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6R 13 Rayleigh-Ritz Procedure for the Eulerian Vlasov-Poisson Equations.* P. J. MORRISON, Princeton U. -- Recently it was shown that the Vlasov-Poisson equations possess underlying Hamiltonian structure.[1] The canonical version of these equations can readily be shown to arise from a variational principle, which serves as a starting point for the Rayleigh-Ritz numerical procedure. The variational integral is discretized by a selection of finite elements with nodal density prearranged for accuracy. Such a discretization is singled out; there is no arbitrariness in the selection of basis functions as for the Galerkin method. The evolution equations for the system, a finite set of ordinary differential equations, are obtained by variation of this discretized integral. We anticipate that this variational principle will also be useful for multiple time scale approximation techniques.

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1. P. J. MORRISON, Princeton Plasma Lab. Report #1788 (1981).

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RAYLEIGH - RITZ PROCEDURE FOR THE EULERIAN VLASOV-POISSON EQUATIONS

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Rayleigh-Ritz Procedure for the Eulerian Vlasov-Poisson Equations.* P. J. MORRISON, Princeton U. -- Recently it was shown that the Vlasov-Poisson equations possess underlying Hamiltonian structure.[1] The canonical version of these equations can readily be shown to arise from a variational principle, which serves as a starting point for the Rayleigh-Ritz numerical procedure. The variational integral is discretized by a selection of finite elements with nodal density prearranged for accuracy. Such a discretization is singled out; there is no arbitrariness in the selection of basis functions as for the Galerkin method. The evolution equations for the system, a finite set of ordinary differential equations, are obtained by variation of this discretized integral. We anticipate that this variational principle will also be useful for multiple time scale approximation techniques.

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Definition : A system is Hamiltonian if it can be written in the form

$$\frac{\partial \chi^i}{\partial t} = F^i(\chi) = \underline{[\chi^i, H]} \quad ,$$

where [,] operates on functionals (e.g. $\int \frac{1}{2}\rho v^2 d\tau$) and satisfies

* bilinear
* Antisymmetric $[A, B] = -[B, A]$
* Jacobi Condition $[[A,B], C] + [[B,C], A] + [[C,A], B] = 0$

Maxwell - Vlasov System

$$\underline{B}_t = -\nabla\times\underline{E} \qquad \underline{E}_t = \nabla\times\underline{B} - \sum_\alpha e_\alpha \int \underline{v}\, f_\alpha \, d\underline{v}$$

$$f_{\alpha_t} = -\underline{v}\cdot\frac{\partial f_\alpha}{\partial \underline{x}} - \frac{e_\alpha}{m_\alpha}(\underline{E} + \underline{v}\times\underline{B})\cdot\frac{\partial f_\alpha}{\partial \underline{v}}$$

$$H(f, \underline{E}, \underline{B}) = \sum_\alpha \int \tfrac{1}{2} m_\alpha v^2 f_\alpha \, dz + \int \frac{E^2 + B^2}{2}\, d\underline{x}$$

$$\boxed{\begin{aligned}[A, B] = \sum_\alpha \int \frac{f_\alpha}{m_\alpha}\left\{\frac{\delta A}{\delta f_\alpha}, \frac{\delta B}{\delta f_\alpha}\right\} dz &+ \frac{e_\alpha}{m_\alpha}\int\left(\frac{\delta A}{\delta \underline{E}}\cdot\frac{\partial f_\alpha}{\partial \underline{v}}\frac{\delta B}{\delta f_\alpha} - \frac{\delta B}{\delta \underline{E}}\cdot\frac{\partial f_\alpha}{\partial \underline{v}}\frac{\delta A}{\delta f_\alpha}\right) dz \\ + \frac{e_\alpha}{m_\alpha}\int \underline{B}\cdot\left(\frac{\partial}{\partial \underline{v}}\frac{\delta A}{\delta f_\alpha}\times\frac{\partial}{\partial \underline{v}}\frac{\delta B}{\delta f_\alpha}\right) dz &+ \int\left(\frac{\delta A}{\delta \underline{E}}\cdot\nabla\times\frac{\delta B}{\delta \underline{B}} - \frac{\delta B}{\delta \underline{E}}\cdot\nabla\times\frac{\delta A}{\delta \underline{B}}\right) d\underline{x}\end{aligned}}$$

where $\{f, g\} = \frac{\partial f}{\partial \underline{x}}\cdot\frac{\partial g}{\partial \underline{v}} - \frac{\partial g}{\partial \underline{x}}\cdot\frac{\partial f}{\partial \underline{v}}$

Vlasov-Poisson

$$[A,B] = \sum_\alpha \int \frac{f_\alpha}{m_\alpha} \left\{ \frac{\delta A}{\delta f_\alpha}, \frac{\delta B}{\delta f_\alpha} \right\} dz$$

where $\{f,g\} = f_x \cdot g_v - f_v \cdot g_x$

$$H = \sum_\alpha \int \frac{1}{2} m_\alpha v^2 f_\alpha \, dz - \frac{1}{2} \sum_{\alpha,\beta}{}' e_\alpha e_\beta \iint K(x|x') f_\alpha(z) f_\beta(z') \, dz \, dz'$$

ONE Dynamical Eq. (per species)

$$\frac{\partial f_\alpha}{\partial t} = -\underline{W}_\alpha \cdot \frac{\partial f_\alpha}{\partial \underline{z}} \qquad \underline{z} = (\underline{x}, \underline{v})$$

where $\underline{W}_\alpha = \left(\underline{v}, \frac{e_\alpha}{m_\alpha} \frac{\partial}{\partial \underline{x}} \sum_\beta \int K(x|x') f_\beta(z') \, dz' \right)$

$$\nabla_p \cdot \underline{W}_\alpha = 0$$

CANONICAL VARIABLES

Let $$f_\alpha = \{\Psi_\alpha, \Upsilon_\alpha\} = \frac{\partial \Psi_\alpha}{\partial x} \cdot \frac{\partial \Upsilon_\alpha}{\partial v} - \frac{\partial \Upsilon_\alpha}{\partial x} \cdot \frac{\partial \Psi_\alpha}{\partial v}$$

obtain

$$\frac{\partial \Psi_\alpha}{\partial t} = \frac{\delta H}{\delta \Upsilon_\alpha} \quad \& \quad \frac{\partial \Upsilon_\alpha}{\partial t} = - \frac{\delta H}{\delta \Psi_\alpha}$$

$$\Longleftrightarrow$$

$$\frac{\partial \Psi_\alpha}{\partial t} = - \underline{W}_\alpha \cdot \nabla_p \Psi_\alpha \qquad \frac{\partial \Upsilon_\alpha}{\partial t} = - \underline{W}_\alpha \cdot \nabla_p \Upsilon_\alpha$$

Recall

$$\underline{W}_\alpha = \left(v, \ \frac{e_\alpha}{m_\alpha} \frac{\partial}{\partial x} \sum_\beta e_\beta \int K(x|x') f_\beta(z') \, dz' \right)$$

Action

$$J = \sum_{\alpha} \int dt \Big\{ \int \Upsilon_\alpha \frac{\partial \Psi_\alpha}{\partial t}\, dz$$

$$- \int \frac{1}{2} m_\alpha v^2 \left[\frac{\partial \Psi_\alpha}{\partial x} \cdot \frac{\partial \Upsilon_\alpha}{\partial v} - \frac{\partial \Upsilon_\alpha}{\partial x} \cdot \frac{\partial \Psi_\alpha}{\partial v} \right] dz$$

$$+ \frac{1}{2} \sum_{\beta} e_\alpha e_\beta \iint K(x|x') \left[\frac{\partial \Psi_\alpha}{\partial x} \frac{\partial \Upsilon_\alpha}{\partial v} - \frac{\partial \Upsilon_\alpha}{\partial x} \frac{\partial \Psi_\alpha}{\partial v} \right] \left[\frac{\partial \Psi_\beta}{\partial x'} \frac{\partial \Upsilon_\beta}{\partial v'} - \frac{\partial \Psi_\beta}{\partial v'} \frac{\partial \Upsilon_\beta}{\partial x'} \right] dz\, dz' \Big\}$$

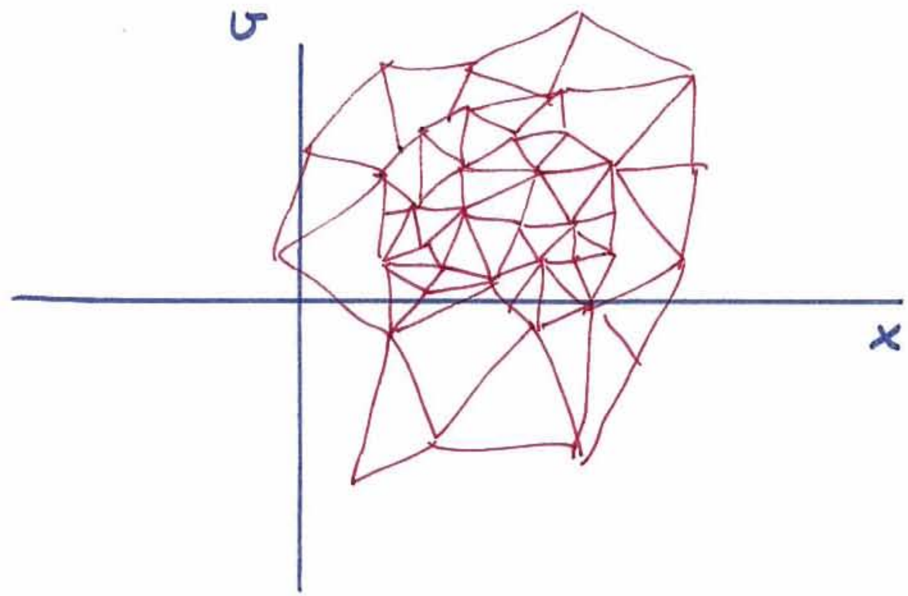
Variation

$$\delta J = \frac{d}{d\epsilon} J(\Psi_\alpha + \epsilon w) \Big|_{\epsilon = 0} = 0$$ Yields equations Above.

$w = 0$ on $\partial D \subset \mathbb{R}^4$

Rayleigh - Ritz

Break phase space up into finite elements



Tailor element size for accuracy.

Define Ψ_α & Γ_α at each Node.

$$\Psi_\alpha = \sum_i N_i^\alpha(t)\, G(z - z_i) \quad \& \quad \Upsilon_\alpha = \sum_i \eta_i^\alpha(t)\, G(z - z_i)$$

(e.g. $G(z - z_i) = G(x - x_i)\, G(v - v_i)$
where G is a tent function)

J becomes

$$J = \sum_\alpha \int dt \left\{ \sum_{i,j} \eta_j^\alpha \frac{dN_i^\alpha}{dt} A_{ij} - \sum_{i,j} \eta_j^\alpha N_i^\alpha B_{ij} \right.$$

$$\left. + \sum_\beta \sum_{i,j,k,l} C_{ijkl}\, N_i^\alpha N_k^\beta \eta_j^\alpha \eta_l^\beta \right\}$$

Variation of J yields a finite set of ordinary differential equations

$$\frac{d\eta_j^\alpha}{dt} = \cdots \qquad \frac{dN_j^\alpha}{dt} = \cdots$$

$$A_{ij} = \int dz \; G(z-z_i)\, G(z-z_j)$$

$$B_{ij} = \int \frac{1}{2} m_\alpha v^2 \left\{ G(z-z_i), G(z-z_j) \right\} dz$$

$$C_{ijkl} = \int dz\, dz' \; \frac{e_\alpha e_\beta}{2} \; K(x|x') \left\{ G(z-z_i), G(z-z_j) \right\}$$

$$\times \left\{ G(z'-z_k), G(z'-z_l) \right\}$$

Caveats

* Equations are Hyperbolic $\Longrightarrow$
  Extremal $\neq$ Extremum
  (May still be stable)

* The Potentials Ψ & Υ will probably wind up which will require relabeling.

* will $\int f\, dz$ be conserved