TOKAMAKS and LIE-POISSON BRACKETS

P. J. Morrison

Institute for Fusion Studies
& Department of Physics
University of Texas at Austin

Overview

I. Tokamaks

II. Reduced MHD

III. Generlized Hamilitonian Field Theory

IV. Lie-Poisson Bracket for RMHD

Collaborator R. D. Hazeltine

V. Semi-direct Product Interpretation

acknowledgement J. E. Marsden

VI. Summary

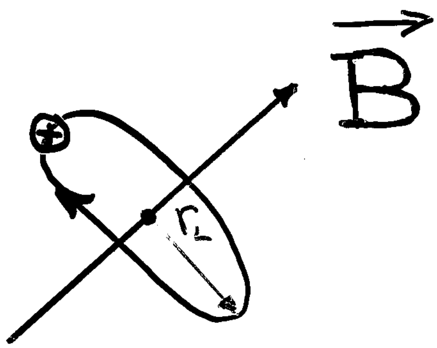
I. Tokamaks

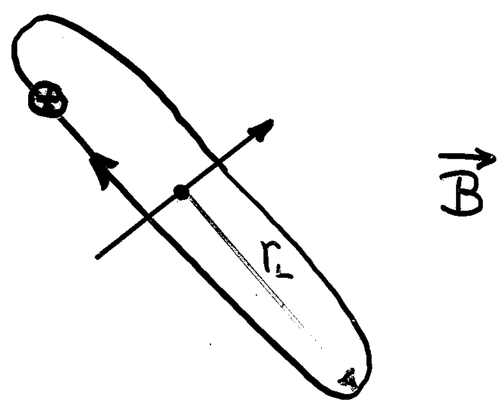
Goal: hold hot enough plasma that is dense enough long enough for fusion energy gain

Method: magnetic field confinement

Tokamak: a leading magnetic field configuration candidate for fusion reactor

Confinement: envolves dynamics of charged particle





$$r_L = \frac{m v_\perp}{e B}$$

• ⇔ guiding center

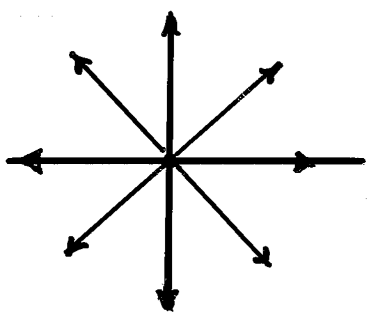
Theorem (Poincaré). Let M be a compact surface on which there is a vector field $\vec{V}$ with a finite number of zeros, then

$$\sum \text{indicies of } \vec{V} = \chi(M) = 2(1-g)$$

Euler characteristic

genus of M (# of handles)

e.g.



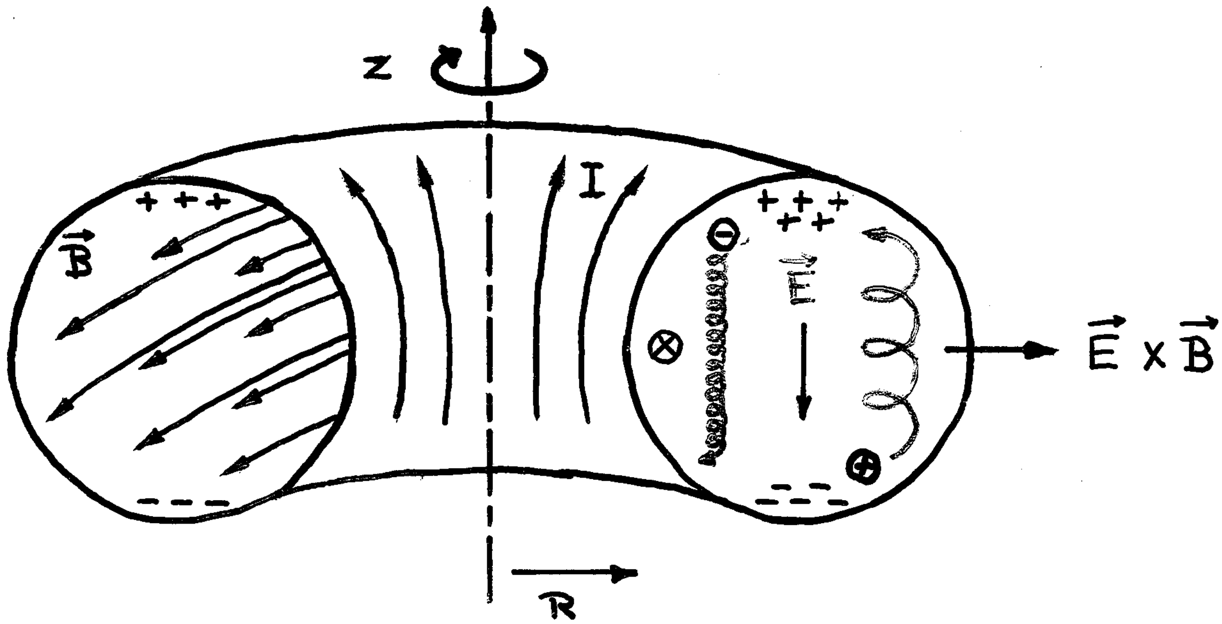
Lemma (Cowlick). For S^2 $g=0$.

$$\text{index} = \frac{\Delta\theta}{2\pi} = +1$$

Observation. Since $r_L \sim \frac{1}{B}$ at zero

$r_L \to \infty \implies$ possible leak.

Toroidal Solenoid

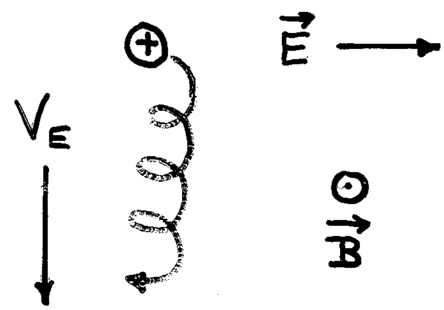


$$B \sim \frac{1}{R}$$

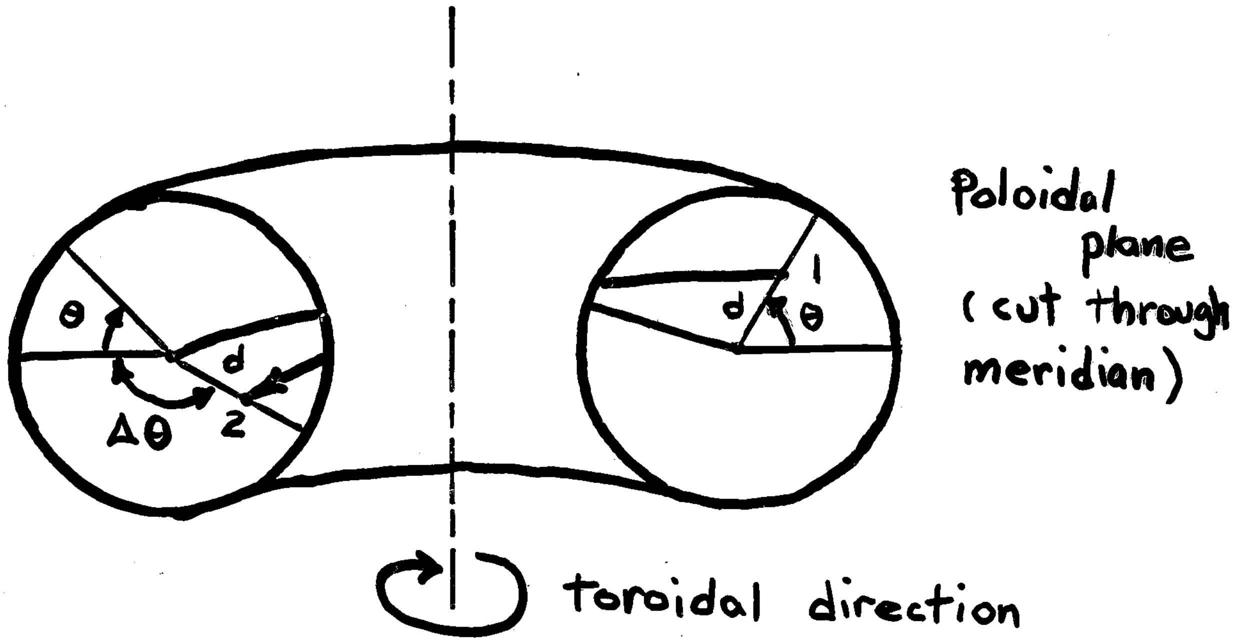
Drifts

$$\vec{V}_{\nabla B} = \pm v_\perp r_L \frac{\vec{B} \times \nabla B}{B^2}$$

$$\vec{V}_E = \frac{\vec{E} \times \vec{B}}{B^2}$$



Equilibrium with Rotational Transform



— minor toroidal axis

— field line

— $\theta = 0$

d distance from minor axis

Equilibrium : field lines in nested toroidal surfaces with rotational transform

Tokamak : achieves rotational transform by internal toroidal plasma current

II. Reduced MHD

Derivation : H. R. Strauss

Low β : Physics of Fluids __19__, 134 (1976).

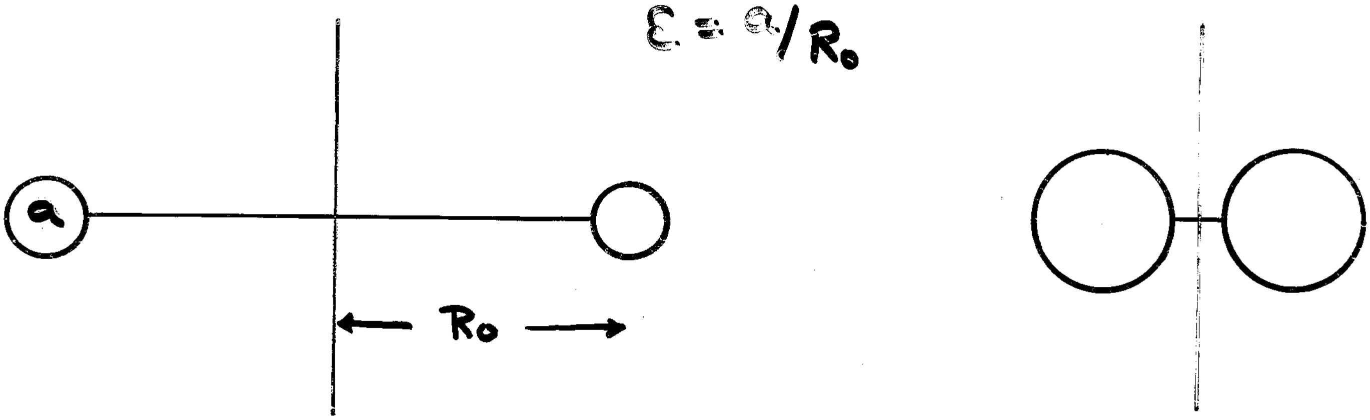
High β : Physics of Fluids __20__, 1354 (1977).

Purpose : Interpretation and control of tokamak experiments

B. Carreras, et al., *Nuclear Fusion* __19__, 1423 (1979).

$$\boxed{\text{Ideal MHD}} \xrightarrow[\text{“}\varepsilon \ll 1\text{”}]{} \boxed{\text{RMHD}}$$

$\varepsilon = a/R_0$



ε_1 $\qquad$ $\varepsilon_1 < \varepsilon_2$ $\qquad$ ε_2

Derivation

Ideal MHD Scaling

$\vec{v}_\perp$	with	$v_p = B_p/\sqrt{\rho}$
t	with	$\tau_p = a/v_p$
$\vec{B}$	with	$B_0 \equiv$ Vacuum on axis toroidal field
$\nabla_\parallel$	with	$R_0 \equiv$ major radius
$\nabla_\perp$	with	$a \equiv$ minor radius

Ad Hoc Scalings

$v_\parallel \equiv 0 \quad (\mathcal{O}(\varepsilon))$

$$\beta \equiv \frac{8\pi p}{B_0^2} \left\langle \begin{array}{ll} \mathcal{O}(\varepsilon^2) & \textbf{Low } \beta \textbf{ Version} \\ \mathcal{O}(\varepsilon) & \textbf{High } \beta \textbf{ Version} \end{array} \right.$$

Incompressibility

$\nabla_\parallel v_\parallel + \nabla_\perp \cdot \vec{v}_\perp = \mathcal{O}(\varepsilon) \implies$ stream function φ s.t.

$$\underline{\vec{v}_\perp = \hat{z} \times \nabla_\perp \varphi}$$

Scaled Magnetic Field

$$\vec{B} = \frac{\hat{z}}{1+\varepsilon x} + \varepsilon \nabla_\perp \psi \times \hat{z} + \varepsilon h \hat{z} + \mathcal{O}(\varepsilon^2)$$

Vacuum toroidal field — Poloidal field — Plasma diamagnetism

$\psi \sim A_z$ (z component of vector potential)

Reduced MHD

$$\frac{\partial \psi}{\partial t} + \frac{\partial \varphi}{\partial z} = \hat{z} \cdot \nabla_\perp \psi \times \nabla_\perp \varphi$$

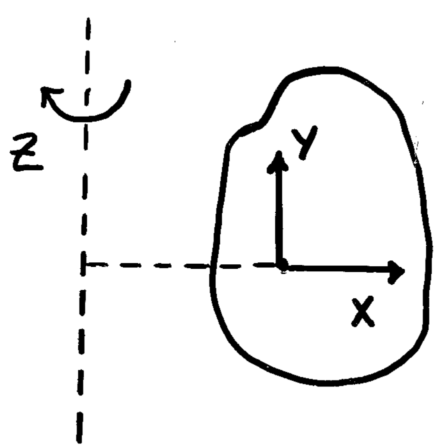
$$\frac{\partial U}{\partial t} + \frac{\partial J}{\partial z} = \hat{z} \cdot \nabla_\perp U \times \nabla_\perp \varphi + \hat{z} \cdot \nabla_\perp \psi \times \nabla_\perp J - \partial \beta / \partial y$$

$$\frac{\partial \beta}{\partial t} = \hat{z} \cdot \nabla_\perp \beta \times \nabla_\perp \varphi$$

where

$J = \nabla_\perp^2 \psi$ toroidal current

$U = \nabla_\perp^2 \varphi$ scalar vorticity



Integral Constants

TABLE 1

MHD Invariant	Comments	RMHD Remnant	Comments
$M = \int \rho \, d\tau$	Casimir Invariant		
$\vec{P} = \int \rho\vec{v} \, d\tau$			
$\vec{L} = \int \vec{x}\times\rho\vec{v}$		$\vec{L}=\hat{z}\int \phi \, d\tau$	
$\vec{H} = \int (\frac{1}{2}\rho v^2+\rho U+\frac{B^2}{2}) d\tau$	$U(\rho,s)$ is the internal energy per unit mass. S is the entropy per unit mass.	$H=\int \frac{1}{2}(\lvert\nabla_\perp\phi\rvert^2+\lvert\nabla_\perp\psi\rvert^2-\beta x) d\tau$	For low-β version $\beta\to 0$. x is Cartesian coordinate in poloidal plane.
$\vec{G} = \int (\rho\vec{x}-\rho\vec{v}t) d\tau$	Center of mass constant. Appeared in Ref. 14.		
$S = \int \rho S(s) d\tau$	S arbitrary. Casimir Invariant.	$P_r = \int g(\beta) d\tau$	g arbitrary. This should be opposite $\int \rho g(p\rho^{-\gamma})\partial\tau$. We write the eq. of state more generally using S.
$\vec{B} = \int \vec{B} \, d\tau$			
$V = \int \vec{v}\cdot\vec{B} \, d\tau$	Barotropic flow or $B\cdot\nabla s=0$.	$V=\int \nabla_\perp\phi\cdot\nabla_\perp\psi \, d\tau$	
$A = \int \vec{A}\cdot\vec{B} \, d\tau$		$A=\int \psi \, d\tau$	
$Q = \int \rho f(\frac{B}{\rho}\cdot\nabla\Phi) d\tau$	f arbitrary. Φ any advected quantity. Appeared in Ref. 21.	$Q=\int f(\frac{\partial\Phi}{\partial z}-[\psi,\Phi]) d\tau$	The special case where ϕ is pressure and f is the identity function was given in Ref. 2.
		$C=\int Uh(\psi)$	h arbitrary. Single helicity. Casimir Invariant.

III. Generalized Hamiltonian Field Theory

Goal: find a "Poisson bracket" for a system of evolution equations of the form:

$$\frac{\partial u^i}{\partial t}(t, \vec{x}) = F^i(\vec{u}, \vec{x}) \qquad i = 1, 2, \cdots m$$

where $\underline{\vec{x} \in V}$ & $\underline{u^i \in \mathbb{L}^2(V)}$ with inner product

$$\langle f | g \rangle = \int_V f g \, d\tau$$

The $\underline{F^i \in \mathcal{L}}$ where $\mathcal{L}$ contains general nonlinear partial differential or integral operators

(i) $\vec{u}$ & $\vec{x}$

(ii) $D_{\vec{k}} u^i \equiv \dfrac{\partial^{|k|} u^i}{\partial x_1^{k_1} \partial x_2^{k_2} \cdots \partial x_n^{k_n}}$

(iii) $\displaystyle\int_V K(\vec{x} \,|\, \vec{x}') \, f(D_{\vec{k}} \vec{u}) \, d\tau$

Functionals: Field theories envolve things like integrals over energy density — "Poisson brackets" act on such things, so we define vector space, Ω, of differentiable functionals of the form

$$G[\vec{u}] = \int_V G(\vec{u}, \vec{x})\, d\tau$$

where $G \in \mathcal{L}$; $\mathcal{L} : \mathbb{L}^2(V) \to \mathbb{L}^2(V)$.

Differentiation: The functional derivative is given by

$$\frac{d}{d\varepsilon} F[u^i + \varepsilon w] \Big|_{\varepsilon=0} = \left\langle \frac{\delta F}{\delta u^i} \middle| w \right\rangle$$

where $w(\partial V) = 0$, $\frac{\delta F}{\delta u^i} \in \mathbb{L}^2(V)$.

Traditionally, Poisson brackets for field theories have the form

$$\{F, G\} = \sum_k \int_V \left[\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right] d\tau$$

and the equations of motion for some Hamiltonian functional H are

$$\frac{\partial \eta_k}{\partial t} = \{\eta_k, H\} \quad , \quad \frac{\partial \pi_k}{\partial t} = \{\pi_k, H\} .$$

Generalize, by maintaining the algebraic properties of $\{ , \}$ and defining the following generic bilinear product on Ω :

$$\{F, G\} = \left\langle \frac{\delta F}{\delta u^i} \middle| O^{ij} \frac{\delta G}{\delta u^j} \right\rangle$$

where $O^{ij} \in \mathcal{L}$.

Require, O^{ij} endow $\{ , \}$ with the following:

(i) $\{F, G\} = -\{G, F\}$ $\qquad F, G \in \Omega$

$\Rightarrow$ O^{ij} anti-self-adjoint

(ii) Jacobi identity

$\{E, \{F, G\}\} + \circlearrowleft = 0$ $\qquad E, F, G \in \Omega$

Definition. A system of equations is Hamiltonian in the generalized sense if there exists an operator $O^{ij} \in \mathcal{L}$ and a functional H such that the system can be cast into the form

$$\frac{\partial u^i}{\partial t} = \{u^i, H\}$$

where $\{ , \}$ makes Ω a Lie algebra.

IV. Brackets for RMHD

overview

- A. Low Beta
  1. 2-D
  2. 3-D
  3. single helicity
- B. High Beta

Three Principles for Guessing Brackets

1. Use physical energy as Hamiltonian

   may not be unique; e.g. KdV, RLW
   reason obvious if equations come from fluid equations

2. Guess correct variables

   wrong variables can make operator complicated. optimize!

3. Dimensional Analysis

   can determine nonlinearity of O^{ij} as well as form.

A. Low β

$$H = \frac{1}{2}\int \left(|\nabla_\perp \varphi|^2 + |\nabla_\perp \psi|^2\right) dz$$

$$\frac{\delta H}{\delta U} = -\varphi \qquad\qquad \frac{\delta H}{\delta \psi} = -J$$

1. Two Dimensions

$$\frac{\partial}{\partial z} \longrightarrow 0$$

Equations of Motion

$$\frac{\partial U}{\partial t} = [U, \varphi] + [\psi, J]$$

$$\frac{\partial \psi}{\partial t} = [\psi, \varphi]$$

where

$$[f, g] = \hat{z} \cdot \nabla_\perp f \times \nabla_\perp g = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} .$$

2-D Bracket

$$\boxed{\{F, G\}_2 = \int d\tau \left\{ U \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U} \right] + \psi \left(\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \psi} \right] - \left[\frac{\delta G}{\delta U}, \frac{\delta F}{\delta \psi} \right] \right) \right\}}$$

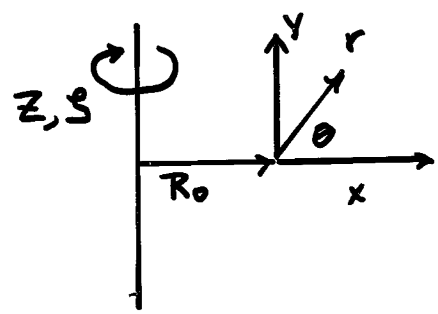
2. Three Dimensions

Equations of Motion

$$\frac{\partial U}{\partial t} = [U, \varphi] + [\psi, J] - \frac{\partial J}{\partial \zeta}$$

$$\frac{\partial \psi}{\partial t} = [\psi, \varphi] - \frac{\partial \varphi}{\partial \zeta}$$

Coordinate system



$z = \zeta$ $\qquad x = r\cos\theta$

$\qquad\qquad y = r\sin\theta$

$$[f, g] = \frac{1}{r}\left(\frac{\partial f}{\partial r}\frac{\partial g}{\partial \theta} - \frac{\partial f}{\partial \theta}\frac{\partial g}{\partial r} \right)$$

3-D Bracket

$$\boxed{\{F, G\}_3 = \{F, G\}_2 + \int d\underline{x} \left(\frac{\delta F}{\delta U} \frac{\partial}{\partial \zeta} \frac{\delta G}{\delta \psi} - \frac{\delta G}{\delta U} \frac{\partial}{\partial \zeta} \frac{\delta F}{\delta \psi} \right)}$$

3. Single Helicity

$\psi(\theta - \zeta/q_0)$ — q_0 determines pitch of helix.

Define

$$\psi_h = \psi + r^2/2q_0 \qquad (\vec{B} \cdot \nabla \psi_h = 0)$$

Equations of Motion Become

$$\left.\begin{aligned} \frac{\partial \psi_h}{\partial t} &= [\psi_h, \varphi] \\ \frac{\partial U}{\partial t} &= [U, \varphi] + [\psi_h, J_h] \end{aligned}\right\} \text{2-D Poisson Bracket}$$

Conjecture. $\dfrac{\partial \bar{\Psi}_n}{\partial \zeta} + [\psi, \bar{\Psi}_n] = 0$ will

Simplify in the general case.

B. High Beta

$$H = \int \frac{1}{2} \left(|\nabla_\perp \varphi|^2 + |\nabla_\perp \psi|^2 - 2\beta x \right)$$

Equations of Motion

$$\frac{\partial U}{\partial t} = [U, \varphi] + [\psi, J] - \frac{\partial J}{\partial \zeta} - \frac{\partial \beta}{\partial y}$$

$$\frac{\partial \psi}{\partial t} = [\psi, \varphi] - \frac{\partial \varphi}{\partial \zeta}$$

$$\frac{\partial \beta}{\partial t} = [\beta, \varphi]$$

High β Bracket

$$\boxed{\{F, G\}_4 = \{F, G\}_3 + \int \beta \left\{ \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \beta} \right] - \left[\frac{\delta G}{\delta U}, \frac{\delta F}{\delta \beta} \right] \right\} d\underline{x}}$$

V. Semi-direct Product Interpretation

Lie; Berezin; Vinogradov and Kupershmidt; Guillemin and Sternberg; Marsden Weinstein, Ratiu and others.

Overview

A. Poisson Structures

EXAMPLE. Part of RMHD bracket

B. Semi-direct Products

EXAMPLE. 2-D or single helicity RMHD

A. Poisson Structures (Lie-Poisson bracket on $\mathfrak{g}^*$)

Definitions

G : a Lie group

$\mathfrak{g}$: Lie algebra of G (left invariant vector fields on G with $[\,,\,]$, commutator product)

$\mathfrak{g}^*$: algebra dual to $\mathfrak{g}$

$\langle\,,\,\rangle$: pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$

Ω : $C^\infty(\mathfrak{g}^*)$, functionals $F: \mathfrak{g}^* \longrightarrow \mathbb{R}$

$\frac{\delta F}{\delta U}$: functional derivative $\frac{\delta F}{\delta U} \in \mathfrak{g}$, $U \in \mathfrak{g}^*$. defined by

$$DF(U)\cdot \nu = \langle \nu, \frac{\delta F}{\delta U} \rangle \quad \forall\, \nu \in \mathfrak{g}^*$$

$\{F, G\}$: Lie-Poisson bracket (P.S. on $\mathfrak{g}^*$)

$$\{F, G\} = -\langle U, [\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}] \rangle$$

EXAMPLE

$$\{F, G\}_U = \int_V U \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U} \right] dx\,dy$$

where $[f, g] = f_x g_y - g_x f_y$

G = group of canonical transformations on $\mathbb{R}^2$

$\mathfrak{g}$ = Lie algebra of Hamiltonian vector fields $\leftarrow$ generating functions $C^\infty(\mathbb{R}^2)$ $[f, g] = g_x f_y - g_y f_x$

$\mathfrak{g}^*$ = densities. Note $U \in \mathfrak{g}^*$

$$\langle h, f \rangle = \int_V h f \, dx\,dy \quad \text{where } h \in \mathfrak{g} \text{ \& } f \in \mathfrak{g}^*$$

$$\{F, G\}_u = \left\langle U, \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U} \right] \right\rangle$$

B. Semi-direct Products

General

G and V two groups

$\rho : G \longrightarrow \mathrm{Aut}(V)$ realization of G

group by extension $S = G \times_{\rho} V$

for (g_1, u_1) & $(g_2, u_2) \in S$ have product

$$(g_1, u_1)(g_2, u_2) = (g_1 g_2 , u_1 (\rho(g_1) u_2))$$

Particular

G a Lie group

V a vector space

ρ a left representation of G

$$(g_1, u_1)(g_2, u_2) = (g_1 g_2 , u_1 + \rho(g_1) u_2)$$

$$\begin{tikzcd} G \arrow[r, "\rho"] & \mathrm{Aut}(V) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\rho'"'] & \mathrm{End}(V) \arrow[u, "\exp"'] \end{tikzcd}$$

Lie algebra of S is

$\mathfrak{s} = \mathfrak{g} \underset{\rho'}{\times} V$ with bracket

$$[(\xi_1, v_1), (\xi_2, v_2)] = \left([\xi_1, \xi_2], \rho'(\xi_1)v_2 - \rho'(\xi_2)v_1\right)$$

Lie Poisson bracket for functionals

$F, G : \mathfrak{s}^* \longrightarrow \mathbb{R}$

$$\{F, G\} = -\left\langle U, \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right]\right\rangle - \left\langle \psi, \rho'\left(\frac{\delta F}{\delta U}\right)\cdot \frac{\delta G}{\delta \psi}\right\rangle$$
$$+ \left\langle \psi, \rho'\left(\frac{\delta G}{\delta U}\right)\cdot \frac{\delta F}{\delta \psi}\right\rangle$$

$\psi \in V^*$ $\quad$ $U \in \mathfrak{g}^*$

$\dfrac{\delta F}{\delta \psi} \in V$ $\quad$ $\dfrac{\delta F}{\delta U} \in \mathfrak{g}$

EXAMPLE

G = group of canonical transformations

V = vector space of functions

$\mathfrak{g}$ = Lie transform

$\mathfrak{g}'$ = action of $\mathfrak{g}$ on V; poisson bracket in plane

$\xi \in \mathfrak{g}$, $v \in V$ $\quad \mathfrak{g}'(\xi)\cdot v = [\xi, v]$

Lie Poisson bracket

$$\{F, G\}[U, \psi] = \int dx\,dy \left(U\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U}\right] + \psi\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \psi}\right] - \psi\left[\frac{\delta G}{\delta U}, \frac{\delta F}{\delta \psi}\right] \right)$$