HAMILTONIAN STRUCTURE
AND
STABILITY
IN
PLASMA PHYSICS

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I. Overview

Structure $\begin{cases} \text{II. Finite Systems} \\ \text{III. Field Theory} \end{cases}$

Stability $\begin{cases} \text{IV. Stability} \\ \text{V. Perturbed Energy } (\delta^2 F) \\ \text{VI. Examples} \end{cases}$

<u>Simple Harmonic Oscillator</u>

Energy: $H = \frac{1}{2}(p^2 + q^2)$ <u>quadratic</u>

Dynamics: $\ddot{q} = -kq$ <u>linear</u>

<u>Eulerian Variable Field</u>

Energy: $H = \int \frac{1}{2}\rho v^2 \, dz$ <u>quadratic</u>

Dynamics: $\vec{v}_t \sim \vec{v} \cdot \nabla \vec{v}$ ~~linear~~ <u>quadratic</u>

General feature of media described by Eulerian variables: Noncanonically Hamilitonian

invicid fluids, ideal magnetohydrodynamics, ideal two-fluid eqs, Maxwell-Vlasov, Liouville eq., . . .

Noncanonical or Generalized Hamiltonian Mechanics:

Definition. A system of ordinary differential equations is Hamiltonian in the generalized sense if it can be cast into the form

$$\dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} = [z^i, H] \qquad i, j = 1, 2, \ldots M$$

need not be even

where

$$[f, g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

has bracket properties.

Generalized Phase Space:

Since definition allows $\underline{\det(J^{ij}) = 0}$ the structure of phase space is changed.

Corank of (J^{ij}) = dimension of null space

Null space spanned by gradients: $\underline{\frac{\partial C}{\partial z^i} J^{ij} = 0}$

Lie's distinguished functions

The quantities C are <u>Casimirs</u> – phase space <u>constants</u>; built into phase space

$$[C, g] = \frac{\partial C}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j} = 0 \qquad \text{for all } g$$

Bracket Properties:

(i) bilinear $[g+h, f] = [g,f] + [h,f]$

(ii) $-[f,g] = [g,f]$

(iii) Jacobi $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = 0$

(iv) $[fg, h] = f[g,h] + [f,h]g$

Lie Algebra Poisson structure

Transformations:

$z^i \longrightarrow z'^i$ coordinate change

$J_c^{ij} \longrightarrow J^{ij}(z')$ contravariant tensor

$J_c^{ij} \longrightarrow J_c^{ij}$ canonical transformation

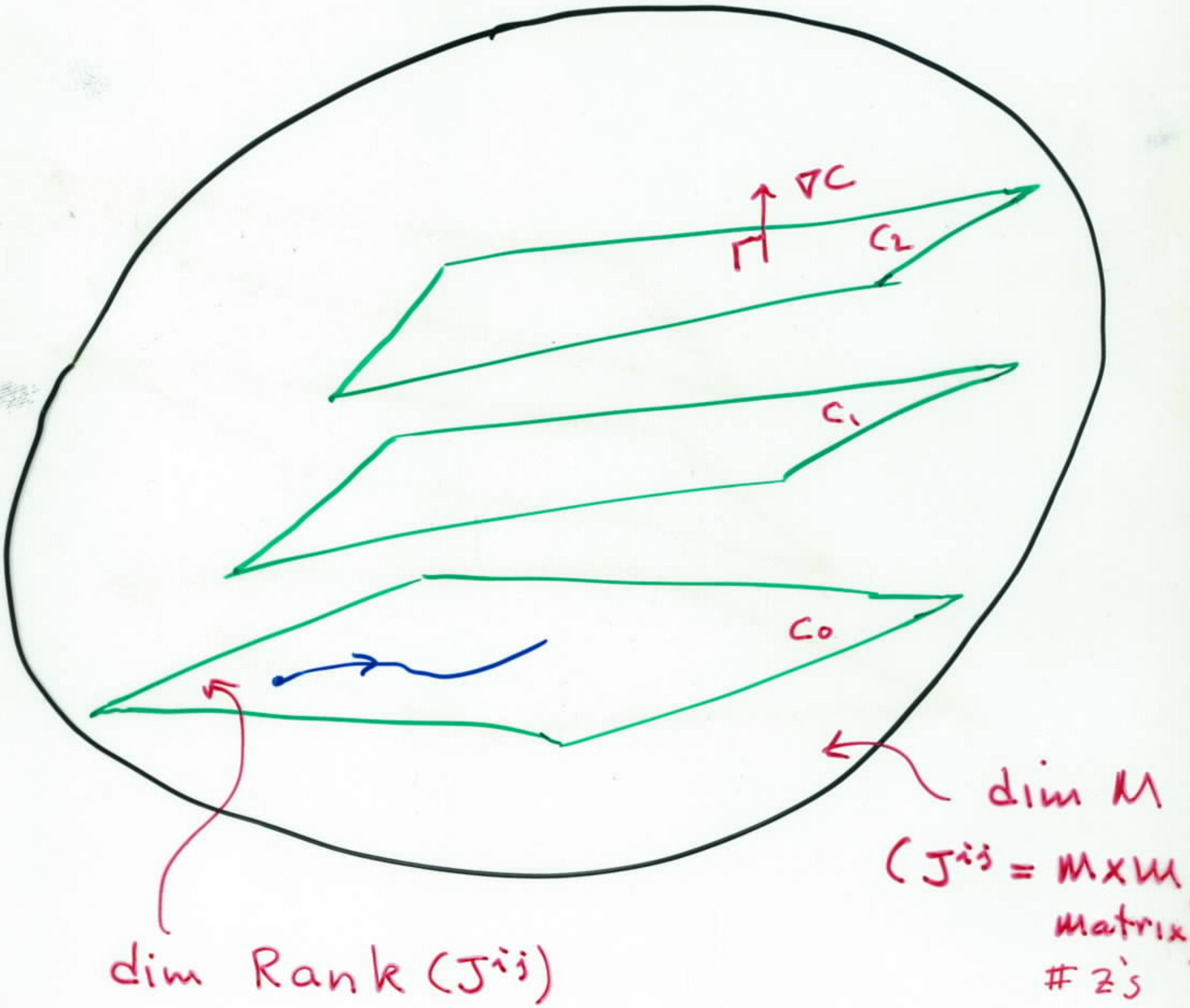
bracket properties are invariant

Converse outlook:

bracket properties $\Rightarrow$ $z'^i \longrightarrow z^i$
$J^{ij} \longrightarrow J_c^{ij}$

Darboux (1882) (local , $\det J^{ij} \neq 0$)

Phase Space (Poisson Manifold) :



For any hamiltonian the trajectory is confined to symplectic leaf.

Poisson manifolds foliate into symplectic leaves

III. Field Theory

Canonical bracket :

$$\{F, G\} = \sum_{k=1}^{L} \int \left(\frac{\delta F}{\delta \eta_k} \frac{\delta G}{\delta \pi_k} - \frac{\delta G}{\delta \eta_k} \frac{\delta F}{\delta \pi_k} \right) d\underline{x}$$

bracket acts on functionals of the field variables, η_k, π_k ; e.g.

$$H = \int \mathcal{H} \, d\underline{x}$$

Hamiltonian density $\left(\frac{1}{2}\rho v^2\right)$

phase space derivatives become functional derivatives

$$\frac{\partial}{\partial q_k} \longrightarrow \frac{\delta}{\delta \eta_k}$$

defined by

$$\delta F = \frac{d}{d\varepsilon} F[\eta + \varepsilon \delta\eta] \Big|_{\varepsilon=0} = DF \cdot \delta\eta = \left\langle \frac{\delta F}{\delta \eta}, \delta\eta \right\rangle$$

$$= \int \frac{\delta F}{\delta \eta} \delta\eta \, d\underline{x}$$

Noncanonical Field Brackets

$$\{F, G\} = \int \frac{\delta F}{\delta \psi^\lambda} \, \mathcal{O}^{\lambda\mu} \, \frac{\delta G}{\delta \psi^\mu} \, dz \qquad \lambda, \mu = 1, \dots M$$

Cosymplectic operator

- Antisymmetry $\Leftrightarrow$ $\mathcal{O}^{\lambda\mu}$ Anti-Selfadjoint
- Jacobi $\rightarrow$ Stiff requirement

Eqs. of Motion:

$$\frac{\partial \psi^\lambda}{\partial t} = \{\psi^\lambda, H\} = \mathcal{O}^{\lambda\mu} \frac{\delta H}{\delta \psi^\mu}$$

Canonical Case:

$$(\mathcal{O}^{\lambda\mu}) = \begin{bmatrix} 0 & I_M \\ -I_M & 0 \end{bmatrix}$$

Examples

KdV Equation

$$u_t = u u_x + u_{xxx} \qquad H[u] = \int \left(\frac{u^3}{6} - \frac{u_x^2}{2} \right) dx$$

$$\{F, G\} = \int \frac{\delta F}{\delta u} \frac{d}{dx} \frac{\delta G}{\delta u} \, dx$$

Gardner (1971)

Ideal MHD

PJM & J. Greene (1980)

$$\{F, G\} =$$

$$-\int d\tau \left\{ \rho \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \rho} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \rho} \right] + \right.$$

$$\vec{M} \cdot \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \vec{M}} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \vec{M}} \right] + \sigma \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \sigma} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \sigma} \right]$$

$$+$$

$$\left. \vec{B} \cdot \left[\frac{\delta F}{\delta \vec{M}} \cdot \nabla \frac{\delta G}{\delta \vec{B}} - \frac{\delta G}{\delta \vec{M}} \cdot \nabla \frac{\delta F}{\delta \vec{B}} + \left(\nabla \frac{\delta F}{\delta \vec{B}} \right) \cdot \frac{\delta G}{\delta \vec{M}} - \left(\nabla \frac{\delta G}{\delta \vec{B}} \right) \cdot \frac{\delta F}{\delta \vec{M}} \right] \right\}$$

Canonical Fields:

Klein - Gordon etc.

$$(\mathcal{O}^{ij}) = \begin{bmatrix} 0 & I_M \\ -I_M & 0 \end{bmatrix}$$

Continuous Media Fields:

Ideal MHD, Vlasov, etc.

$$(\mathcal{O}^{ij}) = \left(\psi^k c_k^{ij} \right)$$

linear in the field variables

c_k^{ij} are structure operators for some Lie algebra on functions

Lie-Poisson Brackets:

$$\{F, G\} = \int \psi^k \left[\frac{\delta F}{\delta \vec{\psi}}, \frac{\delta G}{\delta \vec{\psi}} \right]_k dz$$

outer algebra on functionals

inner algebra on functions

Explains missing nonlinearity

Lie - Poisson Brackets

Lie (1890)

Kirillov (1962)
Kostant (1965)
Souriau (1966)

$G \equiv$ Lie group

$\mathfrak{g} \equiv$ Lie algebra ($\mathfrak{g} = T_e G$)

$[\,,\,] \equiv$ bracket of $\mathfrak{g}$

$\mathfrak{g}^* \equiv$ dual space of linear functionals on $\mathfrak{g}$

$\langle\,,\,\rangle \equiv$ pairing between $\mathfrak{g}$ & $\mathfrak{g}^*$

$\mathfrak{g}^* =$ space of dynamical variables
$\psi \in \mathfrak{g}^*$

$\frac{\delta F}{\delta \psi} \in \mathfrak{g} \quad : \quad F[\psi] \in \mathbb{R}$

$$\boxed{\{F, G\} = \langle \underbrace{\psi}_{\in \mathfrak{g}^*}, \underbrace{\left[\frac{\delta F}{\delta \psi}, \frac{\delta G}{\delta \psi}\right]}_{\in \mathfrak{g}} \rangle}$$

BBGKY HIERARCHY

Liouville $\longrightarrow$ BBGKY

Inner algebra is a filtration on the algebra associated with the gp. of canonical transformations for an n-ptle system.

J. Marsden, PJM, & A. Weinstein (1984)

Four-Field Tokamak Model

Goal: obtain numerically compotable model with "dominante" physics. Restrict to four fields: p, ψ, U, v.

Model has Energy conservation & Casimir conservation: $\underline{A}\cdot\underline{B}$ & $\underline{v}\cdot\underline{B}$.

R. Hazeltine, C. Hsu, & PJM (1986)

CLASSIFICATION

EQUATIONS	HAMILTONIANS	BRACKET	CASIMIRS
KdV MKdV	$\int\left(\frac{u^3}{6} - \frac{1}{2}u_x^2\right)dx$	Gardner	$\int u\,dx$
Liouville Eq. Vlasov-Poisson 2-D Euler Guiding Center	$\int hf\,dz$ $\int h_1 f + \int h_2 ff$ $\int \omega\phi$ $\int \rho\phi$	Canonical Transformations of $\mathbb{R}^{2n}$	$\int F(\psi)\,dz$
RMHD Tokamak Models	$\int \lvert\nabla\phi\rvert^2 + \lvert\nabla\psi\rvert^2$	Above extended by sem-direct pdt.	$\int F(\psi)$ $\int U F(\psi)$
MHD CGL Theory	$\int \frac{1}{2}\rho v^2 + \rho U(\sigma,\rho) + \frac{B^2}{2}$ $U(\sigma,\rho,B)$	Diffeomorphisms of $\mathbb{R}^3 \times$ fns.	$\int A\cdot B$, $\int v\cdot B$ & others

Just as many fields are naturally canonical, there are many equations that have the same generalized Poisson bracket. They have different Hamiltonians.

<u>Casimirs</u> are bracket constants. They are independent of the Hamiltonian. If C is a casimir then $\{C, F\} = 0$ for all F.

Casimirs are useful for obtaining variational principles for equilibria. They are an ingredient in the algorithm for constructing Liapunov functionals.

Hamiltonian Systems Have "Built In" Liapunov Functions (candidates).

Standard Hamiltonian:

$$H = \frac{1}{2} \sum_i p_i^2 + V(q)$$

δW

Lagrange Condition: $\dfrac{\partial^2 V}{\partial q_i \partial q_j} > 0$

Arbitrary Hamiltonian:

Dirchelet Condition: $\dfrac{\partial^2 H}{\partial q_i \partial q_j} > 0 \quad (<0)$

Noncanonical Hamiltonian:

$$F = H + C$$

"free energy"

$$\frac{\partial^2 F}{\partial z^i \partial z^j} > 0 \quad (<0)$$

I.B Liapunov's Theorem

dynamical system:

$$\dot{z}^i = F^i(z) \qquad i = 1, 2, \cdots N$$

equilibrium point:

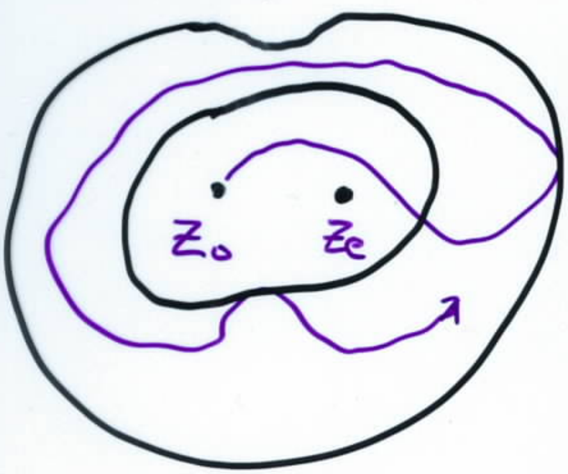
$$F^i(z_e) = 0 \quad \forall i$$

Liapunov's theorem: If there is a function $L(z) \in \mathbb{R}$ such that

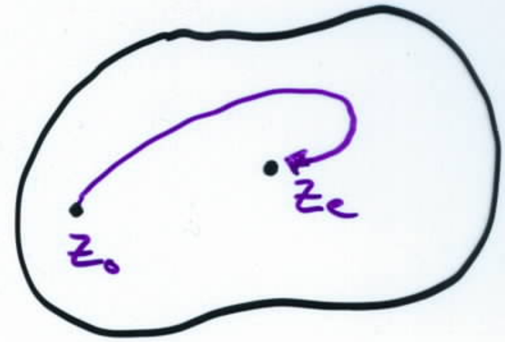
(i) $L(z_e) = 0$ and $L(z) > 0$ if $z \neq z_{eq}$

(ii) $\dot{L}(z) \leq 0$ and $\dot{L}(z) = 0$ iff $z = z_{eq}$

Stability



Asymptotic Stability



V. PERTURBED ENERGY - What is neg. energy wave?

Linear Theory :

$$\dot{z}^i = J^{ij}(z)\,\frac{\partial F}{\partial z^j}$$

$$F = H + C$$

$$z = z_e + \delta z$$

Equil:

$$\frac{\partial F(z_e)}{\partial z^i} = 0$$

$$\delta \dot{z}^i = J^{ij}(z_e)\,\frac{\partial^2 F(z_e)}{\partial z^j \partial z^k}\,\delta z^k$$

$$= \left\{ \delta z^i , \frac{\delta^2 F}{2} \right\}_L$$

perturbed Hamiltonian

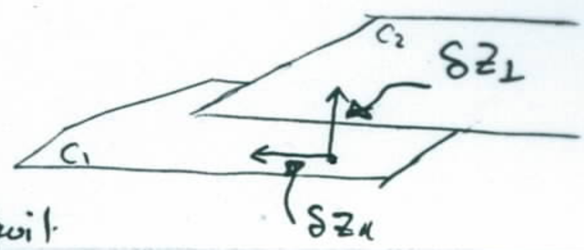
Not! $\delta^2 H$.

$\frac{\delta^2 F}{2}$ = linearized Hamiltonian

Should it be the energy?

Why should the linearized energy depend on C?

Add Source term } $\delta z_{\parallel}$ relevant



Analogy} $dW = dU + TdS$

work done on ↑ input E. ↑ heat change in Equil.

Arbitrary source $\Rightarrow$ $\delta z_\perp$

$H \to H + H_{ext}(t)$ $\Rightarrow$ $\delta z_\parallel$ only

$J^{ij} \dfrac{\partial H_{ext}}{\partial z^j}$ in leaf

Assume:

$$H_{ext} = z^j S_j(t)$$

linear in z

Examples (many)

(i) $-q\, F_{ext}$ 1-D. Freedom System

(ii) $-f\, \Phi_{ext}$ Vlasov-Poisson

Energy Input

$$\Delta H_c = -\int_0^\tau \dot{z}^j S_j(t)\, dt$$

Can Show

$$\boxed{\Delta H_c = \frac{\delta^2 F}{2}}$$

Casimir Constrained Source:

$$H \to H + H_{ext}$$

guarantees motion on the leaf

$\Leftrightarrow$ Sommerfeld / Brillouin
V. Laue

Having identified the perturbed energy we can transform to action angle variables (sometimes) and obtain:

$$\frac{\delta^2 F}{2} = \sum_i \omega_i J_i$$

freq.

action

A <u>negative energy wave</u> (mode) occurs when $\omega_i < 0$. The sign of the ω_i's cannot be changed by transformation (Sylvester's theorem).

This <u>basic definition</u> agrees with usual case when comparison can be made, i.e. when $\exists\ \varepsilon(k,\omega)$.

GENERAL DEFINITION OF NEGATIVE ENERGY MODE

* REAL SPECTRUM
* INDEFINITE $\delta^2 F$

$\blacksquare$

agrees with V. Laue - Brillouin but more general. Need not have an $\varepsilon(k, \omega)$.

Fourier Transform & "Get on the leaf"

e.g.
$$\delta n_\alpha = \sum_k N_k^{s(\alpha)} \sin kx + N_k^{c(\alpha)} \cos kx$$
etc.

One set of canonical variables →

$$q_{k1}^{(\alpha)} = \frac{\sqrt{\pi}\, V_k^{c(\alpha)} m_\alpha}{v_{T\alpha}} \qquad P_{k_1}^{(\alpha)} = \frac{\sqrt{\pi}}{k} N_k^{s(\alpha)} v_{T\alpha}$$

$$q_{k2}^{(\alpha)} = \frac{\sqrt{\pi}}{k} N_k^{c(\alpha)} \qquad p_{k_2}^{(\alpha)} = \sqrt{\pi}\, V_k^{s(\alpha)} m_\alpha$$

Another set when $v_{Te} < v_D < v_{Ti} + v_{Te}$

$$h = \frac{1}{2}\delta^2 F = \sum_{\substack{k \\ i=1,4}} \underbrace{\omega_i}_{\omega(k)} \frac{(P_j^2 + Q_j^2)}{2}$$

$\underline{\omega_1 > 0}$, $\underline{\omega_2 > 0}$, $\underline{\omega_3 > 0}$

$\underline{\omega_4 < 0}$ ← Negative energy wave

Von Laue 1905

Signature agrees w/ $\boxed{\mathrm{sgn}\left(\omega \frac{\partial \epsilon}{\partial \omega}\right)}$

can show generally

Hamiltonian Version of III. Resonant Three Wave Coupling Via Averaging for Two-Stream

Bracket Perturbation theory yields

$$H = H_0 + H_1$$

$$H_0 = \sum_{k=1}^{\infty} \left(\omega_1^k J_1^k + \omega_2^k J_2^k + \omega_3^k J_3^k + \omega_4^k J_4^k \right)$$

$$q^2 + p^2$$

Many degrees of freedom in plasma make it possible for resonance, i.e. $\exists$ k's s.t.

$$\boxed{\omega_4^{k_4} + \omega_1^{k_1} + \omega_2^{k_2} = 0}$$

recall $\omega_4 < 0$

$$H_1 \sim J_1^{1/2} J_2^{1/2} J_4^{1/2} \cos(\theta_1 + \theta_2 + \theta_4)$$

$$k_4 = k_1 + k_2 \qquad + \langle\, \rangle \to 0$$

Explosive instability; e.g.

$$J_1^{1/2} \sim \frac{A}{t_0 - t}$$ like Cherry's example

Bracket Pertubation Theory (about equil)

Must expand bracket as well as H

$$\dot{z}^i = J^{ij} \frac{\partial F}{\partial z^j} \quad ; \quad z = z_e + \delta z$$

expand to second order

$$\delta \dot{z} = \underbrace{\left[J + \frac{\partial J}{\partial z} \delta z \right.}_{+ \cdots} \left. \vphantom{\frac{\partial J}{\partial z}} \right] \left[\frac{\partial F}{\partial z} + \frac{\partial^2 F}{\partial z^2} \delta z + \frac{1}{2} \frac{\partial^3 F}{\partial z^3} \delta z^2 \right]$$

Media truncates

$$\delta \dot{z} = \{\delta z, h_3\}_L \quad ; \quad \{f, g\} = \frac{\partial f}{\partial \delta z} \left[J + \frac{\partial J}{\partial z} \delta z \right] \frac{\partial g}{\partial \delta z}$$

$$h_3 = \frac{1}{2} \frac{\partial^2 F}{\partial z^2} \delta z^2 + \underbrace{\frac{1}{6} \frac{\partial^3 F}{\partial z^3} \delta z^3}$$

<u>Not all</u> cubic nonlinearity

Still have nonlinearity in bracket

Clean up to $\mathcal{O}(\delta z^2)$ by

$$\delta \bar{z} = A \, \delta z + \frac{1}{2} B \, (\delta z)^2$$

Puts <u>all</u> cubic nonlinearity in bracket