12/18/80

Continuous Hamiltonian Systems: The Equations of Ideal MHD & Vlasov

P. Morrison

OVERVIEW

1. REVIEW OF HAMILTONIAN Systems
2. Generalization
3. KdV Equation
4. MHD
5. Vlasov-Poisson
6. Possibilities & Avenues

1. Review

§ 1.1 Usual Approach

P.E. & K.E. $\Longrightarrow$

$$\mathcal{L}(q, \dot{q}) \xrightarrow[\text{Legendre}]{} \mathcal{H}(q, p)$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \qquad \dot{p}_k = -\frac{\partial H}{\partial q_k}$$

$$k = 1, 2, \dots N$$

§ 1.2 Why Beneficial

— Dynamics contained in H instead of 2N functions

— Transformation theory $Q_k = Q_k(\underline{p}, \underline{q})$ & $P_k = P_k(\underline{p}, \underline{q})$ involves one function — Mixed variable generating function or Lie

"Harmonic Analysis as the Explotation of Sym." G.W. Mackey, Bull. AMS, V.3, #1 (1980) pp. 543 – 696

§1.3 Axiomatic Approach

$$[A,B] = \sum_k \left[\frac{\partial A}{\partial q_k}\frac{\partial B}{\partial p_k} - \frac{\partial B}{\partial q_k}\frac{\partial A}{\partial p_k} \right]$$

$$\dot{q}_k = [q_k, H] \qquad \dot{p}_k = [p_k, H]$$

let $z^i = \begin{cases} q_k & i = k = 1, 2, \cdots N \\ p_k & i = k+N = N+1, \cdots 2N \end{cases}$

$$\dot{z}^i = [z^i, H] \quad \& \quad \underline{[A,B] = \frac{\partial A}{\partial z^i} J^{ij} \frac{\partial B}{\partial z^j}}$$

where $(J^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$

$$\bar{z}^i = \bar{z}^i(\underline{z}) \implies$$

$$\bar{J}^{ij} = J^{rs} \frac{\partial \bar{z}^i}{\partial z^r} \frac{\partial \bar{z}^k}{\partial z^s} \qquad \text{if}$$

$\bar{J} = J$ canonical

Properties of Bracket

(i) $[A,B] = -[B,A]$ Antisymmetry

$[A+B,C] = [A,C] + [B,C]$

(ii) $[\lambda A, B] = \lambda [A,B]$, $\lambda \in \mathbb{R}$ Bilinearity

(iii) $[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$ Jacobi Id.

(4)

$J^{ij} = [z^i, z^j] \Rightarrow$ Jacobi Identity

$$S^{ijk} = J^{il}\frac{\partial}{\partial z^l}J^{jk} + J^{jl}\frac{\partial}{\partial z^l}J^{ki} + J^{kl}\frac{\partial}{\partial z^l}J^{ij}$$

$$\bar{S}^{ijk} = S^{qrs}\frac{\partial \bar{z}^i}{\partial z^q}\frac{\partial \bar{z}^j}{\partial z^r}\frac{\partial \bar{z}^k}{\partial z^s}$$

$$S = 0 \Rightarrow \bar{S} = 0$$

Vector space of functions (over $\mathbb{R}$) w/ Poisson Bracket product defines Lie Algebra. Ex. Vectors in E^3 w/ cross Pdt.

<u>Axiomatic Approach</u>

- Forget about canonical variables concentrate on algebraic properties

- Find Hamiltonian, "Guess" bracket (Easier than finding canonical variables)

- Darbeux Theorem $\Rightarrow$ Canonical variables exist (1882)

— Noncanonical Variables may be better for perturbation theory.

R. Littlejohn, J. Math. Phys. <u>20</u>, 2445 (1979).

§ 1.4 Continuous Hamiltonian Systems

$N \to \infty$, PDE's

H. Goldstein, <u>Classical Mech.</u>, Ch. 11

$$[\hat{A}, \hat{B}] = \sum_k \int \left(\frac{\delta \hat{A}}{\delta \eta_k} \frac{\delta \hat{B}}{\delta \pi_k} - \frac{\delta \hat{B}}{\delta \eta_k} \frac{\delta \hat{A}}{\delta \pi_k} \right) dz$$

$$= \int \frac{\delta \hat{A}}{\delta u^i} O^{ij} \frac{\delta \hat{B}}{\delta u^j} \, dz = \left\langle \frac{\delta \hat{A}}{\delta u^i} \middle| O^{ij} \frac{\delta \hat{B}}{\delta u^j} \right\rangle$$

$$(O^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$ cosymplectic density

— Darboux for ∞ dimensions?

J. Marsden, private communication

6-A

2. References

Y. Manin, J. Soviet Math., 11 (1979) p. 1

B. Kupershmidt, Lect. Notes in Math, 755.

P.R. Chernoff & J. Marsden, Lect. Notes in Math., 425, S. Verlag 1974

Abraham & J. Marsden, ch. 4., Foundations of Mech. Benjamin.

I. Gelfand & I. Dorfman, Func. Anal. & App., 13 (1979) p. 248

2. Generalization

§2.1 Preliminary Specification

$$\frac{\partial u^i(\underline{x}, t)}{\partial t} = F^i(\underline{u}, \underline{x}) \qquad i = 1, 2, \cdots N$$

u^i is defined on $T \times V$
where $T \subset \mathbb{R}$ & $V \subset \mathbb{R}^n$

Suppose F^i is an operator composed of C^∞ functions of the following:

a) $\underline{u}$ & $\underline{x}$

b) $u^i_k \triangleq \dfrac{\partial^k u^i}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_m^{k_m}}$; $k = |\underline{k}| = \sum_{i=1}^{m} k_i$,
$k_i \in \mathbb{Z}^+$

c) $u^i_{-k} \triangleq \int^{x_1^{k_1}} \int^{x_2^{k_2}} \cdots \int^{x_m^{k_m}} u^i \, (dx_1)^{k_1} (dx_2)^{k_2} \cdots (dx_m)^{k_m}$

d) $\int_V K(\underline{x} | \underline{x}') \, f(\underline{u}_k, \underline{u}_{-k}) \, d\tau'$

Call Class of Operators $\mathcal{L}$

⑦

§ 2.2 Two Vector Spaces

a) Suppose solutions $u^i(\underline{x}, t)$ are elements of ω, vector space of functions of $\underline{x}$.

Define inner product by

$$\langle f \mid g \rangle = \int_V f g \, d\tau$$

b) Vector space, Ω, whose elements are functionals of the form

$$\hat{F}\{\underline{u}\} = \int_V F(\underline{u}) \, d\tau$$

where $F \in \mathcal{L}$

EXAMPLES

a) if $f, g(\underline{u})$

$\langle f \mid g \rangle \in \Omega$

b) $\hat{H}\{\rho, v\} = \int_V \left(\frac{1}{2}\rho v^2 + \rho \bar{U}\right) d\tau$

$\in \Omega$

§2.3 DEFINITION.

a system is Hamiltonian if the following hold:

a) $\exists$ an integral invariant

$$\hat{H}\{\underline{u}\} = \int_V H(\underline{u})\, dz \in \Omega$$

s.t.

$$\frac{\partial H}{\partial t} + \sum_i^n \frac{\partial}{\partial x_i} J^i = 0$$

b) Can write equations in the form

$$\frac{\partial u^i}{\partial t} = O^{ij} \frac{\delta \hat{H}}{\delta u^j} \qquad i,j = 1,2,\cdots N$$

where $\frac{\delta \hat{H}}{\delta u^j}$ is the functional derivative defined by

9 $$\frac{d}{d\epsilon}\hat{F}\{u^i+\epsilon w\}\Big|_{\epsilon=0} \triangleq \left\langle \frac{\delta \hat{F}}{\delta u^i} \middle| w \right\rangle$$

EXAMPLES

a) $$\hat{\mathcal{L}}\{q\} = \int \mathcal{L}(q,\dot{q})\,dt$$

$$\frac{d}{d\epsilon}\hat{\mathcal{L}}\{q+\epsilon w\}\Big|_{\epsilon=0} = \int \underbrace{w\left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right)}_{w\frac{\delta \hat{\mathcal{L}}}{\delta q}} dt$$

b) $$\hat{F}\{u\} = \int F(u, u_x, u_{xx}, \cdots)\,dx$$

$$\frac{\delta \hat{F}}{\delta u} = \frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial F}{\partial u_{xx}} - \cdots$$

c) $$\hat{F}\{u\} = u(x) = \int u(x')\,\delta(x-x')\,dx'$$

$$\frac{\delta \hat{F}}{\delta u(x')} = \delta(x-x')$$

(10)

c) $\mathcal{O}^{ij}$ via bracket makes Ω a Lie Algebra

Recall!
$$[\hat{F}, \hat{G}] = \left\langle \frac{\delta \hat{E}}{\delta u^i} \middle| \mathcal{O}^{ij} \frac{\delta \hat{G}}{\delta u^j} \right\rangle$$

$$[\hat{F}, \hat{G}] : \Omega \times \Omega \longrightarrow \Omega$$

i) $[\hat{F}, \hat{G}] = -[\hat{G}, \hat{F}] \quad \forall \hat{F}, \hat{G} \in \Omega$

$\Longleftrightarrow \quad \mathcal{O}^{ij}$ is anti-self-adjoint

ii) Jacobi Identity

$$[\hat{E}, [\hat{F}, \hat{G}]] + [\hat{F}, [\hat{G}, \hat{E}]] + [\hat{G}, [\hat{E}, \hat{F}]] = 0$$

$$\forall \hat{F}, \hat{G}, \hat{E} \in \Omega$$

If the above hold then can write system in Heisenberg form

$$\frac{\partial u^i}{\partial t} = [u^i, \hat{H}]$$

§2.4 Jacobi Identity

P. Lax, Comm. Pure Appl. Math., 28 (1975) 141.

Recall $\frac{d}{d\epsilon} \hat{F}\{u^i + \epsilon w\}\Big|_{\epsilon=0} = \left\langle \frac{\delta \hat{F}}{\delta u^i} \Big| w \right\rangle \triangleq \hat{G} \in \Omega$

Do it again

$$\frac{d}{d\eta} \hat{G}\{u^j + \epsilon z\}\Big|_{\eta=0} = \left\langle \frac{\delta \hat{G}}{\delta u^j} \Big| z \right\rangle$$

$$\triangleq \left\langle \frac{\delta^2 \hat{F}}{\delta u^j \delta u^i} z \Big| w \right\rangle$$

By symmetry operator $\frac{\delta^2 \hat{F}}{\delta u^j \delta u^i}$ has property

$$\left\langle \frac{\delta^2 \hat{F}}{\delta u^i \delta u^j} z \Big| w \right\rangle = \left\langle z \Big| \frac{\delta^2 \hat{F}}{\delta u^j \delta u^i} w \right\rangle$$

(12)

EXAMPLE

$$\hat{F}\{u\} = \int F(u, u_x)\, dx$$

$$\hat{G} = \frac{d}{d\epsilon} \hat{F}\{u + \epsilon w\}\Big|_{\epsilon=0} = \left\langle \frac{\delta \hat{F}}{\delta u} \Big| w \right\rangle = \left\langle \frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x} \Big| w \right\rangle$$

$$= \left\langle \frac{\partial F}{\partial u} - \frac{\partial^2 F}{\partial u \partial u_x} u_x - \frac{\partial^2 F}{\partial u_x^2} u_{xx} \Big| w \right\rangle$$

$$\frac{d}{d\eta} \hat{G}\{u + \eta z\}\Big|_{\eta=0} = \Big\langle \frac{\partial^2 F}{\partial u^2} z + \cancel{\frac{\partial^2 F}{\partial u \partial u_x}} z_x$$

$$- \frac{\partial^3 F}{\partial u^2 \partial u_x} u_x z - \frac{\partial^3 F}{\partial u \partial u_x^2} u_x z_x - \cancel{\frac{\partial^2 F}{\partial u \partial u_x}} z_x$$

$$- \frac{\partial^3 F}{\partial u \partial u_x^2} u_{xx} z - \frac{\partial^3 F}{\partial u_x^3} u_{xx} z_x - \frac{\partial^2 F}{\partial u_x^2} z_{xx} \Big| w \Big\rangle$$

$$= \left\langle \frac{\delta^2 \hat{F}}{\delta u^2} z \Big| w \right\rangle = \left\langle z \Big| \frac{\delta^2 \hat{F}}{\delta u^2} w \right\rangle$$

(13)

$$\frac{d}{d\epsilon}[\hat{F},\hat{G}]\{u^{\alpha}+\epsilon w\}\Big|_{\epsilon=0} = \left\langle \frac{\delta[\hat{F},\hat{G}]}{\delta u^{\alpha}} \Big| w \right\rangle$$

$$= \frac{d}{d\epsilon}\left\langle \underset{(1)}{\frac{\delta\hat{F}}{\delta u^{i}}} \Big| \underset{(3)}{O^{ij}} \underset{(2)}{\frac{\delta\hat{G}}{\delta u^{j}}} \right\rangle (u^{\alpha}+\epsilon w)\Big|_{\epsilon=0}$$

$$= \left\langle \frac{\delta^2\hat{F}}{\delta u^{\alpha}\delta u^{i}} w \Big| O^{ij}\frac{\delta\hat{G}}{\delta u^{j}} \right\rangle_{(1)} + \left\langle \frac{\delta\hat{F}}{\delta u^{i}} \Big| O^{ij}\frac{\delta^2\hat{G}}{\delta u^{\alpha}\delta u^{j}} w \right\rangle_{(2)}$$

$$+ \left\langle \frac{\delta\hat{F}}{\delta u^{i}} \Big| \frac{\delta O^{ij}}{\delta u^{\alpha}}(w)\frac{\delta\hat{G}}{\delta u^{j}} \right\rangle_{(3)}$$

ISOLATING W $\Longrightarrow$

$$\frac{\delta[\hat{F},\hat{G}]}{\delta u^{\alpha}} = \underset{(1)}{\frac{\delta^2\hat{F}}{\delta u^{i}\delta u^{\alpha}} O^{ij}} - \underset{(2)}{\frac{\delta^2\hat{G}}{\delta u^{j}\delta u^{\alpha}} O^{ji}\frac{\delta\hat{F}}{\delta u^{i}}}$$

$$+ \underset{(3)}{K_{\alpha}^{ij}}\left(\frac{\delta\hat{F}}{\delta u^{i}}, \frac{\delta\hat{G}}{\delta u^{j}}\right)$$

Obtain for Jacobi

$$S = \left\langle \frac{\delta \hat{E}}{\delta u^m} \,\middle|\, O^{m\ell} \frac{\delta [\hat{E}, \hat{c}]}{\delta u^\ell} \right\rangle + \text{cyc.}$$

using operator symmetries

$$S = \left\langle \frac{\delta \hat{E}}{\delta u^m} \,\middle|\, O^{m\ell} K_\ell^{ij} \left(\frac{\delta \hat{F}}{\delta u^i}, \frac{\delta \hat{G}}{\delta u^j} \right) \right\rangle + \text{cyc.}$$

compare with

$$S^{ijk} = J^{i\ell} \frac{\partial}{\partial z^\ell} J^{jk} + \text{cyc.}$$

for F. D. F. case

<u>Th.</u> If operator C^{ij} does not depend upon the u^i, then Jacobi is immediate.

(15)

3. KdV Equation

C. S. Gardner, J. Math. Phys. 12, 1548 (1971).

$$u_t = -u u_x - \beta u_{xxx}$$

The following is conserved

$$\hat{H}\{u\} = \int \left(\beta \frac{u_x^2}{2} - \frac{u^3}{6}\right) dx$$

Gardner bracket

$$[\hat{F}, \hat{G}] = \int \frac{\delta \hat{F}}{\delta u} \frac{d}{dx} \frac{\delta \hat{G}}{\delta u} \, dx$$

Lie properties, including Jacobi, are immediate.

$$u_t = [u, \hat{H}] = -\frac{d}{dx}\left(\frac{u^2}{2} + \beta u_{xx}\right)$$

Note form of bracket.

4. Ideal MHD

P. J. Morrison & J. M. Greene, Phys. Rev. Lett. 45, 790 (1980).

§4.1 Equations

$$\underline{v}_t = -\nabla \frac{\underline{v}^2}{2} + \underline{v} \times (\nabla \times \underline{v}) - \frac{1}{\rho}\nabla(\rho^2 U_\rho) + \frac{1}{\rho}(\nabla \times \underline{B}) \times \underline{B}$$

$$\rho_t = -\nabla \cdot (\rho \underline{v})$$

$$\underline{B}_t = \nabla \times (\underline{v} \times \underline{B})$$

$$S_t = -\underline{v} \cdot \nabla S$$

$$U(\rho, S) \triangleq \frac{\text{Internal Energy}}{\text{Mass}}$$

$$S \triangleq \frac{\text{Entropy}}{\text{Mass}}$$

$$p = \rho^2 U_\rho \qquad T = U_S$$

17

§4.2 Bracket

$$\hat{H} = \int_V \left(\frac{1}{2}\rho v^2 + \rho U(\rho, S) + \frac{B^2}{2} \right) d\tau$$

$$[\hat{F}, \hat{G}] = -\int_V \Big\{ \Big[\frac{\delta \hat{F}}{\delta \rho} \nabla \cdot \frac{\delta \hat{G}}{\delta \underline{v}} + \frac{\delta \hat{F}}{\delta \underline{v}} \cdot \nabla \frac{\delta \hat{G}}{\delta \rho} \Big]$$

$$+ \Big[\frac{1}{\rho} \frac{\delta \hat{F}}{\delta \underline{v}} \cdot (\nabla \times \underline{v}) \times \frac{\delta \hat{G}}{\delta \underline{v}} \Big]$$

$$+ \Big[\frac{\nabla S}{\rho} \cdot \Big(\frac{\delta \hat{F}}{\delta S} \frac{\delta \hat{G}}{\delta \underline{v}} - \frac{\delta \hat{G}}{\delta S} \frac{\delta \hat{F}}{\delta \underline{v}} \Big) \Big]$$

$$+ \Big[\Big(\frac{1}{\rho} \frac{\delta \hat{F}}{\delta \underline{v}} \cdot \underline{B} \times \big(\nabla \times \frac{\delta \hat{G}}{\delta \underline{B}}\big) \Big) + \Big(\frac{\delta \hat{F}}{\delta \underline{B}} \cdot \nabla \times \big(\frac{\underline{B}}{\rho} \times \frac{\delta \hat{G}}{\delta \underline{v}} \big) \Big) \Big] \Big\} d\tau$$

let $\rho = \chi^0$, $S = \chi^1$, $\underline{v} = (\chi^2, \chi^3, \chi^4)$
& $\underline{B} = (\chi^5, \chi^6, \chi^7)$ then

$$= \int_V \frac{\delta \hat{F}}{\delta \chi^i} O^{ij} \frac{\delta \hat{G}}{\delta \chi^j} = \Big\langle \frac{\delta \hat{F}}{\delta \chi^i} \Big| O^{ij} \frac{\delta \hat{G}}{\delta \chi^j} \Big\rangle$$

(18)

§ 4.3 Canonical Variables

let $\underline{v} = \dfrac{\underline{B} \times (\underline{\nabla} \times \underline{A})}{\rho} + \underline{\nabla}\phi$

$+ \dfrac{\psi}{\rho} \underline{\nabla} s \quad \Longrightarrow$

$$[\hat{F}, \hat{G}] = \int_V \left\{ \left[\frac{\delta \hat{G}}{\delta \phi} \frac{\delta \hat{F}}{\delta \rho} - \frac{\delta \hat{F}}{\delta \phi} \frac{\delta \hat{G}}{\delta \rho} \right] \right.$$

$$\left. + \left[\frac{\delta \hat{F}}{\delta s} \frac{\delta \hat{G}}{\delta \psi} - \frac{\delta \hat{F}}{\delta \psi} \frac{\delta \hat{G}}{\delta s} \right] + \left[\frac{\delta \hat{G}}{\delta \underline{B}} \cdot \frac{\delta \hat{F}}{\delta \underline{A}} - \frac{\delta \hat{F}}{\delta \underline{B}} \cdot \frac{\delta \hat{G}}{\delta \underline{A}} \right] \right\} d\tau$$

§ 4.4 Conservation Form

let $\underline{M} = \rho \underline{v}$ and $\sigma = \rho s$

momentum density

specific Entropy

$\Longrightarrow$ 8 conservation Eqs.

Bracket Becomes

$$
\begin{aligned}
\{\hat{F},\hat{G}\} = -\int dx \Big\{ & \rho\Big[\frac{\delta \hat{F}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{G}}{\delta \rho} - \frac{\delta \hat{G}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{F}}{\delta \rho}\Big] \\
& + \underline{M}\cdot\Big[\frac{\delta \hat{F}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{G}}{\delta \underline{M}} - \frac{\delta \hat{G}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{F}}{\delta \underline{M}}\Big] \\
& + \sigma\Big[\frac{\delta \hat{F}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{G}}{\delta \sigma} - \frac{\delta \hat{G}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{F}}{\delta \sigma}\Big] \\
& + \underline{B}\cdot\Big(\Big[\frac{\delta \hat{F}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{G}}{\delta \underline{B}} - \frac{\delta \hat{G}}{\delta \underline{M}}\cdot\nabla\frac{\delta \hat{F}}{\delta \underline{B}}\Big] \\
& \qquad + \Big[\Big(\nabla\frac{\delta \hat{F}}{\delta \underline{B}}\Big)\cdot\frac{\delta \hat{G}}{\delta \underline{M}} - \Big(\nabla\frac{\delta \hat{G}}{\delta \underline{B}}\Big)\cdot\frac{\delta \hat{F}}{\delta \underline{M}}\Big]\Big)\Big\}
\end{aligned}
$$

(20)

5. Vlasov - Poisson

P. J. Morrison, PPPL R-1702 (1980).
To appear in Phys. Lett. A.

§5.1 Equations

$$\frac{\partial f(\underline{x},\underline{v},t)}{\partial t} = -\underline{v}\cdot\frac{\partial f}{\partial \underline{x}} - \underline{E}\cdot\frac{\partial f}{\partial \underline{v}} = 0$$

$$\frac{\partial}{\partial \underline{x}}\cdot\underline{E} = \int f\, d\underline{v}$$

$$\Longleftrightarrow$$

$$\frac{\partial f}{\partial t} = -\underline{v}\cdot\frac{\partial f}{\partial \underline{x}} + \frac{\partial}{\partial \underline{x}}\cdot\int \frac{f(\underline{x}',\underline{v}',t)}{|\underline{x}-\underline{x}'|}\, d\underline{x}'\, d\underline{v}'\, \frac{\partial f}{\partial \underline{v}}$$

let $\underline{z} = (\underline{x},\underline{v})$

$$\hat{H}\{f\} = \int H_1(\underline{z}) f(\underline{z})\, d\underline{z} + \frac{1}{2}\iint f(\underline{z}) f(\underline{z}')\, H_2(\underline{z},\underline{z}') \times d\underline{z}\, d\underline{z}'$$

$$H_1 = \frac{1}{2} m v^2 \; ; \quad H_2 = \frac{1}{|\underline{x}-\underline{x}'|}$$

§ 5.2 Bracket

$$[\hat{F}, \hat{G}] = \int f \left\{ \frac{\delta \hat{F}}{\delta f}, \frac{\delta \hat{G}}{\delta f} \right\} dz$$

where $\{f, g\} = \frac{\partial f}{\partial \underline{x}} \cdot \frac{\partial g}{\partial \underline{v}} - \frac{\partial g}{\partial \underline{x}} \cdot \frac{\partial f}{\partial \underline{v}}$

Observe

$$\int f \left\{ \frac{\delta \hat{F}}{\delta f}, \frac{\delta \hat{G}}{\delta f} \right\} dz = - \int \frac{\delta \hat{F}}{\delta f} \left\{ f, \frac{\delta \hat{G}}{\delta f} \right\} dz$$

$$\frac{\delta \hat{H}}{\delta f} = H_p$$

§ 5.3 Jacobi Identity

$$\frac{\delta [\hat{F}, \hat{G}]}{\delta f} = \left\{ \frac{\delta \hat{F}}{\delta f}, \frac{\delta \hat{G}}{\delta f} \right\} + \text{operator terms}$$

$$S = [\hat{E}, [\hat{F}, \hat{G}]] + cyc = \int f \left\{ \frac{\delta \hat{E}}{\delta f}, \frac{\delta [\hat{F}, \hat{G}]}{\delta f} \right\} dz + cyc$$

$$= \int f \left\{ \frac{\delta \hat{E}}{\delta f}, \left\{ \frac{\delta \hat{F}}{\delta f}, \frac{\delta \hat{G}}{\delta f} \right\} \right\} + cyc. \equiv 0$$

6. Possibilities & Avenues

- Maxwell - Vlasov
- Relativistic Maxwell - Vlasov
- 1-D Fluid, ∞ conservation laws, dual Hamiltonian
- Approx. schemes which preserve Hamiltonian form. Discretization & Truncation.
- Unending search for conservation laws
- 1-D Fluid w/ Entropy
- 2-D MHD