CLASSICAL FIELD THEORY OF PLASMAS

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Basic Idea

Eulerian plasma fields are put into Hamiltonian form by generalizing the Poisson bracket. For continuous media the Poisson bracket has a special form.

Motivation

Conceptually organize, add insight, decrease the effort for calculation.

Applications

Classification of plasma fields, origin of variational principles, stability, tokamak model equations

OVERVIEW

I. Introduction

II. Generalized Hamiltonian Mechanics

III. Field Theory

IV. Applications

V. Examples

II. Generalized Hamiltonian Mechanics

Hamiltons Eqs. :

$$\dot{q}_k = \frac{\partial H}{\partial p_k} = [q_k, H]$$

$$k = 1, 2, \ldots N$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} = [p_k, H]$$

Poisson Bracket :

$$[f, g] = \sum_{k=1}^{N} \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q_k} \frac{\partial f}{\partial p_k} \right)$$

Cosymplectic Form :

let

$$z^i = \begin{cases} q_k & i = 1, 2, \ldots N = k \\ p_k & i = k+N = N+1, \ldots 2N \end{cases}$$

obtain

$$[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}$$

$$(J_c^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

$$\dot{z}^i = J_c^{ij} \frac{\partial H}{\partial z^j}$$

dynamics

kinematics or phase space

Bracket Properties:

(i) bilinear

(ii) $-[f,g]=[g,f]$

(iii) Jacobi $\quad [f,[g,h]]+[g,[h,f]]+[h,[f,g]]=0$

(iv) $[fg,h]=f[g,h]+[f,h]g$

Lie Algebra

Transformations:

$z^i \longrightarrow z'^i$ coordinate change

$J_c^{ij} \longrightarrow J'^{ij}$ contravariant tensor

$J_c^{ij} \longrightarrow J_c'^{ij}$ canonical transformation

bracket properties are invariant

Converse outlook:

bracket properties $\Rightarrow$ $z'^i \longrightarrow z^i$, $J^{ij} \longrightarrow J_c^{ij}$

Darboux (local , $\det J^{ij} \neq 0$)

Generalized Hamiltonian Mechanics :

Definition. A system of ordinary differential equations is Hamiltonian in the generalized sense if it can be cast into the form

$$\dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} = [z^i, H] \qquad i,j = 1,2, \dots M$$

need not be even

where

$$[f, g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

has bracket properties.

Generalized Phase Space :

Since definition allows <u>$\det(J^{ij}) = 0$</u> the structure of phase space is changed.

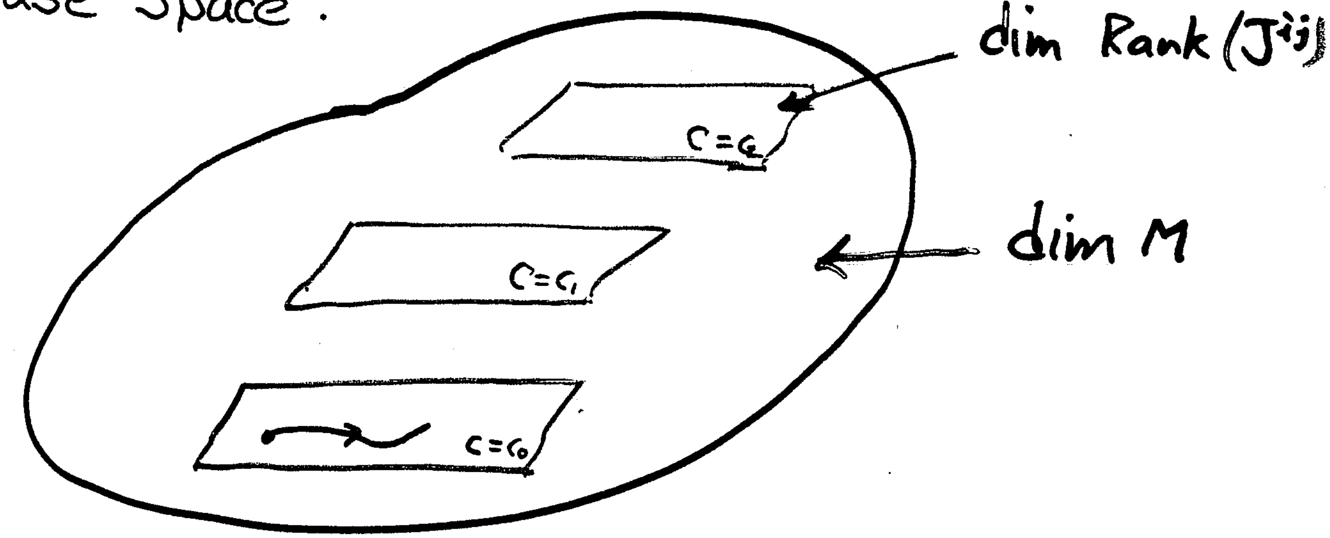
Corank of (J^{ij}) = dimension of null space

Null space spanned by gradients : $\underline{\frac{\partial C}{\partial z^i} J^{ij} = 0}$

The quantities C are <u>Casimirs</u> – phase space <u>constants</u>; built into phase space

$$[C, g] = \frac{\partial C}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} = 0 \qquad \text{for all } g$$

Phase Space :



For any hamiltonian the trajectory is confined to symplectic leaf.

II. Field Theory

Canonical bracket :

$$\{F, G\} = \sum_{k=1}^{L} \int \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right) d\underline{x}$$

bracket acts on functionals of the field variables, η_k, π_k ; e.g.

$$H = \int \mathcal{H} \, d\underline{x}$$

Hamiltonian density $\left(\frac{1}{2}\rho v^2\right)$

phase space derivatives become functional derivatives

$$\frac{\partial}{\partial q_k} \longrightarrow \frac{\delta}{\delta \eta_k}$$

defined by

$$\delta F = \left. \frac{d}{d\varepsilon} F[\eta + \varepsilon \delta \eta] \right|_{\varepsilon=0} = DF \cdot \delta \eta = \left\langle \frac{\delta F}{\delta \eta}, \delta \eta \right\rangle$$

$$= \int \frac{\delta F}{\delta \eta} \delta \eta \, d\underline{x}$$

Generalization - Noncanonical Brackets

$$\{F, G\} = \int \frac{\delta F}{\delta \psi^i} O^{ij} \frac{\delta G}{\delta \psi^j} \, d\tau$$

$$= \left\langle \frac{\delta F}{\delta \psi^i}, O^{ij} \frac{\delta G}{\delta \psi^j} \right\rangle$$

cosymplectic operator

(1) Antisymmetry $\Rightarrow$ O^{ij} anti-self-adjoint

(2) Jacobi - stiff requirement!

(Bracket must be Lie product for algebra of functionals)

Equations of Motion:

$$\frac{\partial \psi^i}{\partial t} = \{\psi^i, H\} = O^{ij} \frac{\delta H}{\delta \psi^j}$$

Canonical Case $\quad (O^{ij}) = \begin{pmatrix} 0 & I_M \\ -I_M & 0 \end{pmatrix}$

Canonical Fields:

Klein - Gordon etc.

$$(O^{ij}) = \begin{bmatrix} 0 & I_N \\ -I_N & 0 \end{bmatrix}$$

Continuous Media Fields:

Ideal MHD, Vlasov, etc.

$$(O^{ij}) = \left(\psi^k c_k^{ij}\right)$$

linear in the field variables

c_k^{ij} are structure operators for some Lie algebra on functions

Lie - Poisson Brackets:

$$\{F, G\} = \int \psi^k \left[\frac{\delta F}{\delta \vec{\psi}}, \frac{\delta G}{\delta \vec{\psi}} \right]_k dz$$

outer algebra on functionals

inner algebra on functions

IV. Applications

A. Classification of Fields

B. Variational Principles for Equil.

C. Stability

D. Derivation of Model Equations

CLASSIFICATION

EQUATIONS	HAMILTONIAN	BRACKET	CASIMIRS
KdV MKdV	$\int \left(\frac{u^3}{6} - \frac{1}{2} u_x^2\right) dx$	Gardner	$\int u\, dx$
Liouville Eq. Vlasov-Poisson 2-D Euler Guiding Center	$\int h f\, dz$ $\int h_1 f + \int h_2 f f$ $\int \omega \phi$ $\int \rho \phi$	Canonical Transformations of $\mathbb{R}^{2n}$	$\int F(\psi) dz$
RMHD Tokamak Models	$\int \lvert\nabla\phi\rvert^2 + \lvert\nabla\psi\rvert^2$	Above extended by sem-direct pdt.	$\int F(\psi)$ $\int U F(\psi)$
MHD CGL Theory	$\int \frac{1}{2}\rho v^2 + \rho U(\sigma,\rho) + \frac{B^2}{2}$ $U(\sigma, \rho, B)$	Diffeomorphisms of $\mathbb{R}^3 \times$ fns.	$\int A \cdot B$, $\int v \cdot B$ & others

Just as many fields are naturally canonical, there are many equations that have the same generalized Poisson bracket. They have different Hamiltonians.

<u>Casimirs</u> are bracket constants. They are independent of the Hamiltonian. If C is a casimir then $\{C, F\} = 0$ for all F.

Casimirs are useful for obtaining variational principles for equilibria. They are an ingredient in the algorithm for constructing Liapunov functionals.

Variational Principles

For ordinary Hamiltonian systems equilibria correspond to critical points of the Hamiltonian. For plasma fields the Hamiltonian is the energy. Variation of the energy gives trivial equilibria. Nontrivial equilibria arise when the energy is varied at constant Casimirs.

Example

I = Energy + Mag. Helicity

$\uparrow$

Casimir

Thermodynamic Variational Principles

Fowler, Newcomb, Oberman & Kruskal, Rosenbluth, Gardner

Taylor

approach: Energy is minimized subject to some constraint like constant entropy or helicity. It is then noted that the Euler-Lagrange eq. thus obtained corresponds to the equation for equilibria.

comments: This approach is ad hoc. No connection between the dynamics and energy minimization is made. Why does this approach yield the correct equilibria?

The noncanonical Hamiltonian formalism fills in this gap. To see this note that

$$\frac{\partial \psi^i}{\partial t} = \{\psi^i, H\} = \{\psi^i, I\} = O^{ij}\frac{\delta(H+C)}{\delta\psi^j}$$

$$I = H + C \,, \quad \{\psi^i, C\} = 0$$

Therefore

$$\frac{\delta(H+C)}{\delta\psi^j} = 0 \implies \frac{\partial\psi^i}{\partial t} = 0 \qquad !$$

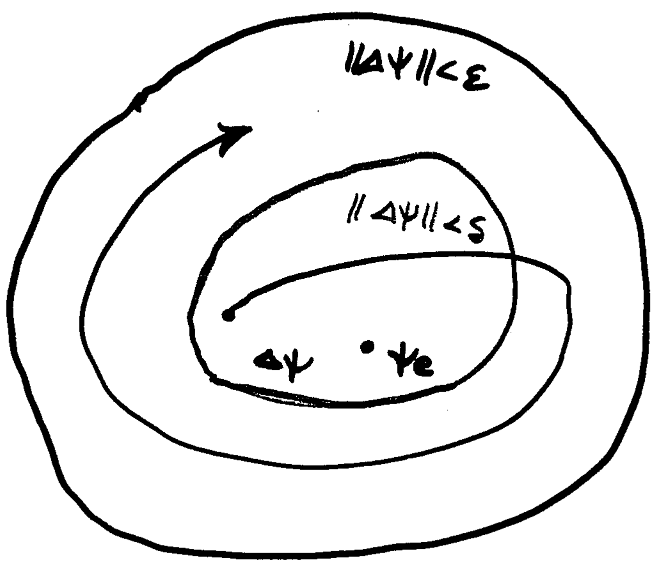
STABILITY

The variational principle

$$\frac{\delta}{\delta\psi^i}(H+C)=0 \Rightarrow \text{Equil.}$$

is useful for proving stability. In particular nonlinear stability:

Definition. An equilibrium is Liapunov stable if for all $\varepsilon>0$ there is a $\delta>0$ such that for a $\psi=\psi_e+\Delta\psi$ with $\|\Delta\psi\|<\delta$ at $t=0$, then $\|\Delta\psi\|<\varepsilon$ for all time.

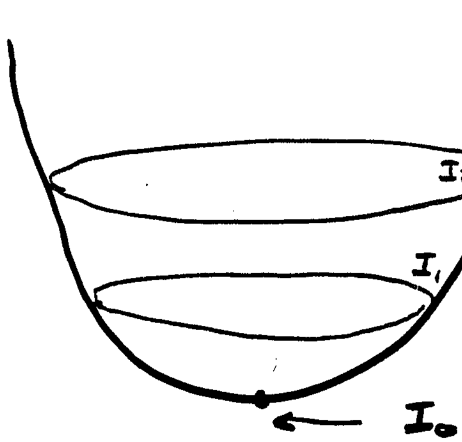


$\Delta\psi$ finite

$\|\ \|$ comes from $\delta^2 I = D^2 I[\psi_e]\cdot\delta\psi^2$
Quadratic form in $\delta\psi$

In practice easy step from definite

$\delta^2 I \longrightarrow$ " ".

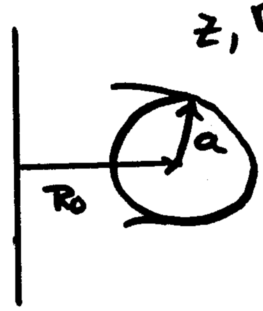


V. Examples

A. RMHD

B. CRMHD

Reduced MHD - Low β - single helicity



z, r, θ polar coords

$$\hat{z} \cdot \nabla f \times \nabla g = [f, g]$$

$\frac{a}{R_0} \ll 1$

Scalar vorticity $\quad \dot{U} = [U, \phi] + [\psi, J]$

Ohm's Law $\quad \dot{\psi} = [\psi, \phi]$

$J = \nabla_\perp^2 \psi$ pol. flux

$U = \nabla_\perp^2 \phi$ stream fn.

Bracket:

$$\{F, G\} = \int \left\{ U \left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta U} \right] + \psi \left(\left[\frac{\delta F}{\delta U}, \frac{\delta G}{\delta \psi} \right] + \left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta U} \right] \right) \right\}$$

inner algebra is semi-direct product extension

Casimirs:

$$C_1 = \int F(\psi)\, dz \qquad C_2 = \int U\, G(\psi)\, dz$$

$\vec{A} \cdot \vec{B}$ $\qquad$ $\vec{v} \cdot \vec{B}$

Variational Principle :

$$I = H + C_1 + C_2$$

$$= \int \frac{|\nabla\phi|^2}{2} + \frac{|\nabla\psi|^2}{2} + F(\psi) + U\,G(\psi)$$

$$\frac{\delta I}{\delta \psi} = -\nabla^2\psi + F'(\psi) + U\,G'(\psi) = 0$$

$$\frac{\delta I}{\delta U} = -\phi + G(\psi) = 0$$

Equilibria with flow

$$\left[\, G'^2(\psi) - 1 \,\right] \nabla^2\psi + G'G''(\psi)\,|\nabla\psi|^2 + F'(\psi) = 0$$

$$\phi = G(\psi)$$

Special Cases

(1) $\phi = \psi$ Alfven Waves

(2) $\nabla^2\psi = F'(\psi)$ Grad Shafranov Eq.

Stability :

$$\delta^2 I = \int \Big\{ |\nabla\delta\phi - \nabla(G'\delta\psi)|^2 + |\nabla\delta\psi|^2(1-G'^2)$$
$$+(\delta\psi)^2 \underbrace{\left[|\nabla\psi|^2 (G'G'')' + \nabla^2\psi \, 2G'G'' + F'' \right]}_{\mathcal{L}} \Big\}$$

(i) $G'^2(\psi) < 1$

(ii) $\mathcal{L} > 0$

Alfven Waves : $(F=0, \; G(\psi)=\psi)$

$$\delta^2 I = \int |\nabla\delta\phi - \nabla\delta\psi|^2 \, dz = \| \delta\vec{\Psi} \|$$

ϕ remains near ψ but both may grow.

Kink Mode : $(G=0)$

$F''(\psi) > 0$ monotonic current profile

(strong requirement that F'' not have pole - no resonance ; case with resonance has subtlely)

Compressible Reduced MHD (2-D)

Scalar Vorticity $\quad \dot{U} = [U, \phi] + [\psi, J] + 2[p, h]$

Ohm's Law $\quad \dot{\psi} = [\psi, \phi]$

//-motion $\quad \dot{v} = [v, \phi] + [\psi, p]$

Pressure $\quad \dot{p} = [p, \phi] + \beta[\psi, v] + 2\beta[h, \phi]$

Casimirs:

$$C_1 = \int F(\psi)\, dz$$

$$C_2 = \int v\, N(\psi)\, dz$$

$$C_3 = \int L(\psi)\,(p/\beta + 2h)\, dz$$

$$C_4 = \int \left(G(\psi)\, U - v\, G'(\psi)\,(p/\beta + 2h) \right) dz$$

What about the other constants?

Suppose $\hat{I} = H + C + P$

momentum

$$\frac{\delta \hat{I}}{\delta \psi^i} = 0 \not\Longleftrightarrow \dot{\psi}^i = 0$$

$$\dot{\psi} \overset{?}{=} \{\psi, \hat{I}\} = \{\psi, H + P\}$$

since $\{C, H\} = 0$

but

$$\{\psi, P\} \neq 0$$

thus the connection

$$\dot{\psi} = 0 \Longleftrightarrow \frac{\delta \hat{I}}{\delta \psi^i} = 0$$

don't get equil

Can make connection to Equil

$\psi(x - u_0 t)$.