APS INVITED TALK N.Y.

Hidden Hamiltonian Structure in Infinite Dimensional Systems of Plasma Physics

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OVERVIEW

I. Introduction

II. Background

III. Vlasov - Noncanonical

IV. Vlasov - Canonical

My Poster 6R13 Wed.

Rel. M-V

Bialynicki - Birula
Hubbard
Amend

} 1S2 & 3

2-D M-V

Spencer & Kaufman } 8T8

Marsden & Weinstein } 8T9

II. Background

A. ORDINARY DIFF. EQS.

$$\mathcal{L}(q, \dot{q}) \xrightarrow[\text{LEGENDRE}]{} H(p, q)$$

$$\dot{q}_k = [q_k, H] \qquad \dot{p}_k = [p_k, H]$$

$$k = 1, 2, \cdots N$$

where

$$[f, g] = \sum_k^N \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

f & G are functions of p_k & q_k.

Let $z^i = \begin{cases} q_k & k = i = 1, 2, \cdots N \\ p_k & i = k+N = k+1, \cdots 2N \end{cases}$

$$[f, g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

$$(J^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

Transforms as a contravariant Tensor

Hamiltons Eqs. become

$$\dot{z}^i = [z^i, H] = J^{ij} \frac{\partial H}{\partial z^j}$$

Commutator Properties

* $[f, g]$ is bilinear
* $[f, g] = -[g, f]$
* $[f, [g, h]] + \circlearrowright = 0$

Canonical Transformations Preserve the form of J^{ij}

General Coordinate Transformations Do Not!

$$\bar{J}^{st} = \frac{\partial \bar{z}^s}{\partial z^i} J^{ij} \frac{\partial \bar{z}^t}{\partial z^j}$$

General Coordinate Transformations <u>Do</u> preserve Bracket Properties.

<u>Converse Outlook</u>: $\bar{J}^{ij}$ has properties but not skew-diagonal. Can one find canonical Coordinates?

Yes! ($\det J^{ij} \neq 0$) Darboux (1882)

DEF. A system $\dot{x}^i = F^i(x)$ is Hamiltonian if it can be written in the form $\dot{x}^i = [x^i, H]$ wher $[\ ,\ ]$ makes vector spore of phase functions into a Lie Algebra.

Because locally canonical Variables can be found

B. Field Equations (e.g. P.D.E. or I.E.)

$$\boxed{[F,G] = \sum_{k=1}^{m} \int \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right) dz}$$

F & G are Functionals e.g.

$$H(\rho, v) = \int \frac{\rho v^2}{2} \, dz$$

$$\frac{dF}{d\epsilon}(\eta + \epsilon w)\Big|_{\epsilon=0} = \int \frac{\delta F}{\delta \eta} w \, dz = \left\langle \frac{\delta F}{\delta \eta} \Big| w \right\rangle$$

$$[F,G] = \left\langle \frac{\delta F}{\delta u^i} \Big| \mathcal{O}^{ij} \frac{\delta G}{\delta u^j} \right\rangle$$

canonical $(\mathcal{O}^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$, in

General Anti-self-Adjoint wrt $\langle | \rangle$

+ Jacobi condition

Def. If can write a system of Field Equations in the form

$$X^i_t = [X^i, H]$$

where $[\, ,\,]$ has properties, then call Hamiltonian.

EXAMPLE K-dV Equation

$$\frac{\partial u}{\partial t} = + u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3}$$

$$H = \int \left(\frac{u^3}{6} - \beta \frac{u_x^2}{2} \right) dz$$

$$\frac{\delta H}{\delta u} = \frac{u^2}{2} + \beta u_{xx}$$

Gardner Bracket

$$[f, g] = \int \frac{\delta f}{\delta u} \frac{d}{dx} \frac{\delta g}{\delta u} dz$$

Note:

$$\frac{\partial u}{\partial t} = [u, H] = \frac{d}{dx} \left(\frac{u^2}{2} + \beta u_{xx} \right)$$

CAN CANONIZE!

III. Vlasov – Noncanonical

A. Vlasov – Poisson ($B=0$)

$$\boxed{\frac{\partial f_\alpha(\underline{z},t)}{\partial t} = -\underline{w}^\alpha \cdot \frac{\partial f_\alpha}{\partial \underline{z}}}$$

$\underline{w}^\alpha ; \underline{z} \in \mathbb{R}^6$

$$\underline{w}^\alpha = \left(\underline{v} \, , \; \frac{e_\alpha}{m_\alpha} \frac{\partial}{\partial \underline{x}} \sum_\beta \int_{\mathbb{R}^6} \nabla^{-2}(\underline{x}|\underline{x}') f_\beta(\underline{z}') d\underline{z}' \right)$$

(Note: $\dfrac{\partial}{\partial \underline{z}} \cdot \underline{w}^{(\alpha)} = 0$)

like 2-D turbulence

$$H(f) = \sum_\alpha \frac{m_\alpha}{2} \int v^2 f_\alpha \, d\underline{z}$$

$$- \frac{1}{2} \sum_{\alpha,\beta} e_\alpha e_\beta \int \nabla^{-2}(\underline{x}|\underline{x}') f_\alpha f_\beta \, d\underline{z} \, d\underline{z}'$$

$$\boxed{[F,G] = \sum_\alpha \int \frac{f_\alpha(z)}{m_\alpha} \left\{ \frac{\delta F}{\delta f_\alpha}, \frac{\delta G}{\delta f_\alpha} \right\} dz}$$

Where

$$\{f,g\} = \frac{\partial f}{\partial \underline{x}} \cdot \frac{\partial g}{\partial \underline{v}} - \frac{\partial f}{\partial \underline{v}} \cdot \frac{\partial g}{\partial \underline{x}}$$

Not difficult to see

$$\frac{\partial f_\alpha}{\partial t} = [f_\alpha, H] \qquad \blacksquare$$

<u>Another Way</u>

$$\frac{\partial f_\alpha}{\partial t} = -\underline{v} \cdot \frac{\partial f_\alpha}{\partial \underline{x}} - \frac{e_\alpha}{m_\alpha} \underline{E} \cdot \frac{\partial f_\alpha}{\partial \underline{v}}$$

$$\frac{\partial \underline{E}}{\partial t} = -\sum_\alpha e_\alpha \int \underline{v} f_\alpha \, d\underline{v}$$

$$H(f, \underline{E}) = \sum_\alpha \int \frac{m_\alpha}{2} v^2 f_\alpha \, dz + \frac{1}{2}\int E^2 \, d\underline{x}$$

$$[F,G] = \sum_\alpha \int_{\mathbb{R}^6} \frac{f_\alpha}{m_\alpha} \left\{ \frac{\delta F}{\delta f_\alpha}, \frac{\delta G}{\delta f_\alpha} \right\} dz$$

$$- \frac{e_\alpha}{m_\alpha} \int \left(\frac{\delta F}{\delta f_\alpha} \frac{\partial f_\alpha}{\partial \underline{v}} \cdot \frac{\delta G}{\delta \underline{E}} - \frac{\delta G}{\delta f_\alpha} \frac{\partial f_\alpha}{\partial \underline{v}} \frac{\delta F}{\delta \underline{E}} \right) dz$$

B. Maxwell-Vlasov

Maxwell's Eqs. in Vacuum

$$H = \int \frac{\underline{E}^2 + \underline{B}^2}{2} \, d\underline{x}$$

$$[F,G]_{VM} = \int \left(\frac{\delta F}{\delta \underline{E}} \cdot \nabla \times \frac{\delta G}{\delta \underline{B}} - \frac{\delta G}{\delta \underline{B}} \cdot \nabla \times \frac{\delta F}{\delta \underline{E}} \right) d\underline{x}$$

Jacobi Automatic

Whole Thing

$$[F,G] = \sum_\alpha \int \frac{f_\alpha}{m_\alpha}\left\{\frac{\delta F}{\delta f_\alpha}, \frac{\delta G}{\delta f_\alpha}\right\} dz$$

$$+ \frac{e_\alpha}{m_\alpha}\int \left(\frac{\delta G}{\delta f_\alpha}\frac{\partial f_\alpha}{\partial \underline{v}}\cdot\frac{\delta F}{\delta \underline{E}} - \frac{\delta F}{\delta f_\alpha}\frac{\partial f_\alpha}{\partial \underline{v}}\cdot\frac{\delta G}{\delta \underline{E}}\right) dz$$

$$+ \frac{e_\alpha}{m_\alpha}\int \underline{B}\cdot\left(\frac{\partial}{\partial \underline{v}}\frac{\delta F}{\delta f_\alpha}\times\frac{\partial}{\partial \underline{v}}\frac{\delta G}{\delta f_\alpha}\right) dz$$

$$+ \int \left(\frac{\delta F}{\delta \underline{E}}\cdot\nabla\times\frac{\delta G}{\delta \underline{B}} - \frac{\delta G}{\delta \underline{B}}\cdot\nabla\times\frac{\delta F}{\delta \underline{E}}\right) dx$$

Can prove Jacobi Directly!! $\blacksquare$

$$\frac{\partial \underline{B}}{\partial t} = [\underline{B}, H] = -\nabla \times \underline{E}$$

$$\frac{\partial \underline{E}}{\partial t} = [E, H] = \nabla \times \underline{B} - \sum_\alpha e_\alpha \int \underline{v} f_\alpha \, dv$$

$$\frac{\partial f_\alpha}{\partial t} = -\underline{v} \cdot \frac{\partial f_\alpha}{\partial \underline{x}} - \frac{e_\alpha}{m_\alpha} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial f_\alpha}{\partial \underline{v}}$$

Note: $\nabla \cdot B = 0$

& $\nabla \cdot E = \rho$

are initial conditions

$$\boxed{[\nabla \cdot B, H] = [\nabla \cdot E - \rho, H] = 0}$$

IV Canonical Form

Let $f_\alpha = \{ \psi_\alpha , \Gamma_\alpha \}$

where

$$\{ \, , \, \} = \frac{\partial}{\partial \underline{x}} \cdot \frac{\partial}{\partial \underline{p}} - \frac{\partial}{\partial \underline{p}} \cdot \frac{\partial}{\partial \underline{x}}$$

Can show

$$[F, G] = \sum_\alpha \int f_\alpha \left(\frac{\partial}{\partial \underline{x}} \frac{\delta F}{\delta f_\alpha} \cdot \frac{\partial}{\partial \underline{p}} \frac{\delta G}{\delta f_\alpha} - \frac{\partial}{\partial \underline{p}} \frac{\delta F}{\delta f_\alpha} \cdot \frac{\partial}{\partial \underline{x}} \frac{\delta G}{\delta f_\alpha} \right)$$

$$\longrightarrow \sum_\alpha \int \left(\frac{\delta F}{\delta \psi_\alpha} \frac{\delta G}{\delta \Gamma_\alpha} - \frac{\delta G}{\delta \psi_\alpha} \frac{\delta F}{\delta \Gamma_\alpha} \right)$$

use $\underline{E}$ & $\underline{A}$ $(\nabla \times \underline{A} = \underline{B})$ for rest.