Metriplectic Dynamics: Magnetofluids & Maxwell-Vlasov – Collisions

Philip J. Morrison

Department of Physics and Institute for Fusion Studies
The University of Texas at Austin
morrison@physics.utexas.edu → my webpage

SIAM online March 18, 2022
Energy-Based Mathematical Methods and Thermodynamics

"There are three stages in scientific discovery. First, people deny that it is true, then they deny that it is important; finally they credit the wrong person." – *Bill Bryson*

Acknowlegement: US Dept of Energy & Humboldt Foundation

Overview

I. Hamiltonian Dynamics

II. Dissipative Dynamics

I. Hamiltonian Dynamics

Classical Field Theory for Classical Purposes

Dynamics of matter described by

- **Fluid models**
  - Euler's equations, Navier-Stokes, ...
- **Magnetofluid models**
  - MHD, XMHD (Hall, electron mass physics), 2-fluid, ...
- **Kinetic theories**
  - Vlasov-Maxwell, Landau-Lenard-Balescu, gyrokinetics, ...
- **Fluid-Kinetic hybrids**
  - MHD + hot particle kinetics, gyrokinetics, ...

Applications:

atmospheres, oceans, fluidics, natural and laboratory plasmas

Hamiltonian and Dissipative structures are <u>organizing principles</u>

Classical Field Theories for Classical Purposes Have Common Structure

Two Dichotomies:

- **Lagrangian vs. Eulerian variables**
  - particle or material vs. spatial or observable
- **Lagrangian vs. Hamiltonian formalisms**
  - Action principle vs. Poisson bracket

Basic procedure of **reduction**:

action principle $\rightarrow$ Hamiltonian $\rightarrow$ noncanonical Poisson bracket

Noncanonical Hamiltonian Structure

Sophus Lie (1890) $\longrightarrow$ PJM (1980) $\longrightarrow$ Poisson Manifolds etc.

Noncanonical Coordinates:

$$\dot{w}^i = \{w^j, H\} = J^{ij}\frac{\partial H}{\partial w^j}\,, \qquad \{A, B\} = \frac{\partial A}{\partial w^i}J^{ij}(w)\frac{\partial B}{\partial w^j}$$

Poisson Bracket Properties:

antisymmetry $\longrightarrow$ $\{A, B\} = -\{B, A\}\,,$

Jacobi identity $\longrightarrow$ $\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$

G. Darboux: $det J \neq 0 \Longrightarrow J \to J_c$ Canonical Coordinates

Sophus Lie: $det J = 0 \Longrightarrow$ Canonical Coordinates plus Casimirs

Flow on Poisson Manifold

Definition. A Poisson manifold $\mathcal{Z}$ is differentiable manifold with bracket

$$\{\,,\,\}: C^{\infty}(\mathcal{Z}) \times C^{\infty}(\mathcal{Z}) \to C^{\infty}(\mathcal{Z})$$

st $C^{\infty}(\mathcal{Z})$ with $\{\,,\,\}$ is a Lie algebra realization, i.e., is

i) bilinear,
ii) antisymmetric,
iii) Jacobi, and
iv) consider only Leibniz, i.e., acts as a derivation.

Flows are integral curves of noncanonical Hamiltonian vector fields, JdH.

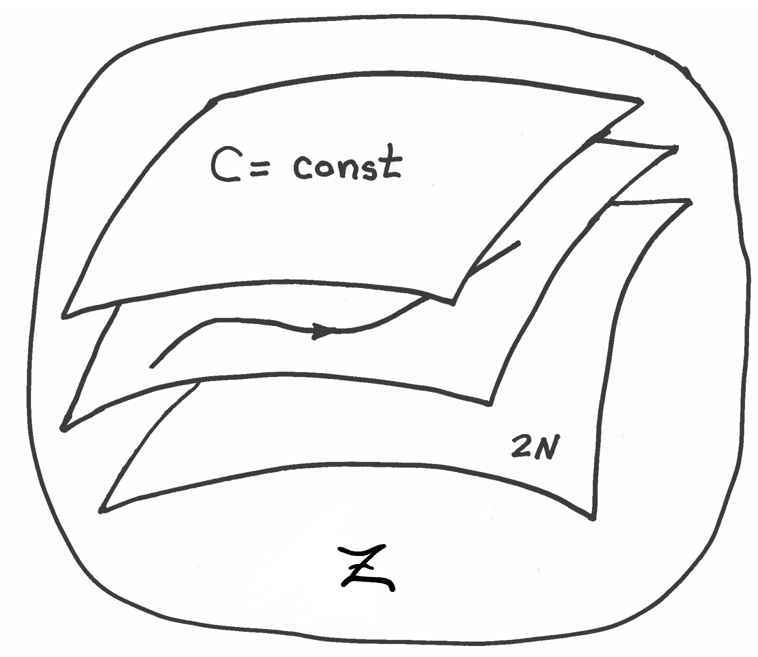
Because of degeneracy, $\exists$ functions C st $\{A, C\} = 0$ for all $A \in C^{\infty}(\mathcal{Z})$. Called Casimir invariants (Lie's distinguished functions!).

Poisson Manifold $\mathcal{Z}$ Cartoon

Degeneracy in $J \;\Rightarrow$ Casimirs:

$$\{A, C\} = 0 \quad \forall\; A : \mathcal{Z} \to \mathbb{R}$$

Lie-Darboux Foliation by Casimir (symplectic) leaves:



Three Plasma Dynamical Systems

- Magnetohydrodynamics
- Extended Magnetohydrodynamics
- Maxwell-Vlasov Equations

Magnetohydrodynamics (MHD)

Equations of Motion:

Force	$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} - \nabla p + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}$
Density	$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{v})$
Entropy	$\frac{\partial s}{\partial t} = -\boldsymbol{v}\cdot\nabla s$
Ohm's Law	$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = \eta\boldsymbol{J} = \eta\nabla\times\boldsymbol{B} \approx 0$
Magnetic Field	$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla\times\boldsymbol{E} = \nabla\times(\boldsymbol{v}\times\boldsymbol{B})$

Energy:

$$H = \int_D d^3x \left(\frac{1}{2}\rho|\boldsymbol{v}|^2 + \rho U(\rho, s) + \frac{1}{2}|\boldsymbol{B}|^2\right)$$

Thermodynamics:

$$p = \rho^2\frac{\partial U}{\partial \rho} \qquad T = \frac{\partial U}{\partial s} \qquad \text{or} \qquad p = \kappa\rho^\gamma$$

Noncanonical Lie-Poisson Bracket (pjm & Greene 1980):

$$
\begin{aligned}
\{F,G\} &= -\int_D d^3x \Bigg[M_i \left(\frac{\delta F}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta G}{\delta M_i} - \frac{\delta G}{\delta M_j} \frac{\partial}{\partial x^j} \frac{\delta F}{\delta M_i} \right) \\
&+ \rho \left(\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right) + \sigma \left(\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right) \\
&+ \boldsymbol{B} \cdot \left[\frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{B}} - \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{B}} \right] \\
&+ \boldsymbol{B} \cdot \left[\nabla \left(\frac{\delta F}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} - \nabla \left(\frac{\delta G}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta F}{\delta \boldsymbol{B}} \right] \Bigg] ,
\end{aligned}
$$

Dynamics:

$$\frac{\partial \rho}{\partial t} = \{\rho, H\}, \quad \frac{\partial s}{\partial t} = \{s, H\}, \quad \frac{\partial \boldsymbol{v}}{\partial t} = \{\boldsymbol{v}, H\}, \text{ and } \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}.$$

Densities:

$$\boldsymbol{M} := \rho \boldsymbol{v} \qquad\qquad \sigma := \rho s$$

Casimir Invariants

Casimir Invariants:

$$\{F, C\}^{MHD} = 0 \qquad \forall \text{ functionals } F.$$

Casimirs Invariant entropies:

$$C_S = \int d^3x \, \rho f(s) \,, \qquad \text{f arbitrary}$$

Casimirs Invariant helicities:

$$C_B = \int d^3x \, \boldsymbol{B} \cdot \boldsymbol{A} \,, \qquad C_V = \int d^3x \, \boldsymbol{B} \cdot \boldsymbol{v}$$

Helicities have topological content, linking etc.

Fortschritte der Physik 7, 503—558 (1959)

Über die Ausbreitung von Wellen in einem Plasma

Von REIMAR LÜST

Max-Planck-Institut für Physik und Astrophysik, München

Inhaltsverzeichnis

Zusammenfassung

Einleitend wird kurz auf die Bedeutung und die Anwendungsmöglichkeiten hingewiesen, die die Wellen in einem Plasma für verschiedene Gebiete der Physik und der Astrophysik haben können (Abschnitt 1). Die Grundgleichungen für ein nicht quasineutrales Plasma werden in Abschnitt 2 abgeleitet, wobei

Extended Magnetohydrodynamics (XMHD)

Ideal Ohm's Law:

$$\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B} = \frac{d_e^2}{\rho}\left(\frac{\partial \boldsymbol{J}}{\partial t} + \nabla \cdot \left(\boldsymbol{V}\boldsymbol{J} + \boldsymbol{J}\boldsymbol{V} - \frac{d_i}{\rho}\boldsymbol{J}\boldsymbol{J}\right)\right) + \frac{d_i}{\rho}\Big(\boldsymbol{J} \times \boldsymbol{B} - \nabla p_e\Big).$$

Momentum:

$$\rho\left(\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right) = -\nabla p + \boldsymbol{J} \times \boldsymbol{B} - d_e^2\, \boldsymbol{J} \cdot \nabla\left(\frac{\boldsymbol{J}}{\rho}\right).$$

Two parameters, $d_e = \frac{c}{\omega_{p_e} L}$ measures electron inertia and $d_i = \frac{c}{\omega_{p_i} L}$ accounts for current carried by electrons mostly ... .

Energy Conservation

Candidate Hamiltonian:

$$H = \int d^3x \left[\rho \frac{|\boldsymbol{V}|^2}{2} + \rho U(\rho) + \frac{|\boldsymbol{B}|^2}{2} + d_e^2 \frac{|\boldsymbol{J}|^2}{2\rho} \right]$$

Kimura and pjm 2014 on energy conservation

H is conserved. Pressure, $p = \rho^2 \partial U / \partial \rho$.

What is the Poisson bracket? Casimirs? Helicities?

XMHD Hamiltonian Structure

Poisson Bracket:

$$\begin{aligned}\{F,G\}^{XMHD} &= \{F,G\}^{MHD} \\ &+ d_e^2 \int_D d^3x \left[\frac{\nabla\times\mathbf{V}}{\rho}\cdot\Big((\nabla\times F_{\mathbf{B}^\star})\times(\nabla\times G_{\mathbf{B}^\star})\Big)\right] \\ &+ d_i \int_D d^3x\, \frac{\mathbf{B}^\star}{\rho}\cdot\Big[(\nabla\times F_{\mathbf{B}}^\star)\times(\nabla\times G_{\mathbf{B}}^\star)\Big]\end{aligned}$$

where we introduce the 'inertial' magnetic field

$$\mathbf{B}^\star = \mathbf{B} + d_e^2\,\nabla\times\left(\frac{\nabla\times\mathbf{B}}{\rho}\right),$$

Hamiltonian:

$$H = \int_D d^3x \left[\frac{\rho|\mathbf{V}|^2}{2} + \rho U(\rho) + \frac{\mathbf{B}\cdot\mathbf{B}^\star}{2}\right].$$

XMHD Hamiltonian Structure (cont)

Casimirs;

$$C^{\pm}_{XMHD} = \int_D d^3x\ (\mathbf{V} + \lambda_\pm \mathbf{A}^\star) \cdot (\nabla \times \mathbf{V} + \lambda_\pm \mathbf{B}^\star)\ ,$$

where

$$\lambda_\pm = \frac{-d_i \pm \sqrt{d_i^2 + 4d_e^2}}{2d_e^2}\,.$$

Jacobi Identity:

Abdelhamid, Kawazura, Yoshida (2015); pjm, Lingam, Miloshevich, D'Avignon (2016)

Maxwell-Vlasov Equations

Maxwell's Equations:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -c\nabla \times \boldsymbol{E} \qquad \frac{\partial \boldsymbol{E}}{\partial t} = c\nabla \times \boldsymbol{B} - 4\pi \boldsymbol{J}_e$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{E} = 4\pi \rho_e$$

Coupling to Vlasov

$$\frac{\partial f_s}{\partial t} = -\boldsymbol{v} \cdot \nabla f_s - \frac{e_s}{m_s} \left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \right) \cdot \frac{\partial f_s}{\partial \boldsymbol{v}}$$

$$\rho_e(\boldsymbol{x}, t) = \sum_s e_s \int f_s(\boldsymbol{x}, \boldsymbol{v}, t)\, d^3 v\,, \qquad \boldsymbol{J}_e(\boldsymbol{x}, t) = \sum_s e_s \int \boldsymbol{v}\, f_s(\boldsymbol{x}, \boldsymbol{v}, t)\, d^3 v$$

$f_s(\boldsymbol{x}, \boldsymbol{v}, t)$ is a phase space density for particles of species s with charge and mass, e_s, m_s.

$$\psi = \Big(\boldsymbol{E}(\boldsymbol{x}, t),\, \boldsymbol{B}(\boldsymbol{x}, t),\, f_s(\boldsymbol{x}, \boldsymbol{v}, t) \Big)$$

Maxwell-Vlasov Hamiltonian Structure

Hamiltonian:

$$H = \sum_s \frac{m_s}{2}\int |\boldsymbol{v}|^2 f_s \, d^3x \, d^3v + \frac{1}{8\pi}\int (|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2) \, d^3x \,,$$

Bracket:

$$\begin{aligned} \{F, G\} \;=\; & \sum_s \int \Big(\frac{1}{m_s} f_s \left(\nabla F_{f_s} \cdot \partial_{\boldsymbol{v}} G_{f_s} - \nabla G_{f_s} \cdot \partial_{\boldsymbol{v}} F_{f_s} \right) \\ & + \frac{e_s}{m_s^2 c} f_s \, \boldsymbol{B} \cdot \left(\partial_{\boldsymbol{v}} F_{f_s} \times \partial_{\boldsymbol{v}} G_{f_s} \right) \\ + \; & \frac{4\pi e_s}{m_s} f_s \left(G_{\boldsymbol{E}} \cdot \partial_{\boldsymbol{v}} F_{f_s} - F_{\boldsymbol{E}} \cdot \partial_{\boldsymbol{v}} G_{f_s} \right) \Big) d^3x \, d^3v \\ + \; & 4\pi c \int (F_{\boldsymbol{E}} \cdot \nabla \times G_{\boldsymbol{B}} - G_{\boldsymbol{E}} \cdot \nabla \times F_{\boldsymbol{B}}) \, d^3x \,, \end{aligned}$$

where $\partial_{\boldsymbol{v}} := \partial/\partial \boldsymbol{v}$, F_{f_s} means functional derivative of F with respect to f_s etc.

pjm 1980,1982; Marsden and Weinstein 1982

Maxwell-Vlasov Structure (cont)

Equations of Motion:

$$\frac{\partial f_s}{\partial t} = \{f_s, H\}\,, \quad \frac{\partial \boldsymbol{E}}{\partial t} = \{\boldsymbol{E}, H\}\,, \quad \frac{\partial \boldsymbol{B}}{\partial t} = \{\boldsymbol{B}, H\}\,.$$

Casimirs invariants:

$$\begin{aligned} \mathcal{C}_s^f[f_s] &= \int \mathcal{C}_s(f_s)\, d^3x d^3v \\ \mathcal{C}^E[\boldsymbol{E}, f_s] &= \int h^{\boldsymbol{E}}(x) \left(\nabla \cdot \boldsymbol{E} - 4\pi \sum_s e_s \int f_s\, d^3v \right) d^3x\,, \\ \mathcal{C}^B[\boldsymbol{B}] &= \int h^{\boldsymbol{B}}(x)\, \nabla \cdot \boldsymbol{B}\, d^3x\,, \end{aligned}$$

where $\mathcal{C}_s$, $h^{\boldsymbol{E}}$ and $h^{\boldsymbol{B}}$ are arbitrary functions of their arguments. These satisfy the degeneracy conditions

$$\{F, C\} = 0 \quad \forall F\,.$$

- GEMPIC Sonnendrücker's Wednesday talk – Poisson integrator

Summary

Poisson brackets defined by $\mathbb{J}$, dynamics $\partial\psi/\partial t = \{\psi, H\}$:

$$\begin{aligned} \mathbb{J}_{MHD} &\quad\rightarrow\quad \text{Casimirs} \\ \mathbb{J}_{XMHD} &\quad\rightarrow\quad \text{Casimirs} \\ \mathbb{J}_{M-V} &\quad\rightarrow\quad \text{Casimirs} \end{aligned}$$

Good theories in their ideal limit ($\nu, \eta, \cdots \rightarrow 0$) conserve energies, H, and have **Poisson brackets**. Bad theories do bad things: unaccounted energy, unphysical instabilities, etc.

Dissipation? Casimirs are candidates for entropies!

Dissipation: Metriplectic Dynamics $\equiv$ Generic

“History”

- Rayleigh Dissipation (Theory of Sound, Ch. IV §81 1873)

- 20th Century gradient flows (Cahn-Hilliard, Otto, Ricci Flows, Poincarè conjecture on S^3, ...)

- pjm, pjm & Kaufman 1982

- pjm 1984, Kaufman 1984, ... Grmela 1984

- pjm 1986 Metriplectic Dynamics

- Grmela & Oetttinger 1997, Generic $\equiv$ Metriplectic Dynamics

- Many works since ... e.g. Bloch, et al. 2013, pjm & Coquinot

- Double brackets: Brockett, Vallis et al., Flierl &pjm,. . .

Bracket Formulations circa 1980

$$\{\{F,G\}\} = \{F,G\} + (F,G)$$

Generator: $\mathcal{F} = H + S$: Kaufman (1984), pjm (1984), Grmela (1984) .... H: Kaufman & pjm (1982), Brockett, Carnevale, et al. ...

Symmetry: Kaufman & pjm (1982), Kaufman (1984), pjm (1984), Brockett, Vallis et al. (1989)

Degeneracy: Kaufman & pjm (1982), pjm (1984), Grmela (1984)

Metriplectic Dynamics $\equiv$ Generic: pjm (1984,1986)

Entropy, Degeneracies, and 1st and 2nd Laws

- Casimirs of $\{,\}$ are 'candidate' entropies. Election of particular $S \in \{\text{Casimirs}\} \Rightarrow$ thermal equilibrium (relaxed) state.

- Generator (free energy): $\mathcal{F} = H + S$

- 1st Law: identify energy with Hamiltonian, H, then

$$\dot{H} = \{H, \mathcal{F}\} + (H, \mathcal{F}) = 0 + (H, H) + (H, S) = 0$$

  Degeneracy such that $(H, f) = 0 \ \forall \ f$

- 2nd Law: entropy production

$$\dot{S} = \{S, \mathcal{F}\} + (S, \mathcal{F}) = (S, S) \geq 0$$

  Lyapunov relaxation to the equilibrium state: $\delta\mathcal{F} = 0$.

Two examples of pjm 1984

Vlasov with Collisions

$$\frac{\partial f}{\partial t} = -v \cdot \nabla f - a \cdot \nabla_v f + \left.\frac{\partial f}{\partial t}\right)_c$$

where

$$\text{Collision term} \quad \rightarrow \quad \left.\frac{\partial f}{\partial t}\right)_c$$

could be , Landau, Lenard Balescu, etc.

Conserves, mass, momentum, energy,

$$\frac{dH}{dt} = \frac{d}{dt}\int \frac{1}{2} m v^2 f + \text{interaction} = 0$$

and makes entropy

$$\frac{dS}{dt} = -\frac{d}{dt}\int f \ln(f) \geq 0$$

Landau Collision Operator

Metriplectic bracket:

$$(A,B) \;=\; \int dz \int dz' \left[\frac{\partial}{\partial v_i}\frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_i'}\frac{\delta A}{\delta f(z')}\right] T_{ij}(z,z') \times \left[\frac{\partial}{\partial v_j}\frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_j'}\frac{\delta B}{\delta f(z')}\right]$$

$$T_{ij}(z,z') = w_{ij}(z,z') f(z) f(z')/2$$

Conservation and Lyapunov:

$w_{ij}(z,z') = w_{ji}(z,z') \quad w_{ij}(z,z') = w_{ij}(z',z) \quad g_i w_{ij} = 0$ with $g_i = v_i - v_i'$

Landau kernel:

$$w_{ij}^{(L)} = (\delta_{ij} - g_i g_j/g^2)\delta(\mathbf{x}-\mathbf{x}')/g$$

Entropy:

$$S[f] = \int dz\, f \ln(f)$$

Ideal fluid with viscous heating and thermal conductivity.

$$\frac{\partial v_i}{\partial t} = \{v_i, \mathcal{H}\} \tag{18}$$

$$\frac{\partial \rho}{\partial t} = \{\rho, \mathcal{H}\} \tag{19}$$

$$\frac{\partial s}{\partial t} = \{s, \mathcal{H}\} \tag{20}$$

where the GPB, {,}, is given by

$$\{F,G\} = -\int\Bigg(\frac{\delta F}{\delta \rho}\vec{\nabla}\cdot\frac{\delta G}{\delta \vec{v}} + \frac{\delta F}{\delta \vec{v}}\cdot\vec{\nabla}\frac{\delta G}{\delta \rho} + \frac{\delta F}{\delta \vec{v}}\cdot\left[\frac{(\vec{\nabla}\times\vec{v})}{\rho}\times\frac{\delta G}{\delta \vec{v}}\right] + \frac{\vec{\nabla}s}{\rho}\cdot\left[\frac{\delta F}{\delta s}\frac{\delta G}{\delta \vec{v}} - \frac{\delta F}{\delta \vec{v}}\frac{\delta G}{\delta s}\right]\Bigg)d^3x\ . \tag{21}$$

Upon inserting the quantities shown on the right hand side of Eqs. (18)-(20), into Eq. (21) and performing the indicated operations one obtains, as noted, the invicid adiabatic limit of Eqs. (10)-(12).

The Casimirs for the bracket given by Eq. (21) are the total mass $M = \int \rho\, d^3x$ and a generalized entropy functional $\mathcal{S}_f = \int \rho f(s)\, d^3x$, where f is an arbitrary function of s. The latter quantity is added to the energy [Eq. (17)] to produce the generalized free energy of Eq. (4): $\mathcal{F} = \mathcal{H} + \mathcal{S}_f$.

In order to obtain the dissipative terms, we introduce the following symmetric bracket:

$$(F,G) = \frac{1}{\lambda}\int\Bigg\{\frac{1}{\rho}\frac{\delta F}{\delta v_i}\frac{\partial}{\partial x_k}\left[\frac{\sigma_{ik}}{\rho}\frac{\delta G}{\delta s}\right] + \frac{1}{\rho}\frac{\delta G}{\delta v_i}\frac{\partial}{\partial x_k}\left[\frac{\sigma_{ik}}{\rho}\frac{\delta F}{\delta s}\right] + \frac{\sigma_{ik}}{T}\frac{\partial v_i}{\partial x_k}\left[\frac{1}{\rho^2}\frac{\delta F}{\delta s}\frac{\delta G}{\delta s}\right] + T^2\kappa\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta F}{\delta s}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho T}\frac{\delta G}{\delta s}\right] + T\,\Lambda_{ikmn}\frac{\partial}{\partial x_m}\left[\frac{1}{\rho}\frac{\delta F}{\delta v_n}\right]\frac{\partial}{\partial x_k}\left[\frac{1}{\rho}\frac{\delta G}{\delta v_i}\right]\Bigg\}d^3x\ , \tag{23}$$

FIG. 10. This . ...

General Metriplectic Form

$$(F,G) = \int d^n z \int d^n z' \, \mathcal{L}' \left(\frac{\delta F}{\delta \chi}\right) \cdot g(z,z';\chi) \, \cdot \mathcal{L} \left(\frac{\delta G}{\delta \chi}\right)$$

$\mathcal{L}$ a formally self-adjoint pseudo-differential operator, g a symmetric operator, $z = (z^1, \ldots, z^n)$, and $\chi = (\chi^1, \ldots \chi^m)$.

Degeneracies can appear from kernel of $\mathcal{L}$ and g

Nonlocal type occurs e.g. for the Landau collision operator and for Lagrangian fluid mechanics.

Thermodynamics

Entropy/volume: $\sigma(\boldsymbol{x}, t)$

Density of Extensive variable: $\zeta_a(\boldsymbol{x}, t)$ $\qquad a = 1, 2, \ldots$

$$d\sigma = \sum_a \frac{\partial \sigma}{\partial \zeta_a d\zeta_a} =: \sum_a X^a d\zeta_a$$

$$\frac{\partial \zeta_a}{\partial t} + \nabla \cdot \boldsymbol{J}_T = \sum_a \boldsymbol{J}_a \cdot \nabla X^a \,,$$

$$\boldsymbol{J}_T = \sum_a X^a \boldsymbol{J}_a \,, \qquad \boldsymbol{J}_a = \text{unknown flux?}$$

Near Equilibrium Assumption:

$$\boldsymbol{J}_a = \sum_b L_{ab} \nabla X^b$$

Onsager for Affnity ∇X^a:

$$L_{ab} = L_{ba} \qquad \Rightarrow \qquad \text{Second Law}$$

Magnetofluid (F,G)

The Dissipative Bracket:

$$(F,G) = \frac{1}{\mathcal{T}} \int d^3x \, \nabla \frac{\delta F}{\delta \zeta_a} \cdot L_{ab}[\zeta] \cdot \nabla \frac{\delta G}{\delta \zeta_b}$$

Natural Variable $\mathcal{E}$:

$$H = \int d^3x \, \mathcal{E} \qquad \Rightarrow \qquad (F,H) = 0 \quad \forall F$$

Hamiltonian $(\boldsymbol{M}, \boldsymbol{B}^*, \rho, \sigma)$ vs. Metriplectic $(\boldsymbol{M}, \boldsymbol{B}^*, \rho, \mathcal{E})$

Onsager Pairs (Force/Flux):

- Current $\leftrightarrow$ Temp, etc., in particular
- Viscosity $\leftrightarrow$ Current

XMHD, Coquinot & pjm J. Plasma Phys. (2020) complicated Onsager reciprocity

$$
\begin{aligned}
(f,g) = \int_\Omega \mathrm{d}^3y\, \frac{T}{\mathcal{T}} \Bigg[& \nabla\left(\frac{1}{T}\frac{\delta f}{\delta\sigma(\mathbf{y})}\right) \cdot \Bigg\{ T^2\eta_{TT}\nabla\left(\frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\right) \\
& + T\eta_{Tj}\left(\nabla\times\left(\frac{\delta g}{\delta\mathbf{B}^*(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\mathbf{j}\right) \Bigg\} \\
& + \left(\nabla\times\left(\frac{\delta f}{\delta\mathbf{B}^*(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta f}{\delta\sigma(\mathbf{y})}\mathbf{j}\right) \cdot \Bigg\{ T\eta_{jT}\nabla\left(\frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\right) \\
& + \eta_{jj}\left(\nabla\times\left(\frac{\delta g}{\delta\mathbf{B}^*(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\mathbf{j}\right) \Bigg\} \\
& + \left(\nabla\left(\frac{\delta f}{\delta\mathbf{m}(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta f}{\delta\sigma(\mathbf{y})}\nabla\mathbf{v}\right) : \Bigg\{ \Lambda_{vv}\left(\nabla\left(\frac{\delta g}{\delta\mathbf{m}(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\nabla\mathbf{v}\right) \\
& + \Lambda_{vj}\left(\nabla\left(\frac{1}{\rho}\nabla\times\frac{\delta g}{\delta\mathbf{B}^*(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\nabla\left(\frac{\mathbf{j}}{\rho}\right)\right) \Bigg\} \\
& + \left(\nabla\left(\frac{1}{\rho}\nabla\times\frac{\delta f}{\delta\mathbf{B}^*(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta f}{\delta\sigma(\mathbf{y})}\nabla\left(\frac{\mathbf{j}}{\rho}\right)\right) \\
& : \Bigg\{ \Lambda_{jv}\left(\nabla\left(\frac{\delta g}{\delta\mathbf{m}(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\nabla\mathbf{v}\right) \\
& + \Lambda_{jj}\left(\nabla\left(\frac{1}{\rho}\nabla\times\frac{\delta g}{\delta\mathbf{B}^*(\mathbf{y})}\right) - \frac{1}{T}\frac{\delta g}{\delta\sigma(\mathbf{y})}\nabla\left(\frac{\mathbf{j}}{\rho}\right)\right) \Bigg\} \Bigg] .
\end{aligned}
$$

Metriplectic Dynamics References (live links):

- A. N. Kaufman and P. J. Morrison, Algebraic Structure of the Plasma Quasilinear Equations, Physics Letters A **88**, 405–406 (1982).
- P. J. Morrison, Some Observations Regarding Brackets and Dissipation, Center for Pure and Applied Mathematics Report PAM–228, University of California, Berkeley (1984).
- P. J. Morrison and R. D. Hazeltine, Hamiltonian Formulation of Reduced Magnetohydrodynamics, Physics of Fluids **27**, 886–897 (1984).
- P. J. Morrison, A Paradigm for Joined Hamiltonian and Dissipative Systems, Physica D **18**, 410–419 (1986).
- P. J. Morrison,Thoughts on Brackets and Dissipation: Old and New Journal of Physics: Conference Series **169**, 012006 (2009).
- A.M. Bloch, P. J. Morrison, and T. S. Ratiu, Gradient Flows in the Normal and Kaehler Metrics and Triple Bracket Generated Metriplectic Systems, Recent Trends in Dynamical Systems, eds. A. Johann et al., Springer Procs in Math. & Stats. **35**, (2013) pp. 371–415.
- M. Materassi and P. J. Morrison, Metriplectic Torque for Rotation Control of a Rigid Body, Journal Cybernetics and Physics **7**, 78–86 (2018).
- B. Coquinot and P. J. Morrison, A General Metriplectic Framework with Application to Dissipative Extended Magnetohydrodynamics, Journal of Plasma Physics **86**, 835860302 (2020).

A Few (mostly early) Hamiltonian References (live links):

- P. J. Morrison and J. M. Greene, Noncanonical Hamiltonian Density Formulation of Hydrodynamics and Ideal MHD, Phys. Rev. Lett. **45**, 790 (1980); **48**, 569 (1982).
- P. J. Morrison, Poisson Brackets for Fluids and Plasmas, AIP Conf. Proc. **88**, 13 (1982).
- J. E. Marsden, P. J. Morrison, and A. Weinstein, The Hamiltonian Structure of the BBGKY Hierarchy Equations, Contemporary Mathematics **28**, 115–124 (1984).
- R. D. Hazeltine, D. D. Holm, J. E. Marsden, and P. J. Morrison, Generalized Poisson Brackets and Nonlinear Liapunov Stability-Application to Reduced MHD, in Int. Conf. Plasma Phys. Proc. **1**, eds. M. Tran and M. Sawley, (Ecole Polytech. Federale de Lausanne, 1984) p. 203.
- P. J. Morrison, Hamiltonian Description of the Ideal Fluid, Rev. Mod. Phys. **70**, 467 (1998).
- P. J. Morrison, On Hamiltonian and Action Principle Formulations of Plasma Dynamics AIP Conf. Proc. **1188**, 329–344 (2009).
- P. J. Morrison, A General Theory for Gauge-Free Lifting, Phys. Plasmas **20**, 012104 (2013).