$\left\{\begin{matrix}\text{HAMILTONIAN} \\ \text{SYSTEMS}\end{matrix}\right\} \cup \left\{\begin{matrix}\text{SYSTEMS THAT} \\ \text{RELAX}\end{matrix}\right\}$

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Solitons and Coherent Structures
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OVERVIEW

I. H-Theorems
- A. Stat. Mech.
- B. Liapunov Functions

II. Construction
- A. Three forms of "Collision" Operators
- B. KdV
- C. Plasma Collision operator

III. Flows on Metriplectic Manifolds
- A. Symplectic Manifolds
- B. Gradient Flows
- C. Metriplectic Flows
- D. Infinite Dimensions (Field Theory)

IV. Example - Relaxing Free Rigid Body

V. Example - Plasma Collision Operator ?

I. H - Theorems

I.A Boltzmann's Theorem

phase space probability density:

$$f(\vec{x}, \vec{v}, t)$$

transport equation:

$$\frac{df}{dt} = \underbrace{\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{\vec{F}}{m} \cdot \frac{\partial f}{\partial \vec{v}}}_{\text{Hamiltonian}} = \underbrace{\left(\frac{\partial f}{\partial t}\right)_c}_{\text{Source}}$$

Boltzmann's H-Function:

$$H = \int_{\Omega} f \ln f \, dv$$

Boltzmann's Theorem:

$$\frac{\partial H[f]}{\partial t} \leq 0$$

H decreases until $\frac{\partial H}{\partial t} = 0$

$f \longrightarrow$ Maxwell distribution

I. B Liapunov's Theorem

dynamical system:

$$\dot{z}^i = F^i(z) \qquad i = 1, 2, \cdots N$$

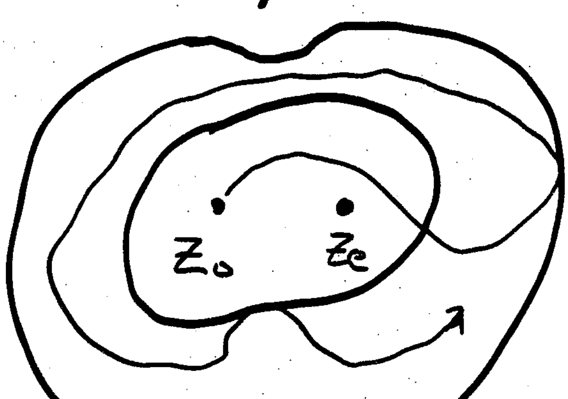
equilibrium point: $F^i(z_e) = 0 \quad \forall i$

Liapunov's theorem: If there is a function $L(z) \in \mathbb{R}$ such that

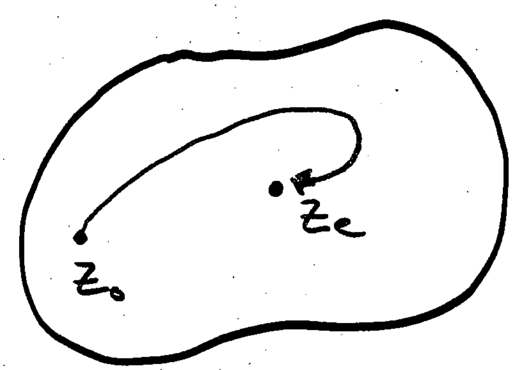
(i) $L(z_e) = 0$ and $L(z) > 0$ if $z \neq z_{eq}$

(ii) $\dot{L}(z) \leq 0$ and $\dot{L}(z) = 0$, iff $z = z_{eq}$

Stability



Asymptotic Stability



II.A Forms For "Collision" Operator

dynamical eq. : $u_t - F(u) = S(u)$

$u(x,t), \quad t \geq 0$

$x \in \Omega$

Consts. w/ $S = 0$: $I = \lambda_0 I_0 + \lambda_1 I_1 + \lambda_2 I_2 + \ldots$

effect of $S \neq 0$:

$$\frac{dI}{dt} = \int_\Omega \frac{\delta I}{\delta u} u_t \, dz = \int_\Omega \frac{\delta I}{\delta u} S \, dz$$

forms :

(1) Let $S_1 = -\frac{\delta I}{\delta u} \Rightarrow$

$$\frac{dI}{dt} = -\int_\Omega \left(\frac{\delta I}{\delta u}\right)^2 dz \leq 0$$

equilibria : $\delta I / \delta u = 0$

(2) $S_2 = -A^+ A \frac{\delta I}{\delta u}$ $\quad \frac{dI}{dt} = -\int_\Omega \left(A \frac{\delta I}{\delta u}\right)^2 dz$

Can design A^+A to conserve some I_α's

(3) $S_3 = -A^+ K A \frac{\delta I}{\delta u}$ $\quad K \geq 0$

Generalized Plasma Collision operator is an S

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Generalized Plasma Collision operator is an S_3

II. B KdV Example

dynamical system : $u_t + u u_x + u_{xxx} = S$

consts. : $I = \sum_{i=1}^{\infty} \lambda_i I_i$

equil. : $\frac{\delta I}{\delta u} = 0$ Kruskal - Zabusky Variational Principal

Special Case : $\hat{I} = \lambda_0 \underset{u}{I_0} + \lambda_1 \underset{\frac{u^2}{2}}{I_1} + \lambda_2 \underset{\frac{u^3}{6} - \frac{u_x^2}{2}}{I_2}$

$\frac{\delta \hat{I}}{\delta u} = 0 \Rightarrow$ Single Soliton Solution

choose S : $S_1 = -\eta \frac{\delta \hat{I}}{\delta u}$

$\Rightarrow$

$$u_t + u u_x + u_{xxx} = -\eta \Big(\underbrace{\lambda_0 + \lambda_1 u + \lambda_2 \big(\tfrac{u^2}{2}}_{\text{N.L. Damping/ growth}} + \underbrace{u_{xx}}_{\text{Burgers}} \big) \Big)$$

II.C Plasma Collision Operator

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{x}} + \vec{E}[f; x, t] \cdot \frac{\partial f}{\partial \vec{v}} = \left(\frac{\partial f}{\partial t}\right)_c$$

Generalized Collision Operator

$$\boxed{\left(\frac{\partial f}{\partial t}\right)_c = \frac{\partial}{\partial v_i} \int \omega_{ij} \left[F(f(v)) \frac{\partial f(v')}{\partial v_j'} - F(f(v')) \frac{\partial f}{\partial v_j} \right] d^3v'}$$

- F is arbitrary function of f
- $\omega_{ij}(v, v')$ symmetric
- $(v_i - v_i')\,\omega_{ij} = 0$

$\Rightarrow$ Conservation of Energy, Momentum & Mass

Generalized Entropy Functional

$$\boxed{S[f] = \int \mathcal{S}(f)\, dz}$$ Casimir

H - Theorem

$$\boxed{\frac{\partial^2 \mathcal{S}}{\partial f^2} F = 1}$$

II.c (continued)

Special Cases

(1) pick $\mathcal{S} = f \ln f \Rightarrow F = f$

obtain Landau or Lenard Balescu

(2) pick $\mathcal{S}$ s.t. $F = f(1-f)$

$\Rightarrow$ relaxation to Lynden-Bell (Fermi-Dirac)

Kadomtsev & Pogutse PRL 25 (1970) 1155

Connection between F and Spectrum

$$\langle \delta f \, \delta f \rangle_{k,\omega} = \delta(v-v')\,\delta(\omega - k\cdot v)\,F(f_0)$$

Remark

Can generalize to phase space diffusion & friction

III. A Symplectic Manifolds

Hamilton's Eqs.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = [q_i, H]$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = [p_i, H]$$

Coord. Relabeling

$$z^i = \begin{cases} q_i & i = 1, \cdots N \\ p_i & i = N+1, \cdots 2N \end{cases}$$

$$\Rightarrow \quad \dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j} \qquad J_c = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

Co symplectic form

Noncanonical Poisson Bracket

$$\{f, g\} = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

$\det J^{ij}$ may $= 0$; $J^{ij} = -J^{ji}$, Jacobi

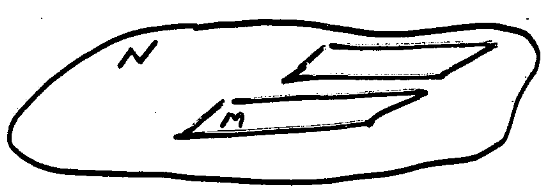
Casimirs

$$\{C, f\} = 0 \quad \forall f \quad \Rightarrow \quad \frac{\partial C}{\partial z^i} J^{ij} \frac{\partial f}{\partial z^j} = 0$$

$$\Rightarrow \quad \frac{\partial C}{\partial z^i} = \text{null eigenvector of } J^{ij}$$

phase space

$M \sim$ Rank J^{ij}



leafs labeled by C's

$\nabla C \perp$ to leaf

III. B Gradient Flows

$$\dot{z}^i = -g^{ij} \frac{\partial L(z)}{\partial z^j}$$

g^{ij} symmetric & has positive eigenvalues

Gradient Flows Have Built In Liapunov Functions

$$\frac{dL}{dt} = \frac{\partial L}{\partial z^i} \dot{z}^i = -\frac{\partial L}{\partial z^i} g^{ij} \frac{\partial L}{\partial z^j} \leq 0$$

If z_e is an isolated min of L (equilibrium) $\Rightarrow$ Asymptotic stability

III.C Metriplectic Flows

metriplectic manifold: Differentiable manifold with a bilinear form defined on functions

$$\{\{f,h\}\} = \{f,h\} + (f,h)$$

in coords

$$= \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial h}{\partial z^j} + \frac{\partial f}{\partial z^i} g^{ij} \frac{\partial h}{\partial z^j}$$

J^{ij} cosymplectic form

det J^{ij} may $= 0$

g^{ij} Metric - Symmetric

det g^{ij} may $= 0$

(any eigenvalues?)

metriplectic flow:

$$\dot{z}^i = J^{ij} \frac{\partial F}{\partial z^j} + g^{ij} \frac{\partial F}{\partial z^j}$$

flow in the leaf

flow out of leaf

III. C (b Continued)

Properties and Concepts

(1) $\{f,h\}$ Antisymmetric, Jacobi etc.

(2) (f,h) symmetric

Can Define Riemann curvature tensor $R^{i}{}_{stu}\,(g^{ij})$

(i) Flat space

(ii) const. curvature

(iii) harmonic, etc.

(3) Metriplectic two form

$$m = J_{ij}\, dz^i \wedge dz^j + g_{ij}\, dz^i \otimes dz^j$$

(4) Metriplectic Group

Can categorize dissipation by curvature tensor.

Operators $A^{+}A$, $A^{+}KA$ (pt. II.A) determine g^{ij}

III C. (Continued)

Dynamical Constraint Surface:
Suppose E, P etc. are isolating integrals. Dynamical Constraint surface is their intersection.

Classical Systems should conserve Energy, Momentum etc while producing Entropy.

Thus null eigenvectors of g^{ij} can be

$$\frac{\partial E}{\partial z^i}, \quad \frac{\partial P}{\partial z^i}, \quad \text{etc.}$$

What should Generate the Flow? (i.e. $F = ?$)

III. C (C_a Continued)

Generalized Free Energy

Recall variational Principle from classical thermodynamics

$$F = H - TS$$

Equilibria arise by varying the energy at constant entropy (T is Lagrange multiplier)

Generalized Free Energy

$$F = H + \sum_i C_i$$

Sum of C's is Generalized Entropy

Natural choice since in Energy-Cas. method critical points of F are equil.

In what sense is $\sum_i C_i$ an entropy?

(1) For Kinetic theory turns out to be the case (c.f. III.C)

(2) Require degeneracy in J^{ij}.

This arises from reduction. Reduction means loss of information; eg. Lagrangian → Eulerian

III.d Infinite Dimensions

$$\{\{F, G\}\} = \{F, G\} + (F, G)$$

$$= \int \frac{\delta F}{\delta \psi^i} O^{ij} \frac{\delta G}{\delta \psi^j} + \int \frac{\delta F}{\delta \psi^i} M^{ij} \frac{\delta G}{\delta \psi^j}$$

$(O^{ij})^{+} = -O^{ij}$ plus Jacobi

$(M^{ij})^{+} = +M^{ij}$ plus Something perhaps

Examples - Fluids, Plasmas, Kinetic Theory

Navier Stokes, II.C, etc.

IV. Relaxing Free Rigid Body

Energy : $E = \frac{1}{2} I_{ij} \omega_i \omega_j$

Angular Mom. : $C = I_{i\ell} I_{it} \omega_\ell \omega_t$

Scale $\Longrightarrow$

$$E = \frac{1}{2} (\alpha_1 \omega_1^2 + \alpha_2 \omega_2^2 + \alpha_3 \omega_3^2)$$

$$C = \frac{1}{2} (\omega_1^2 + \omega_2^2 + \omega_3^2)$$

Euler's Eqs. :

$$\dot{\omega}_i = \varepsilon_{ijk} \frac{\partial E}{\partial \omega_j} \omega_k = \{ \omega_i , E \}$$

$$\dot{\omega}_i = \varepsilon_{ijk} \omega_k (\alpha_j \omega_j)$$

$$\dot{\omega}_1 = (\alpha_2 - \alpha_3) \omega_2 \omega_3 + \lambda \omega_1 \left[(\alpha_1 - \alpha_2) \alpha_2 \omega_2^2 + (\alpha_1 - \alpha_3) \alpha_3 \omega_3^2 \right]$$

$\vdots$

Removes angular momentum at Constant Energy !

$$\dot{\omega}_i = \{\{ \omega_i , F \}\} \; ; \; F = E + C$$

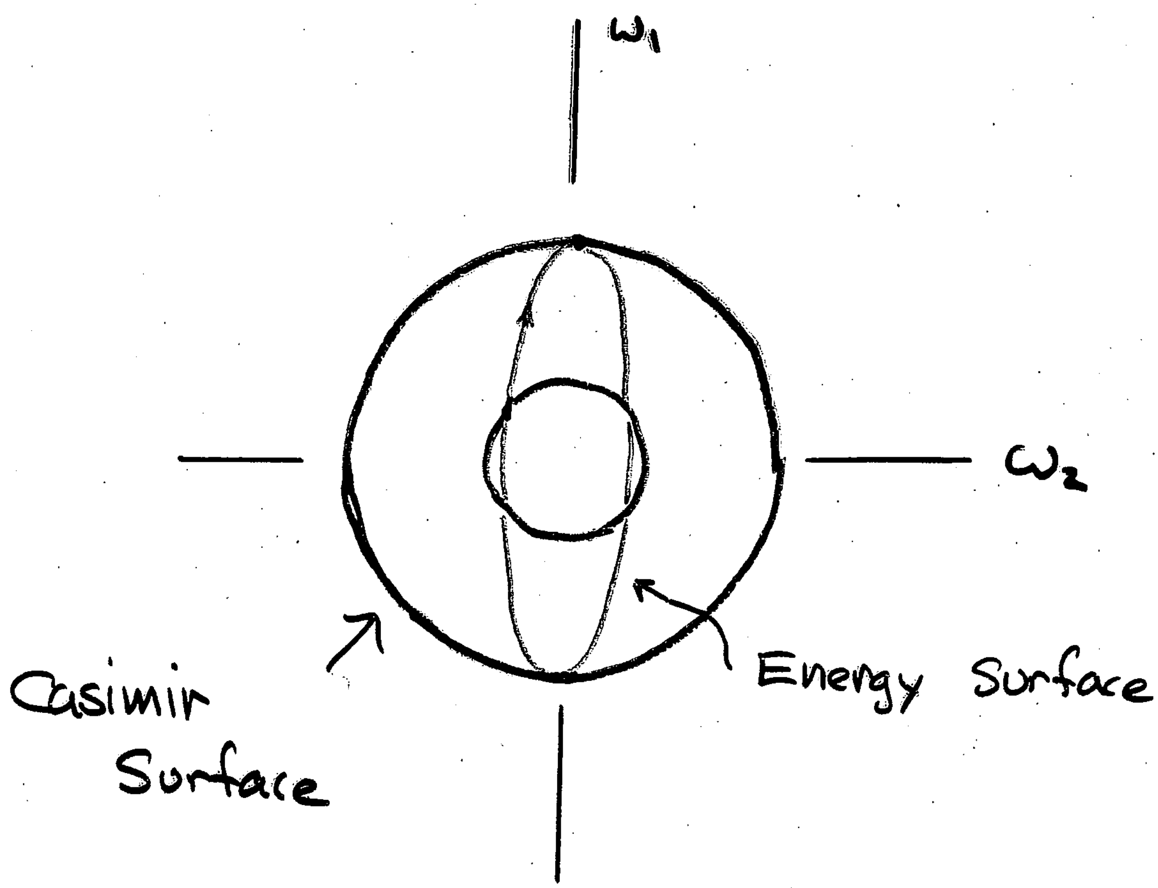
IV. (continued)

Equilibria : $\omega_1 \neq 0$, $\omega_2 = \omega_3 = 0$

Phase Space

Symplectic leaves are spheres

Energy Surface is an ellipsoid



Probably missing last

transparencies? PJM 11/11/08