HIDDEN HAMILTONIAN STRUCTURE IN CLASSICAL FIELD EQUATIONS — EXAMPLES FROM FLUID MECHANICS AND PLASMA PHYSICS

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I. Introduction

A. Hamiltonian Advantage

1. path to <u>quantization</u>

2. <u>statistical mechanics</u>

   a) canonical description guarentees and identifiès invariant measure – Liouville's theorem

   b) can identify Gibbs (canonical) ensemble

3. <u>pertubation theory</u>

   a) dynamics essentially contained in one function H, instead of 2N.

   b) transformation theory $\begin{pmatrix} P \\ Q \end{pmatrix} \to \begin{pmatrix} p \\ q \end{pmatrix}$ involves one function: mixed variable or Lie generator.

B. Background

1. Language
   a) Classical
   b) differential Algebraic
   c) Geometric

2. References
   a) finite degree of freedom systems
   — Arnold
   — Abraham & Marsden
   — Flanders
   — Whittaker

   b) infinite dimensional systems
   — Monin & Kupershmidt
   — Chernoff & Marsden
   — A & M ch. II
   — Gelfand & Dorfman

C. Overview

1. Hamiltonian Systems
   (a) Finite degree of freedom
   (b) ∞ " " "
      i) Example KdV Eq.

2. MHD

3. Vlasov

II. Hamiltonian Systems

A. F. D. F. systems (O.D.E.)

$$\mathcal{L}(q, \dot{q}) \xrightarrow[\text{Legendre}]{} H(p, q)$$

$$\dot{q}_k = [q_k, H] \quad ; \quad \dot{p}_k = [p_k, H]$$

$$k = 1, 2, \cdots N$$

where $$[f, g] = \sum_k^N \left[\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right]$$

$f, g(q, p)$.

Let $$z^i = \begin{cases} q_k & \text{for } k = i = 1, 2, \cdots N \\ p_k & \text{for } i = k + N = k+1, \cdots 2N \end{cases}$$

$$[f, g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

$$(J^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

Contravariant Tensor

$$\dot{z}^i = [z^i, H] = J^{ij} \frac{\partial H}{\partial z^j}$$

Commutator Properties

(i) $[f, g]$ is bilinear

(ii) $[f, g] = -[g, f] \Leftrightarrow J^{ij} = -J^{ji}$

(iii) $[f, [g, h]] + [g, [h, f]] + [h, [f, g]] \equiv S = 0$

$$\Leftrightarrow \quad S^{ijk} = J^{il} \frac{\partial J^{jk}}{\partial z^l} + J^{jl} \frac{\partial J^{ki}}{\partial z^l} + J^{kl} \frac{\partial J^{ij}}{\partial z^l} = 0$$

* Antisymmetry is coordinate independent

* S^{ijk} transforms contravariantly

$\therefore$ $S^{ijk} \equiv 0$ in one frame $\Rightarrow$

$S^{ijk} \equiv 0$ in all frames

<u>Converse Outlook:</u> If J^{ij} has above properties but is not of the form $(J^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$ can one find canonical coordinates. ($\det |J^{ij}| \neq 0$)

Yes! Darboux (1882).

<u>Def.</u> A system $\dot{x}^i = F(\underline{x})$ is Hamiltonian if it can be written in the form $\dot{x}^i = [x^i, H]$ where $[\,,\,]$ makes vector space of phase functions into Lie Algebra.

B. Field Equations (P.D.E., I.E.)

$$[F, G] = \sum_{k=1}^{M} \int \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta F}{\delta \pi_k} \frac{\delta G}{\delta \eta_k} \right) dz$$

F, G Functionals e.g.

$$H(\rho, v) = \int \frac{\rho v^2}{2} \, dz$$

$$\frac{dF}{d\epsilon}(\eta + \epsilon w)\Big|_{\epsilon=0} = \int \frac{\delta F}{\delta \eta} w \, dz \equiv \left\langle \frac{\delta F}{\delta \eta} \Big| w \right\rangle$$

$$[F, G] = \left\langle \frac{\delta F}{\delta u^i} \Big| O^{ij} \frac{\delta G}{\delta u^j} \right\rangle$$

$$(O^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

write O^{ij} s.t.

(i) $[F, G] = -[G, F] \Rightarrow O^{ij}$ Anti-self-adjoint

(ii) Jacobi $\Rightarrow$ $S = \left\langle \frac{\delta E}{\delta u} \Big| O T\left(\frac{\delta F}{\delta u}, \frac{\delta G}{\delta u}\right) \right\rangle + \text{cyc} = 0$

$T \sim \delta O / \delta u$

EXAMPLE K-dV Equation

$$\frac{\partial u}{\partial t} = + u \frac{\partial u}{\partial x} + \beta \frac{\partial^3 u}{\partial x^3}$$

$$H = \int \left(\frac{u^3}{6} - \beta \frac{u_x^2}{2} \right) dz$$

$$\frac{\delta H}{\delta u} = \frac{u^2}{2} + \beta u_{xx}$$

Gardner Bracket

$$[f, g] = \int \frac{\delta f}{\delta u} \frac{d}{dx} \frac{\delta g}{\delta u} \, dz$$

Note:

$$\frac{\partial u}{\partial t} = [u, H] = \frac{d}{dx} \left(\frac{u^2}{2} + \beta u_{xx} \right)$$

CAN CANONIZE!

III. MHD (w/ J. M. Greene)

A. Noncanonical Bracket

$$\frac{\partial \underline{v}}{\partial t} = -\nabla \frac{\underline{v}^2}{2} + \underline{v} \times (\nabla \times \underline{v}) - \frac{1}{\rho} \nabla (\rho^2 U_\rho) + \frac{1}{\rho} (\nabla \times \underline{B}) \times \underline{B}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \underline{v})$$

$$\frac{\partial s}{\partial t} = -\underline{v} \cdot \underline{\nabla} s$$

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B})$$

$$U(\rho, s) \equiv \frac{\text{Internal Energy}}{\text{Mass}}$$

$$s \equiv \frac{\text{Entropy}}{\text{Mass}}$$

$$T = \frac{\partial U}{\partial s} \qquad p = \rho^2 U_\rho$$

$$H = \int \left(\frac{1}{2} \rho v^2 + \rho U(\rho, s) + \frac{B^2}{2} \right) d\tau$$

$$[F, G] = -\int \Big\{ \left[\frac{\delta F}{\delta \rho} \nabla \cdot \frac{\delta G}{\delta \underline{v}} + \frac{\delta F}{\delta \underline{v}} \cdot \nabla \frac{\delta G}{\delta \rho} \right]$$

$$+ \left[\frac{1}{\rho} \frac{\delta F}{\delta \underline{v}} \cdot (\nabla \times \underline{v}) \times \frac{\delta G}{\delta \underline{v}} \right]$$

$$+ \left[\frac{1}{\rho} \underline{\nabla} s \cdot \left(\frac{\delta F}{\delta s} \frac{\delta G}{\delta \underline{v}} - \frac{\delta G}{\delta s} \frac{\delta F}{\delta \underline{v}} \right) \right]$$

$$+ \left[\frac{1}{\rho} \frac{\delta F}{\delta \underline{v}} \cdot \underline{B} \times \left(\nabla \times \frac{\delta G}{\delta \underline{B}} \right) + \frac{\delta F}{\delta \underline{B}} \cdot \nabla \times \left(\frac{\underline{B}}{\rho} \times \frac{\delta G}{\delta \underline{v}} \right) \right] \Big\} d\tau$$

Observe: $\underline{v}_t = [\underline{v}, H]$ etc.

Looks like v's trying to be conjugate to everything in sight.

Not enough v to go around.

Alternate form: $\underline{M} = \rho \underline{v}$, $\sigma = \rho s$ - 8 cons. eqs.

B. Potentials in Hydrodynamics

Euler 1769 $\quad \underline{v} = \nabla\alpha \times \nabla\beta \qquad \nabla\cdot\underline{v} = 0$

Monge 1784 $\quad \underline{v} = \nabla\alpha \times \nabla\beta + \nabla\gamma$

(Like Helmholtz)

$\longleftarrow$ Hamilton 1824 optics
1832 dynamics

Clebsch 1859 $\quad \underline{v} = \alpha\,\nabla\beta + \nabla\phi$

$\nabla\cdot\underline{v} = 0 \implies T(\alpha,\beta) \longrightarrow \phi$

α canonically conjugate to β

Bateman 1929
Davydov 1949 $\quad \underline{v} = \frac{\lambda}{\rho}\nabla\mu + \nabla\phi$

(λ, μ) & (ρ, ϕ)

Whitham, Serrin 1959 $\quad \underline{v} = \frac{\lambda}{\rho}\nabla\mu + \nabla\phi + \frac{\psi}{\rho}\nabla s$

Zakharov
1970 ish

$$\underline{v} = \frac{\underline{B} \times (\nabla \times \underline{T})}{\rho} + \nabla\phi + \frac{\psi}{\rho}\nabla S$$

Chain rule for functional derivatives w/ above transforms,

$$[F,G] = \int \Big\{ \Big[\frac{\delta G}{\delta \phi}\frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta \phi}\frac{\delta G}{\delta \rho} \Big]$$

$$+ \Big[\frac{\delta F}{\delta S}\frac{\delta G}{\delta \psi} - \frac{\delta G}{\delta S}\frac{\delta F}{\delta \psi} \Big] + \Big[\frac{\delta G}{\delta \underline{B}} \cdot \frac{\delta F}{\delta \underline{T}} - \frac{\delta G}{\delta \underline{T}} \cdot \frac{\delta F}{\delta \underline{B}} \Big] \Big\}$$

into our bracket.

IV VLASOV

A. VLASOV - POISSON

1. Noncanonical Description

$$\frac{\partial f_\alpha}{\partial t}(z,t) = -\underline{v}\cdot\nabla f_\alpha + \frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial \underline{x}}\phi(\underline{x},t)\cdot\frac{\partial f_\alpha}{\partial \underline{v}}$$

$$\frac{\partial^2 \phi}{\partial x^2} = -\sum_\alpha e_\alpha \int f_\alpha \, dv \qquad ; \; z = (\underline{x},\underline{v})$$

Like Vorticity

$$\frac{\partial f_\alpha}{\partial t} = -\underline{W}^{(\alpha)}\cdot\frac{\partial f}{\partial \underline{z}}$$

$$\underline{W}^{(\alpha)} = \left(\underline{v}\,,\; \frac{e_\alpha}{m_\alpha}\frac{\partial}{\partial \underline{x}}\sum_\beta \int K(\underline{x}|\underline{x}')\, f_\beta(z')\,dz'\right)$$

$$\frac{\partial}{\partial \underline{z}}\cdot\underline{W}^{(\alpha)} = 0$$

$$H[f] = \sum_\alpha \frac{1}{2} m_\alpha \int v^2 f_\alpha \, dz - \frac{1}{2}\sum_{\alpha,\beta} e_\alpha e_\beta \int K(\underline{x}|\underline{x}')\, f_\alpha f_\beta \, dz\, dz'$$

$$[A,B] = \sum_\alpha \int \frac{f_\alpha(z)}{m_\alpha}\left\{\frac{\delta A}{\delta f_\alpha}, \frac{\delta B}{\delta f_\alpha}\right\} dz$$

$$\{f,g\} = \frac{\partial f}{\partial \underline{x}}\cdot\frac{\partial g}{\partial \underline{v}} - \frac{\partial f}{\partial \underline{v}}\cdot\frac{\partial g}{\partial \underline{x}}$$

2. Canonical Variables

$$f_\alpha = \frac{1}{m_\alpha}\left\{ \Phi^{(\alpha)}, \Upsilon^{(\alpha)} \right\}$$

$$\left\{ \Phi^{(\alpha)}, \frac{\partial \Upsilon^{(\alpha)}}{\partial t} + \underline{W}_\alpha \cdot \frac{\partial}{\partial \underline{z}} \Upsilon^{(\alpha)} \right\} +$$

$$\left\{ \frac{\partial \Phi^{(\alpha)}}{\partial t} + \underline{W}_\alpha \cdot \frac{\partial}{\partial \underline{z}} \Phi^{(\alpha)}, \Upsilon^{(\alpha)} \right\} = 0$$

$$\frac{\partial \Phi^{(\alpha)}}{\partial t} = \frac{\delta H}{\delta \Upsilon^{(\alpha)}} , \quad \frac{\partial \Upsilon^{(\alpha)}}{\partial t} = -\frac{\delta H}{\delta \Phi^{(\alpha)}}$$

3. Discretization & Truncation

$$\Phi^{(\alpha)} = \sum_k \Phi_k^{(\alpha)} \mathcal{M}_k(z)$$

$$\Upsilon^{(\alpha)} = \sum_k \Upsilon_k^{(\alpha)} \mathcal{M}_k(z)$$

$$\langle \mathcal{M}_k | \mathcal{M}_\ell \rangle = \delta_{k,\ell} = \int \mathcal{M}_k^* \mathcal{M}_\ell \, dz$$

$$\psi_k^{(\alpha)} = \frac{\Phi_k^{(\alpha)} + i\,\eta_k^{(\alpha)}}{\sqrt{2}}$$

$$i\,\dot{\psi}_k^{(\alpha)} = \frac{\partial H}{\partial \psi_k^{(\alpha)*}} \qquad i\,\dot{\psi}_k^{(\alpha)*} = -\frac{\partial H}{\partial \psi_k^{(\alpha)}}$$

$$f_k^{(\alpha)} = \sum_{t=k+l} a_{t,l}^{(\alpha)}\, \psi_t^{(\alpha)} \psi_l^{(\alpha)*}$$

$$H = \sum_{\substack{\alpha \\ l_1 = k_1 \\ l_2 \neq k_2}} S_{l,k}^{(2)}\, \psi_l^{(\alpha)*} \psi_k^{(\alpha)}$$

$$+ \sum_{\substack{s_2 = l_2 \\ m_2 = p_2 \\ p_1 - m_1 = l_1 - s_1 \neq 0 \\ \alpha,\beta}} \frac{e_\alpha}{m_\alpha} \frac{e_\beta}{m_\beta} S_{l,s,m,p}^{(4)}\, \psi_l^{(\alpha)*} \psi_s^{(\alpha)} \psi_m^{(\beta)} \psi_p^{(\beta)*}$$

B. Maxwell-Vlasov

1. Noncanonical form

Variables E, B & $f^{(\alpha)}$

2. Canonical form

$$\frac{\partial E}{\partial t} = +\underset{\substack{\uparrow \\ \nabla \times A}}{\nabla \times B} - \sum_\alpha \frac{e_\alpha}{m_\alpha} \int (p - A) f_\alpha \, dp$$

$$\frac{\partial B}{\partial t} = -\nabla \times E \longrightarrow \frac{\partial A}{\partial t} = -E$$

$$\frac{\partial f_\alpha}{\partial t} = -\underline{v} \cdot \frac{\partial}{\partial \underline{x}} f_\alpha - \frac{e_\alpha}{m_\alpha}\left[E + v \times B\right] \cdot \frac{\partial f_\alpha}{\partial v}$$

$$= -\frac{(p - A)}{m_\alpha} \cdot \frac{\partial f_\alpha}{\partial \underline{x}} + \frac{\partial f}{\partial p_i} \frac{\partial A_j}{\partial x_i} (p_j - A_j)$$

$$H = \sum_\alpha \int \frac{1}{2m_\alpha} (p - A)^2 f_\alpha \, d\underline{x} \, d\underline{p}$$

$$+ \frac{1}{2} \int \left[E^2 + (\nabla \times A)^2 \right] dx$$

$$[\hat{A}, \hat{B}] = \sum_\alpha \int \left(\frac{\delta \hat{A}}{\delta \underline{\Phi}^{(\alpha)}} \frac{\delta \hat{B}}{\delta \underline{\Gamma}^{(\alpha)}} - \frac{\delta \hat{B}}{\delta \underline{\Phi}^{(\alpha)}} \frac{\partial \hat{A}}{\delta \underline{\Gamma}^{(\alpha)}} \right) d\underline{x} \, d\underline{p}$$

$$+ \int_x \left(\frac{\delta \hat{A}}{\delta A} \cdot \frac{\delta \hat{B}}{\delta E} - \frac{\delta \hat{A}}{\delta E} \frac{\delta \hat{B}}{\delta A} \right) dx$$

$$f^\alpha = \{ \underline{\Phi}^{(\alpha)}, \underline{\Gamma}^{(\alpha)} \}$$