}{\delta J_k} \right) du\,.$$

Continuum eigenmodes have signature. Finite DOF, Krein-Moser says opposite signature needed for bifurcations: colliding $\omega_i(\lambda) \rightarrow$ instability. Vlasov: unstable modes emerge where signatures meet.

Works for Class of Lie-Poisson Hamiltonian Systems: Recall

Hamiltonian (energy):

$$H[\zeta] = H_1 + H_2 = \int_{\mathcal{Z}} d^2z\, h_1(z)\, \zeta(z) + \frac{1}{2}\int_{\mathcal{Z}} d^2z \int_{\mathcal{Z}} d^2z'\, \zeta(z) h_2(z,z')\, \zeta(z')$$

Lie-Poisson Bracket:

$$\{F,G\} = \int_{\mathcal{Z}} d^2z\ \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta}\right]$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = -\left[\zeta, \frac{\delta H}{\delta \zeta}\right] = -[\zeta, \mathcal{E}]$$

Casimir Invariants:

$$C[\zeta] = \int_{\mathcal{Z}} d^2z\ \mathcal{C}(\zeta)$$

Lie-Poisson Hamiltonian Normal Form

Equilibria:

$$\frac{\partial \zeta}{\partial t} = 0 = \{\zeta, H\} = \left[-\zeta, \frac{\partial H}{\partial \zeta}\right] = -[\zeta_e, \mathcal{E}_e]$$

1 DOF Integrability:

$$z := (q,p) \longleftrightarrow (\theta, J) \quad \Rightarrow (\zeta_e(J), \mathcal{E}_e(J)) \quad \text{or} \quad \zeta_e(\mathcal{E}_e) \Rightarrow$$

Hammerstein IE:

$$\mathcal{E}_e(z) = h_1(z) + \int_{\mathcal{Z}} d^2z'\, h_2(z,z')\, \zeta_e(\mathcal{E}_e(z'))$$

Linear EOM:

$$\frac{\partial \widehat{\zeta}}{\partial t} + [\widehat{\zeta}, \mathcal{E}_e] + [\zeta_e, \widehat{\mathcal{E}}] = 0 \quad \text{or} \quad \frac{\partial \widehat{\zeta}}{\partial t} + \Omega(J)\frac{\partial \widehat{\zeta}}{\partial \theta} = \frac{d\zeta_e}{dJ}\frac{\partial \widehat{\mathcal{E}}}{\partial \theta}\,.$$

where $\Omega(J) := d\mathcal{E}_e/dJ$, $\widehat{\mathcal{E}} = \int_{\mathcal{Z}} d^2z'\, h_2(z,z')\, \widehat{\zeta}(z')$ in terms of (θ, J)

Linear Operator Problem

Fourier series:

$$\hat{\zeta} = \sum_k \zeta_k(J)\; e^{ik\theta - ik\omega t}$$

Eigenvalue problem:

$$\mathcal{L}_k \zeta_k := \Omega(J)\zeta_k - \zeta_e' \,\mathcal{E}_k[\zeta_k] = \omega\,\zeta_k\,,$$

with

$$\mathcal{E}_k(J) = \sum_{k'} \int \mathcal{H}_{k,k'}(J,J')\,\zeta_{k'}(J')\,dJ'\,,$$

where $\mathcal{L}_k : \mathcal{B} \to \mathcal{B}$, Banach space $\mathcal{B}$, eigenvalue ω, and $\mathcal{H}_{k,k'}(J,J')$ comes from h_2.

Partition the spectrum of $\mathcal{L}_k$: $\sigma = \sigma_p \cup \sigma_c \cup \sigma_r$. (i) $\omega \in \sigma_p$, point spectrum, if $\mathcal{L}_k - \omega\mathcal{I}$ is not one-one, where $\mathcal{I}$ is the identity operator. (ii) $\omega \in \sigma_R$, residual spectrum, if the range of $\mathcal{L}_k - \omega\mathcal{I}$ is not dense in $\mathcal{B}$. (iii) $\omega \in \sigma_c$, continuous spectrum, if the inverse of $(\mathcal{L}_k - \omega\mathcal{I})$, defined on its range, is unbounded.

This partition convenient because if σ_r is null, then the approximate or Weyl spectrum corresponds to $\sigma_p \cup \sigma_c$. Assume purely σ_c , via energy-Casimir, e.g.

Generalized G-transform

Associate Integral Equation:

$$\mathcal{E}_k(J, J_\omega) = \sum_{k'} \mathcal{H}_{k,k'}(J, J_\omega) + \sum_{k'} \int \mathcal{E}_{k'}(J', J_\omega)\, \mathcal{F}_{k,k'}(J, J', J_\omega)\, dJ' \,,$$

where

$$\mathcal{F}_{k,k'}(J, J', J_\omega) := \left[\frac{\mathcal{H}_{k,k'}(J, J') - \mathcal{H}_{k,k'}(J, J_\omega)}{\Omega(J') - \Omega(J_\omega)} \right] \zeta_e'(J')$$

well-behaved enough for Fredholm theory.

Transform:

$$G_k[g_k](J, t) := \epsilon_k^R(J)\, g_k(J, t) + \fint \frac{\zeta_e'(J) \mathcal{E}_k(J, J_\omega)}{\Omega(J) - \Omega(J_\omega)}\, g_k(J_\omega, t)\, dJ_\omega \,,$$

with

$$\epsilon_k^R(J_\omega) := 1 - \fint \frac{\zeta_e'(J) \mathcal{E}_k(J, J_\omega)}{\Omega(J) - \Omega(J_\omega)}\, dJ \,.$$

Generalized G-transform: Inverse & Identities

Transform Inverse:

$$\widehat{G}_k[f_k](J_\omega, t) := \frac{1}{|\epsilon_k(J_\omega)|^2}\left[\epsilon_k^R(J_\omega) f_k(J_\omega, t) + \fint \frac{\zeta_e'(J_\omega)\mathcal{E}_k(J, J_\omega)}{\Omega(J) - \Omega(J_\omega)} f_k(J, t)\, dJ\right],$$

where $|\epsilon_k(J)|^2 := (\epsilon_k^R)^2 + (\epsilon_k^I)^2$ and $\epsilon_k^I(J_\omega) := \pi\mathcal{E}_k(J_\omega, J_\omega)\zeta_e'(J_\omega)/\Omega'(J_\omega)$.

That $\widehat{G} \circ G = Id$ follows from Poincaré-Bertrand theorem on the interchange of the order of integration for singular integrals.

Transform Identities:

$$\widehat{G}_k[\Omega\, \zeta_k](J_\omega) = \Omega(J_\omega)\widehat{G}_k[\zeta_k](J_\omega) + \frac{\zeta_e'(J_\omega)}{|\epsilon_k|^2(J_\omega)} \fint \zeta_k(J, t)\, \mathcal{E}_k(J, J_\omega)\, dJ\,,$$

and

$$\widehat{G}_k[\zeta_e' \mathcal{E}_k](J_\omega) = \frac{\zeta_e'(J_\omega)}{|\epsilon_k|^2(J_\omega)} \fint \zeta_k(J)\, \mathcal{E}_k(J, J_\omega)\, dJ\,.$$

Shown by techniques similar to those used for verifying the inverse.

General Canonization and Diagonalization

Hamiltonian:

$$\begin{aligned} H_L &= \delta^2 H + \frac{1}{2}\int dJ d\theta\, \mathcal{C}''(\zeta_e)\,(\delta\zeta)^2 = \delta^2 H - \frac{1}{2}\int dJ d\theta\, \frac{\mathcal{E}_e'(J)}{\zeta_e'(J)}\,(\delta\zeta)^2 \\ &= \frac{1}{2}\sum_{k,k'}\int\!\!\int dJ dJ'\,\zeta_k(J)\,\mathcal{H}_{k,k'}(J,J')\,\zeta_{k'}(J') - \frac{1}{2}\sum_k \int dJ\,\frac{\mathcal{E}_e'(J)}{\zeta_e'(J)}\,\zeta_{-k}\zeta_k\,. \end{aligned}$$

Poisson bracket:

$$\{F,G\}_L = \int d\theta dJ\,\zeta_e(J)\left[\frac{\delta F}{\delta\hat{\zeta}},\frac{\delta G}{\delta\hat{\zeta}}\right] = \sum_{k=1}^{\infty} ik\int dJ\,\zeta_e'\left(\frac{\delta F}{\delta\zeta_k}\frac{\delta G}{\delta\zeta_{-k}} - \frac{\delta G}{\delta\zeta_k}\frac{\delta F}{\delta\zeta_{-k}}\right)\,.$$

Linear dynamics:

$$\frac{\partial\hat{\zeta}}{\partial t} = \{\hat{\zeta}, H_L\}_L\,.$$

General Canonization and Diagonalization Cont.

Canonization:

$$q_k(J,t) := \zeta_k(J,t)\,, p_k(J,t) = \frac{\zeta_{-k}(J,t)}{ik\zeta_e'} \quad \to \quad \{F,G\}_L = \sum_{k=1}^{\infty} \int dJ \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k} \right).$$

Diagonalization:

$$\mathcal{F}[q,P] = \sum_{k=1}^{\infty} \int dJ\, P_k(J)\, \widehat{G}[q_k](J) \qquad (q_k, p_k) \longleftrightarrow (Q_k, P_k)$$

Type-2 mixed variable generating functional again.

$$p_k(J) = \frac{\delta \mathcal{F}[q,P]}{\delta q_k(J)} = \widehat{G}^{\dagger}[P_k](J) \quad \text{and} \quad Q_k(J) = \frac{\delta \mathcal{F}[q,P]}{\delta P_k(J)} = \widehat{G}[q_k](J)\,.$$

Hamiltonian in New Coords:

$$\begin{aligned} H_L &= \sum_{k=1}^{\infty} ik \int dJ\, p_k \left[\zeta_e' \mathcal{E}_k - q_k \mathcal{E}_e'\right] = \sum_{k=1}^{\infty} ik \int dJ\, P_k \left(\widehat{G}[\zeta_e' \mathcal{E}_k] - \widehat{G}[\mathcal{E}_k' G[Q_k]]\right) \\ &= -\sum_{k=1}^{\infty} \int dJ\, ik\, \Omega(J) Q_k(J)\, P_k(J)\,. \end{aligned}$$

Continuation

- Investigation of the consequences of the signature of the continuous spectrum; *i.e.* proof of a kind of Krein-Moser theorem in a Banach space setting where embedded discrete modes emerge from negative σ_c. (George Hagstrom Ph.D. 2011)

- Investigate the theory of adiabatic invariants in this infinite dimensional Hamiltonian context by e.g. adding explicit time dependence to the Hamiltonian.

- Develop analog of Birkhoff's <u>nonlinear</u> normal forms for our class of infinite dimensional Hamiltonian systems with continuous spectra. (Thomas Yudichak Ph.D. 2001)

- Obtain our class of infinite dimensional Hamiltonian mean-field systems by reduction from kinetic theory BBGKY, other.

- Investigate the role played by G-transform in an infinite-dimensional in the setting of functional phase space tangent bundle geometry. Symplectomorphism algebra? $\mathbb{C}$-Morphism?

A Collection of Talk References (P. Morrison Nov. 2022)

Here is a list of papers that contain some of the things I talked about. Unfortunately the material is spread over many papers. All can be downloaded from my web page http://www.ph.utexas.edu/∼morrison

The BBGKY hierarchy paper with Marsden and Weinstein that Matt spoke about is [1].

The G-transform for diagonalizing the Vlasov-Poisson system was introduce for physicists in [2], with embedded point spectrum in [3]. The rigorous version was published in [4].

The technique for assigning a signature to the continuous spectrum was introduced in the refs above, with a rigorous analog of the Hamiltonian-Hopf (Krein-Moser) bifurcation with a signed continuous spectrum given in [5]. A tutorial treatment is in [6, 7].

Application of the G-transform to the 2-dimensional incompressible Euler equation is given in [8, 9].

The general form of the G-transform for a large class of mean-field Lie-Poisson systems is given in [10].

The Lie-Poisson-Moyal description of quantum mechanics using the Wigner function was given in [11].

Mean-field Hamilton-Jacobi theory for Vlasov and other theories appeared early in [12] and continued in [13–15] . A summary intended for mathematicians was given in [16]. A survey of various formulations of Vlasov is given in [17].

A description of Vlasov where the mixed variable generating function is replaced by a Lie series to reduce noncanonical Vlasov to a symplectic leaf is given in [18]

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