MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Tagungsbericht 30/1987

Mathematische Probleme in Strömungen und Plasmen

12.7. bis 18.7.1987

Die Tagung stand unter der Leitung von Herrn Kirchgässner (Stuttgart) und Herrn Marsden (Berkeley). Es nahmen teil 46 Mathematiker und Physiker, davon 15 deutsche. Die wichtigsten Themen waren:

1. Hamiltonsche Beschreibung von Strömungen, insbesondere Wirbeltransport, nichtlineare Stabilität.
2. Dynamische Systeme und Turbulenz. Hier besonders: homokline Verzweigung durch exponentiell kleine Durchdringung von stabiler und instabiler Mannigfaltigkeit, nichtlineare Wellen, Stabilität und resonante äußere Kräfte.
3. Transonische Strömungen, eine neue variationelle Formulierung und Numerik hierzu.
4. Bifurkation und Symmetriebrechung, besonders am Beispiel der Couette-Taylor Instabilität. Im Mittelpunkt standen die Hopf-Bifurkationen bei vorhandener O(3)-Symmetrie. Ferner wurden behandelt neue Methoden zur Bestimmung von Normalformen nichtlinearer Vektorfelder.

Folgende Personen hielten Hauptvorträge (in zeitlicher Reihenfolge):

Busse (Bayreuth), Golubitsky (Houston), Bemelmans (Saarbrücken), Amick (Chicago), Pulvirenti (l'Aquila), Marsden (Berkeley), Newell (Tucson), Batt (München), Tasso (Garching), Marchioro (Rom), Nečas (Prag), Feistauer (Prag).

Seminare über Spezialthemen wurden veranstaltet von:

Iooss (Nizza), Stewart (Warwick): Symmetriebrechung und Bifurkation

Turner (Madison), Wan (Buffalo): Nichtlineare Wellen

Heywood (Vancouver), v. Wahl (Bayreuth): Navier-Stokes

Here $\nabla(j)$ is the Levi-Cività connection of $m(j)$, $R(j)$ is the Ricci tensor of $m(j)$ represented as an operator by $m(j)$ and $\Delta(j)$ is the Laplacian given by $m(j)$. $X(t) \in \Gamma TR$ with $\operatorname{div}_{\mu(j)} X(t) = 0$. F is well defined.

F.H. BUSSE: **The Optimum Theory of Turbulence**

The main goals and methods of the optimum theory of turbulence are reviewed and a new application is outlined. Through the derivation of bounds on average properties of turbulent fluid flows rigorous results can be obtained in contrast to other theories of turbulence which must rely on additional assumptions. The extremalizing vector fields derived as solutions of the variational problems not only provide the upper bounds, but exhibit some interesting properties which can be compared with observed turbulent fluid flow. The newly treated case of an internally heated sphere is described as an application of the theory to a problem with a finite domain. The results of the Optimum Theory could be improved by the imposition of additional integralrelationships (moments) of the basic Navier-Stokes- Equations. Using additional constraints one could also introduce time-dependence into the formulation of the variational problems.

P. J. MORRISON: **Sufficient and "Necessary" Free Energy Conditions for Stability**

There exist a large number of sufficient conditions for the stability of ideal fluid and plasma systems, which depend upon the positive definiteness of some quadratic form. Usually it is believed that these conditions yield no information when indefinite. I argue the contrary. Equilibria away from thermodynamic equilibrium, such as those with mean flow in fluids or bumpy distribution functions in kinetic theory, possess free energy that may not be tapped by the ideal linearized equations. Expressions for the change in free energy upon perturbation from equilibrium can be obtained. If such an expression is definite, stability is proved by Liapunov's theorem. If the expression is indefinite then there are two possibilities: 1) spectral instability or 2) spectral stability with negative energy modes. We argue via examples that the later case is generically unstable to infinitesimal perturbations, due to nonlinear effects, or to the inclusion of appropriate dissipation mechanisms. In this sense the free energy condition is necessary and sufficient for stability. This very general idea we call the free energy principle.

Sufficient and "Necessary" Conditions for Fluid & Plasma Stability

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Free Energy Principle (conjecture)

Somebody's Law: If something can go wrong, it will!

$\exists$ in fluid & Plasma systems many sufficient conditions for stability

$$\langle \psi, O\psi \rangle > 0$$

$\Longrightarrow$ Stability

while if

$\langle \psi, O\psi \rangle$ indefinite

$\Longrightarrow$ Nothing

$\forall$ ideal models

$\langle \psi, O\psi \rangle > 0 \Longleftrightarrow$ Pos. def. Free Energy

Free Energy Principle

If the free energy, $\delta^2 F = \delta^2(H+C)$, is indefinite, then there are two conclusions:

(1) The system has linear (spectral) instability. Bad.

(2) The spectrum is stable, but $\exists$ a negative energy mode. Also Bad.

Case (2):

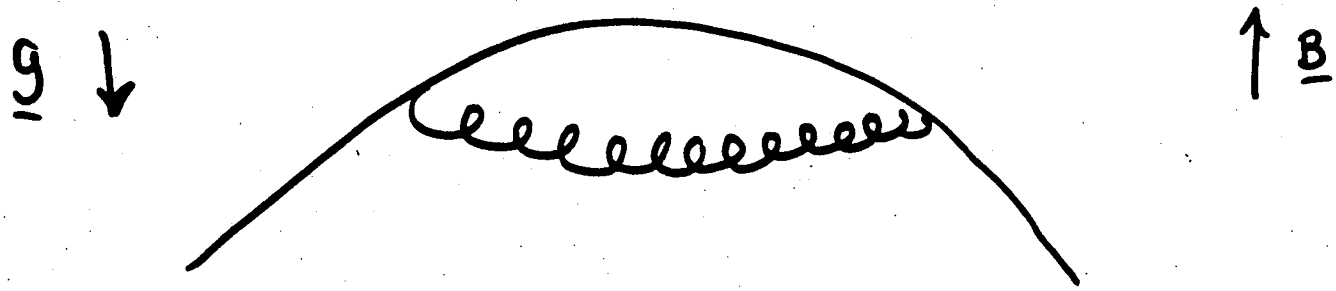
* Nonlinear instability (Not finite amplitude) e.g. explosive growth; slow growth followed by fast; resonance w/ continoous spectrum ...

* Damped Negative energy modes are unstable (structural stability)

[redacted]

Charged Particle on a Mountain

(models FLR stabilization)



Harmonic mountain + . . .

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{eB}{2c} (\dot{y} x - \dot{x} y)$$

$$+ \frac{1}{2} k (x^2 + y^2) + \mathcal{O}(3)$$

anharmonicity

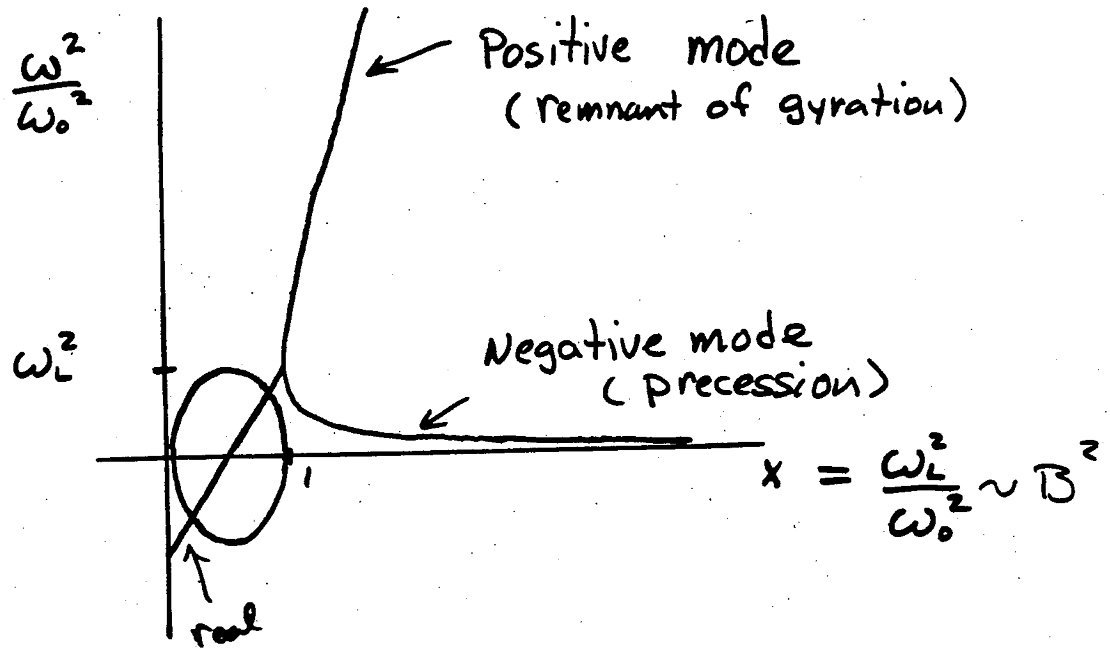
Two frequencies : $\omega_L = \frac{eB}{2mc}$, $\omega_0 = \sqrt{\frac{k}{m}}$

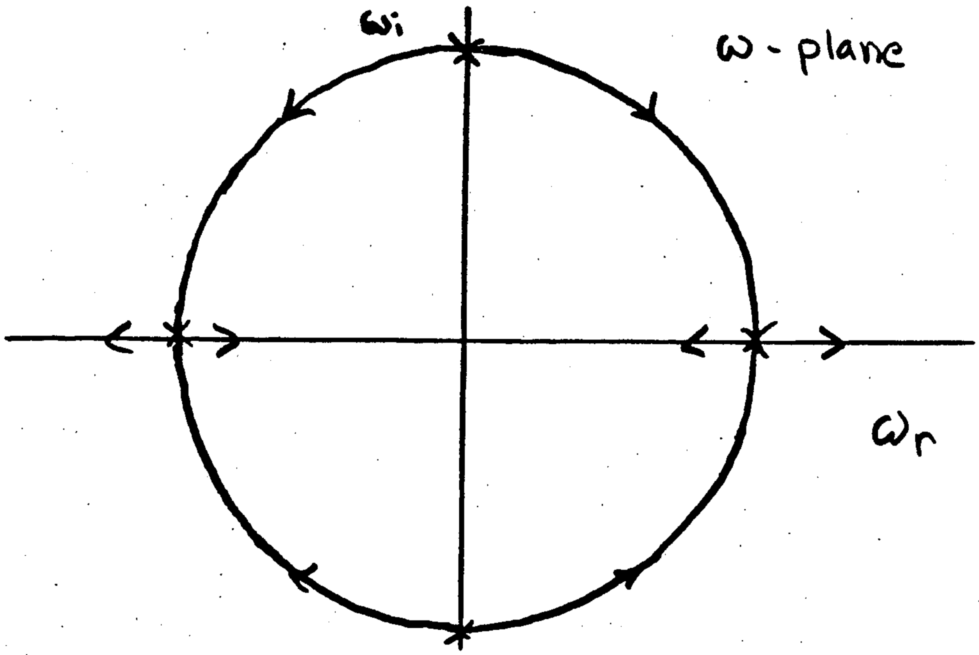
Hamiltonian:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \omega_L (y p_x - x p_y)$$

$$+ \frac{1}{2} m (\omega_L^2 - \omega_0^2)(x^2 + y^2) + \mathcal{O}(3)$$

Eigenfrequencies





Backwards Krein Crash ($\omega_r \Rightarrow$ stable)

STABLE SPECTRUM - UNSTABLE

$$\dot{z}_1 = z_2 - \alpha\,(z_2 z_3 + z_1 z_4)$$

$$\dot{z}_2 = -z_1 + \alpha\,(z_2 z_4 - z_1 z_3)$$

$$\dot{z}_3 = -2 z_4 - \alpha\,(z_1 z_2)$$

$$\dot{z}_4 = 2 z_3 + \frac{\alpha}{2}\,(z_2^2 - z_1^2)$$

equil. & spectral stability:

$$z_i = \underset{\uparrow \text{ equil.}}{0} + \delta z_i\, e^{i\omega t}$$

$\underline{\omega^2 = 1, 4}$ $\Rightarrow$ stable

exact solutions:

$$z_1 = \frac{\sqrt{2}}{\alpha(t-\varepsilon)} \sin(t+\gamma) \qquad z_2 = \frac{\sqrt{2}}{\alpha(t-\varepsilon)} \cos(t+\gamma)$$

$$z_3 = \frac{1}{\alpha(t-\varepsilon)} \sin(2t+\gamma) \qquad z_4 = \frac{-1}{\alpha(t-\varepsilon)} \cos(2t+\gamma)$$

Diverges! Explosive Instability.

Action-Angle Variables:

$$H = H_0 + H_1$$

integrable; nonintegrable perturbation $\mathcal{O}(3)$

$$H_0 = \omega_1 J_1 + \omega_2 J_2 + \bar{\alpha} J_1 J_2^{1/2} \cos(2\theta_1 + \theta_2)$$

resonance occurs when:

$$2|\omega_1| = |\omega_2| \qquad 9\omega_0^2 = 8\omega_L^2$$

$$H_0 = \frac{1}{2}(p_1^2 + q_1^2) - (q_2^2 + p_2^2)$$
$$+ \frac{\alpha}{2}\left[q_2(q_1^2 - p_1^2) - 2q_1 p_1 p_2\right]$$

(Cherry 1927)

Features:

(i) $\omega_1 = 1$, $\omega_2 = -2$

$\mathcal{O}(3)$ resonance: $2\omega_1 + \omega_2 = 0$

(ii) $$\delta^2 H = \left.\frac{\partial^2 H}{\partial z^i \partial z^j}\right|_{\text{equil}} \delta z^i \delta z^j$$

indefinite $\Rightarrow$ negative energy mode

When (i) & (ii) generically unstable

Signature of negative energy:

$$H = \sum_i \omega_i J_i + \cdots$$

J_i ← action

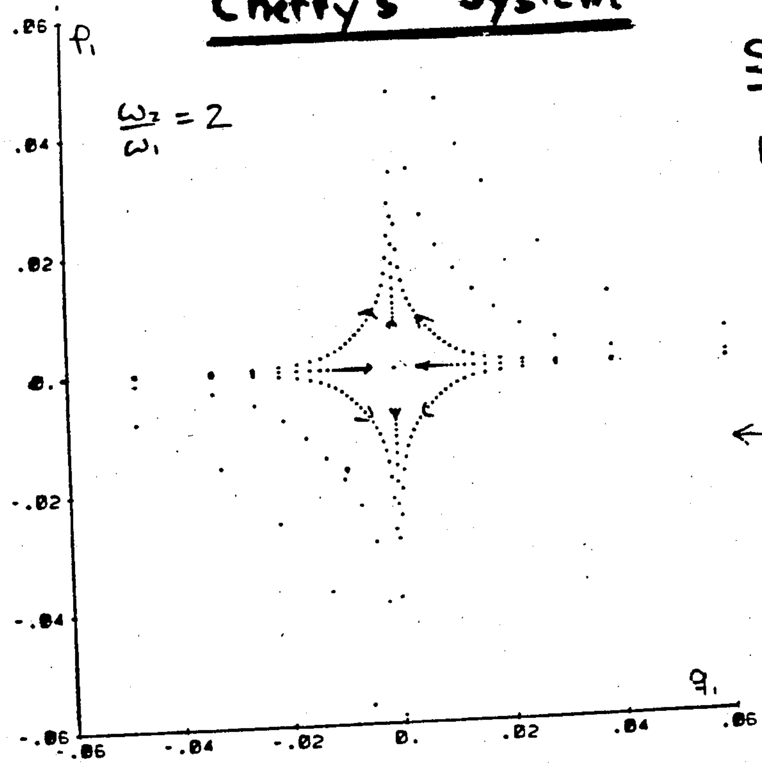
ω_i's not all same sign.

Cherry's System

Courtesy C Kuery

Surfaces of Section

$$H = \frac{\omega_1}{2}\left(p_1^2 + q_1^2\right) - \frac{\omega_2}{2}\left(p_2^2 + q_2^2\right) + \frac{\alpha}{2}\left[q_2\left(q_1^2 - p_1^2\right) - 2 q_1 p_1 p_2\right]$$

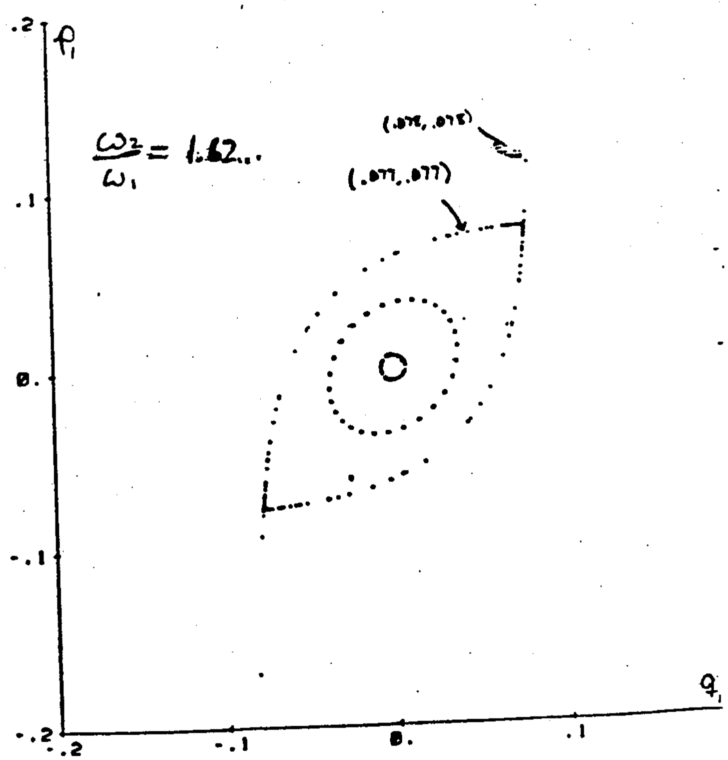


← Tuned O(3) resonance

↖ Surfaces of section

↙ $q_2 = 0$

Detuned resonance →



Orbits outside eyelet → ∞

Both are integrable

Generalization of Cherry

Most systems are not integrable and may be detuned. So we consider the following Hamiltonian:

$$H = \frac{1}{2}\omega_1 (p_1^2 + q_1^2) - \frac{1}{2}\omega_2 (p_2^2 + q_2^2)$$

$$+ \frac{\alpha}{2}\left[q_2(-p_1^2 + q_1^2) - (1+\varepsilon)\, q_1 p_1 p_2\right]$$

non integrable perturbation

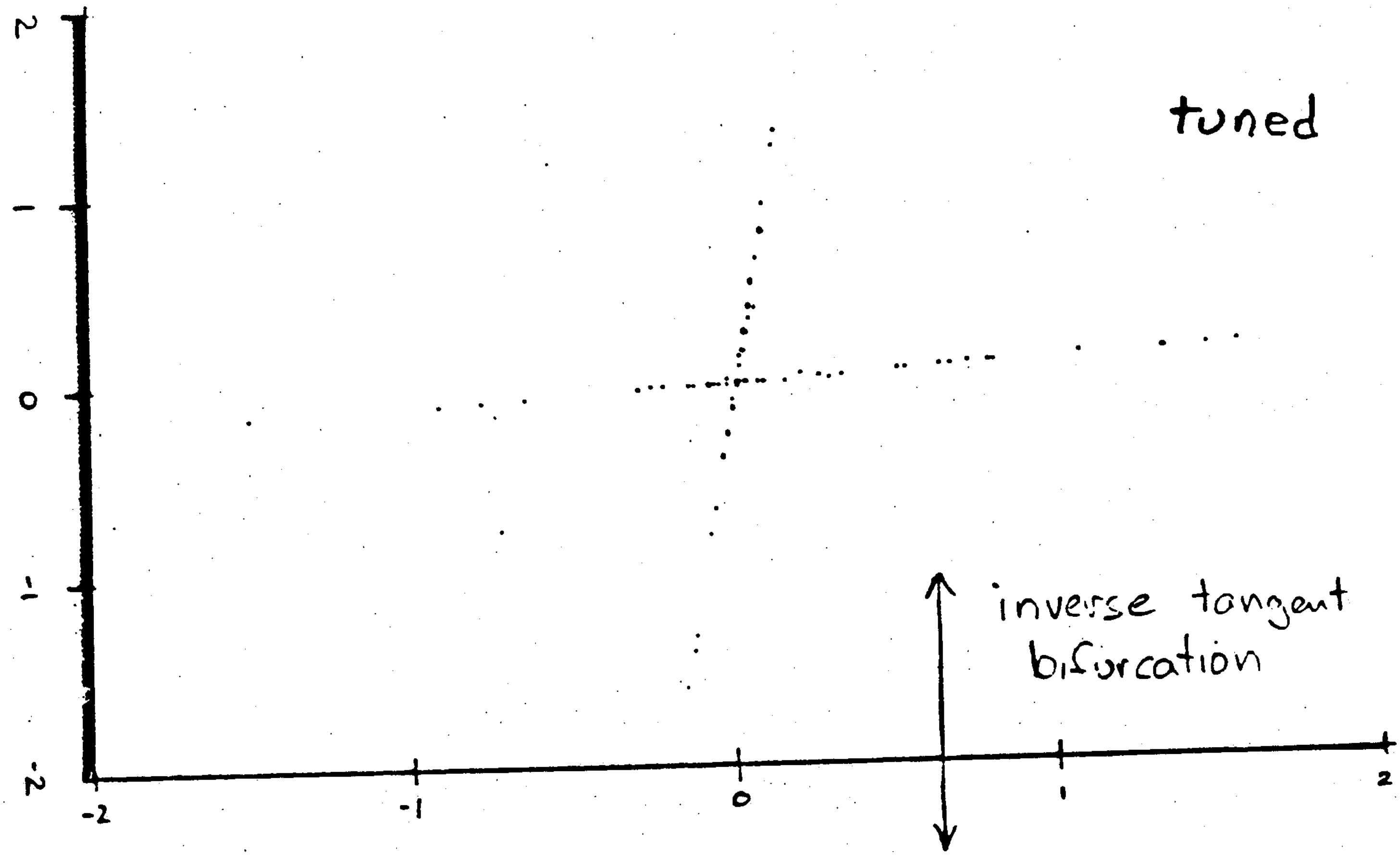
Since differential equations are more difficult to analyze and Arnold diffusion may take a long time we study a map that mimics features of the above system. This can be done with a cubic (not quadratic) map

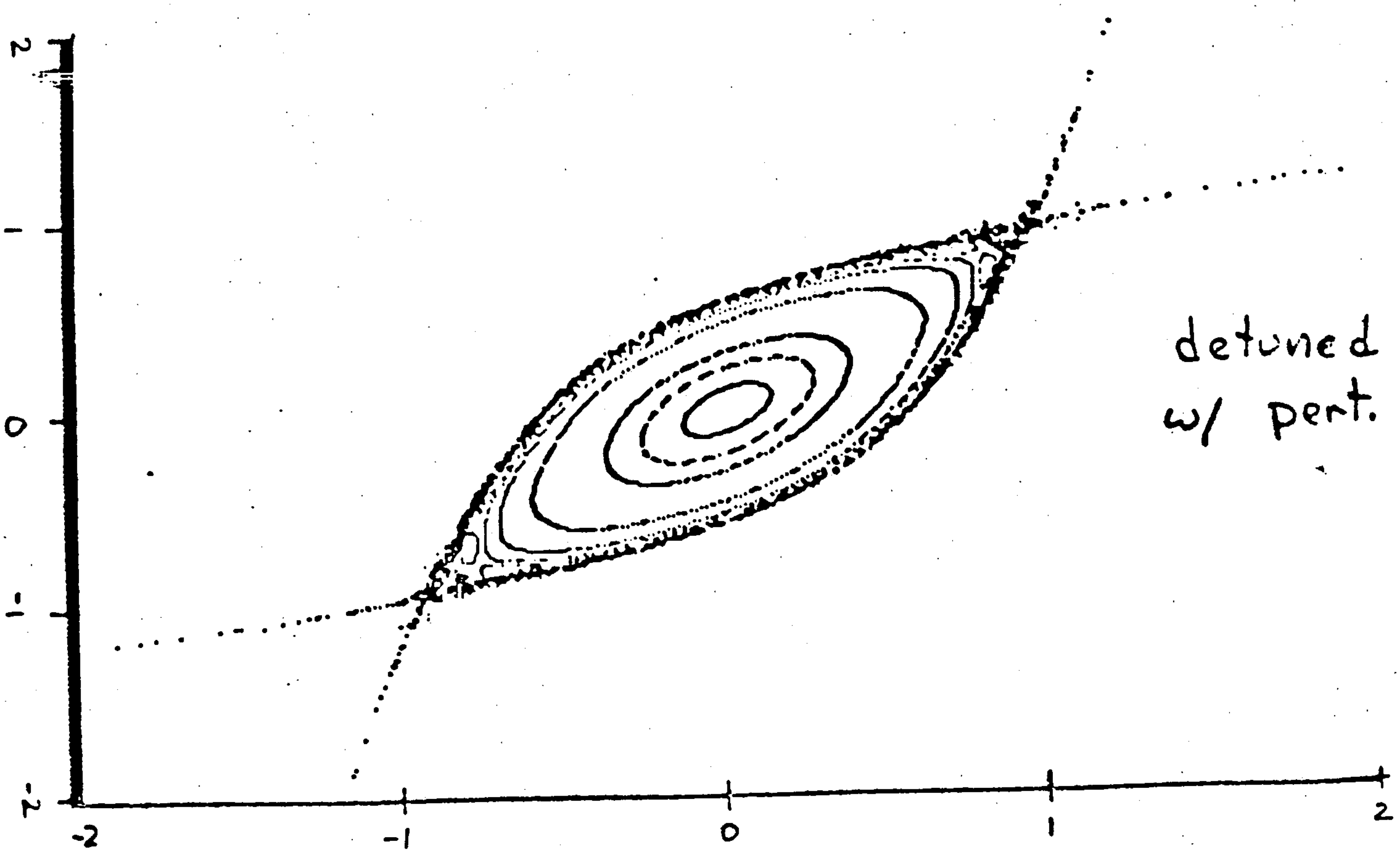
$$\boxed{\begin{aligned} x' &= -y \\ y' &= -t\,y + x - y^3 \end{aligned}}$$

Cubic Area Preserving Map

This area preserving map has an inverse tangent bifurcation at trace $t = -2$ (Res = 0).

Cubic Map (mimic)





Warm Fluid Two-Stream Instability

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha}\frac{dp_\alpha}{dx}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \quad , \quad \frac{dE}{dx} = 4\pi e(n_i - n_e)$$

$$H = \int dx \left\{ \frac{1}{2}\rho_i v_i^2 + \frac{1}{2}\rho_e v_e^2 + \rho_i U_i + \rho_e U_e + \frac{E^2(n_\alpha)}{8\pi} \right\}$$

$$\frac{\delta H}{\delta v_\alpha} = \rho_\alpha v_\alpha = 0$$

$$\frac{\delta H}{\delta \rho_\alpha} = \frac{v_\alpha^2}{2} + h_\alpha + \frac{e_\alpha}{m_\alpha}\phi = 0$$

$$\Longrightarrow \quad \rho_\alpha = \text{const.} \quad v_\alpha = 0$$

$\nabla H = 0 \Rightarrow$ Uninteresting equil.

Need constraints. Why? $\int v\,dx, \int n\,dx$

Understanding comes from Noncanonical Formalism. $\Big\{$ conservation of phase space volume

PERTURBED ENERGY & $\delta^2 F$

$$\dot{z}^i = J^{ij}(z)\frac{\partial F}{\partial z^j} = [z^i, \underset{\underset{F}{\shortparallel}}{H+C}]$$

Linearize : $z = z_e + \delta z$

Equilibrium: $\dfrac{\partial F(z_e)}{\partial z^j} = 0$

Generally $\dfrac{\partial H(z_e)}{\partial z^j} \neq 0$ or yields trivial equilibria.

$$\delta \dot{z}^i = J^{ij}(z_e)\, \frac{\partial^2 F(z_e)}{\partial z^j \partial z^k}\, \delta z^k$$

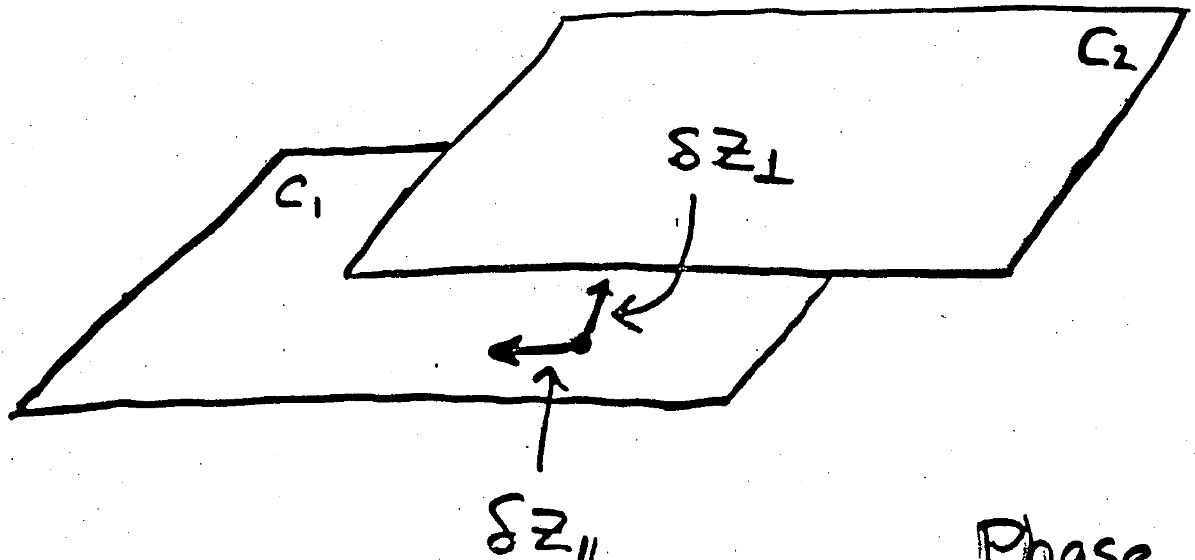
$$= \left\{ \delta z^i, \frac{\delta^2 F}{2} \right\}_L$$

$$\uparrow$$

<u>Perturbed</u> <u>Hamiltonian</u> $\dfrac{\delta^2 F}{2}$

Not $\dfrac{\delta^2 H}{2}$! What is the energy.

$\delta^2 F/2$ = Free Energy



Phase Space

We add a source term and pull the system away from equilibrium.

$\delta z_\parallel$ is the only relevant ~~part~~

Since $\delta z_\perp$ changes the equil.

Thermodynamic analogy: $dW = dU + TdS$

Let $H \to H + H_{ext}(t)$ input $\quad \delta z_\parallel \quad \delta z$

$H_{ext} = z^j S_j(t) \quad (= q\, F_{ext}(t))$

$$\boxed{\Delta H_c = -\int_0^t \dot{z}^j S_j(t)\, dt = \frac{\delta^2 F}{2}}$$

Two-Stream Instability (warm ions & electrons)

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha}\frac{\partial p_\alpha}{\partial x}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \qquad \frac{\partial E}{\partial x} = 4\pi e\,(n_i - n_e)$$

__equil.__ n_{0i}, n_{0e}, v_D ← drifting electrons

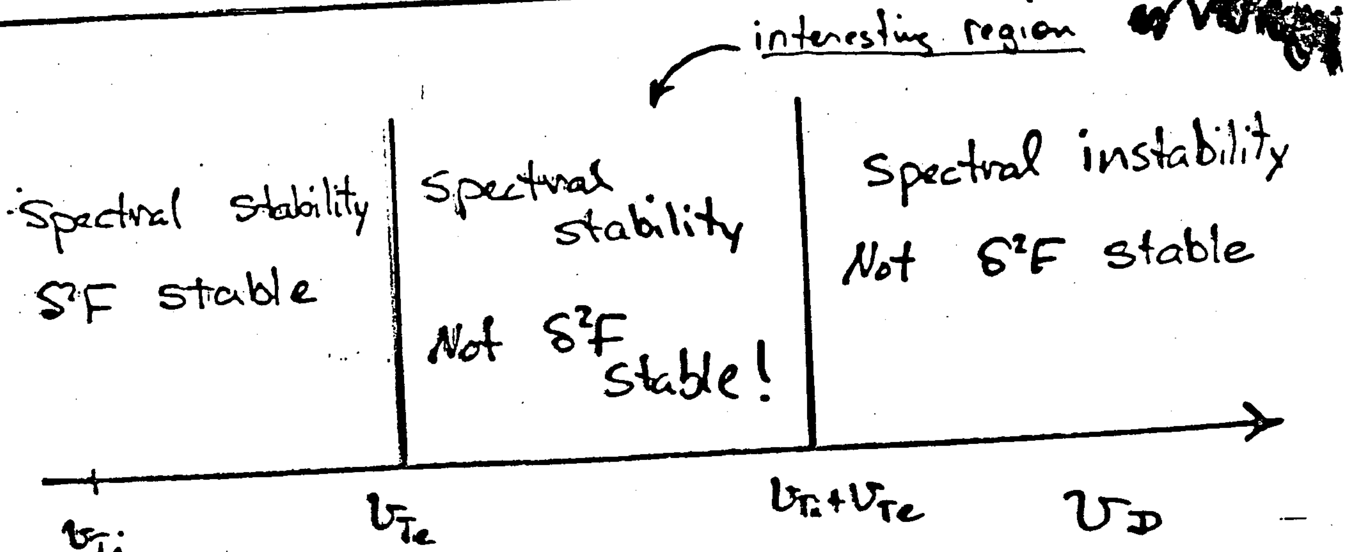
__spectral stability__ condition given via

$$\boxed{0 = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 v_{Ti}^2} - \frac{\omega_{pe}^2}{(\omega - k v_D)^2 - k^2 v_{Te}^2} = \varepsilon(k,\omega)}$$

__Threshold:__ $v_D > v_{Ti} + v_{Te}$ ⇒ __instability__

__$\delta^2 F$:__

__Threshold:__ $v_D < v_{Te}$ ⇒ __$\delta^2 F$ positive definite__



Noncanonical Variables $\longrightarrow$

Canonical Variables + Fourier Trans.

$\Longrightarrow$

$$H = \sum_{k}^{\infty} \omega_k J_k + \mathcal{O}(J^{3/2})$$

In the band $v_{Te} < v_D < v_{Ti} + v_{Te}$

$\exists$ ω_k's < 0.

Pick out "1" resonant triad +
resonant driving term $\Longrightarrow$

$$J \sim \frac{1}{t_0 - t}$$

Explosive Growth.

Detune resonance $\Rightarrow$?

4 Dimensional Symplectic Map (mimic)

(anharmonic mountain with earthquake)

Generating Function: $F = QQ' + qq' + \frac{\tau Q'^2}{2} - \frac{\tau q'^2}{2} + \frac{Q^3}{3} + \frac{q^4}{4}$

Coupling $: \rightarrow + a q Q$

coupled quadratic & cubic
area preserving maps.

$$\frac{\partial F}{\partial Q'} = P' \qquad \frac{\partial F}{\partial Q} = -P \qquad \frac{\partial F}{\partial q'} = -p' \qquad \frac{\partial F}{\partial q} = p$$

$$P' = Q \qquad\qquad p' = -q$$

$$Q' = \tau Q + Q^2 + P + \underline{aq} \qquad\qquad q' = p + \tau q + q^3 + \underline{aQ}$$

<u>Orbit A</u>

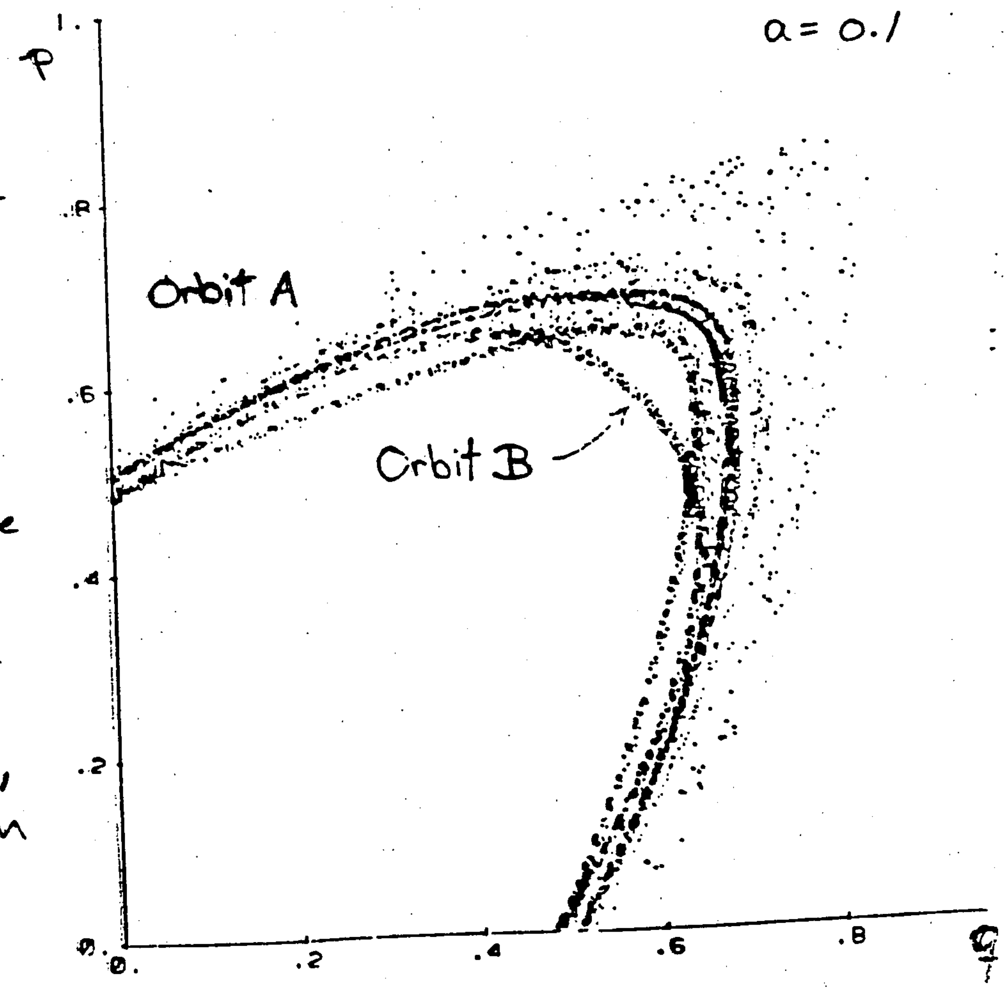
$(q, p, Q, P) = (.65, .65, 0, 0)$

No movement in <u>10 million</u> iterations. (5×10^3 plotted)

<u>Orbit B</u>

$(q, p, Q, P) = (.623\ , .623\ldots, 0, 0)$

<u>2 million</u> iterations. The first 5×10^3 map out separatrix lying <u>completely</u> inside A. Suddenly the orbit jumps outside A, jumps again and then $\longrightarrow \infty$. The last 5×10^3 are plotted.



Courtesy C Kurtz

Dissipation of Negative Energy Modes

Dissipation $\Rightarrow$ $\dfrac{dH}{dt} < 0$

2nd Law of Thermodynamics

$$H = \frac{q_+^2 + p_+^2}{2} - \frac{(q_-^2 + p_-^2)}{2}$$

$$\dot{q}_+ = p_+ \qquad\qquad \dot{q}_- = -p_-$$

$$\dot{p}_+ = -q_+ - \nu_+ p_+ \qquad\qquad \dot{p}_- = q_- - \nu_- p_-$$

$$\sim e^{i\omega t} \qquad\qquad \text{small } \nu_i > 0$$

$$\omega_+ \to \omega_+ + i\frac{\nu_+}{2} \qquad\qquad \omega_- \to \omega_- - i\frac{\nu_-}{2}$$

damping $\qquad\qquad$ growth

Conclusion

$\delta^2 F$ is sufficient &

"Necessary" for stability.

If there is free energy in a system & no reason it cannot be tapped, then it can happen & probably will.