THE FREE ENERGY "PRINCIPLE" AND NEGATIVE ENERGY WAVES

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OVERVIEW

(I) Statement of Principle

(II) Examples of NL instability

(III) Dissipation Destabilization

(V) Objections

THE LAWS OF NATURE

Parkinson's Law: Work expands to fill the time available for its completion.

Parkinson's Law, Modified: The junk you have will expand to fill the available space.

The Peter Principle: In every hierarchy, each employee tends to rise to his level of incompetence.

Murphy's Law: If something can go wrong, it will.

O'Brian's [illegible] Murphy was an optimist

Weiler's Law: Nothing is impossible for the man who does not have to do it himself.

Finagle's Law: Once a job is fouled up, anything done to improve it makes it worse.

→ **Rudin's Law:** A theory is better than its explanation.

Unnamed Law: If it happens, it must be possible.

Clarke's Third Law: Any sufficiently advanced technology is indistinguishable from magic.

Gumperson's Law: The outcome of a given event probability will be inverse to the degree of desirability.

Cutler-Webster Law: There are two sides to every argument unless a man is personally involved, in which case there is only one.

Cropp's Law: The amount of work done varies inversely with the amount of time spent in the office.

May's Law: The quality of correlation is inversely proportional to the level of control (the fewer the facts, the smoother the curves).

Albrecht's Law: Social innovations tend to the level of minimum tolerable well-being.

Sturgeon's Law: 90% of everything is trash.

Energy
Free ‸ Principle (conjecture)

For fluid and plasma systems, describing different geometries, physics etc., there exist many <u>sufficient</u> conditions for stability of the form

$$(\Psi, O\Psi) > 0$$

$\Rightarrow$ stability

Conversely if

$$(\Psi, O\Psi) \quad \text{indefinite}$$

$\Rightarrow$ Nothing !!

For all ideal models

$(\Psi, O\Psi) > 0 \Longleftrightarrow$ positive definite Free Energy

Free Energy Principle

If the free energy, $\delta^2 F = \delta^2 (H+C)$, is indefinite, then there are two conclusions:

(1) The system has linear (spectral) instability. Bad.

(2) The spectrum is stable, but $\exists$ a negative energy mode. Also Bad.

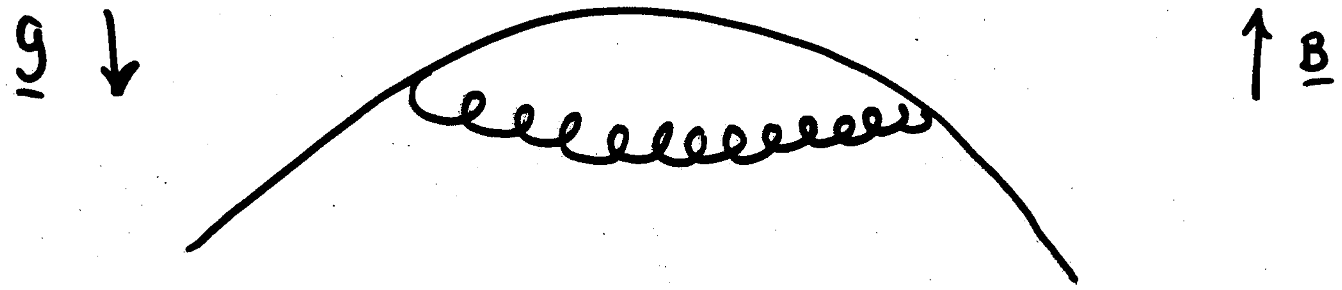
Case (2):

- * Nonlinear instability (Not finite amplitude) e.g. explosive growth; slow growth followed by fast; resonance w/ continous spectrum...
- * ~~Damped~~ Dissipated Negative energy modes are unstable (structural stability)

[redacted]

Charged Particle on a Mountain

(models FLR stabilization)



Harmonic mountain + . . .

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) + \frac{eB}{2c}(\dot{y}x - \dot{x}y)$$

$$+ \frac{1}{2} k (x^2 + y^2) + \underset{\substack{\Uparrow \\ \text{anharmonicity}}}{\mathcal{O}(3)}$$

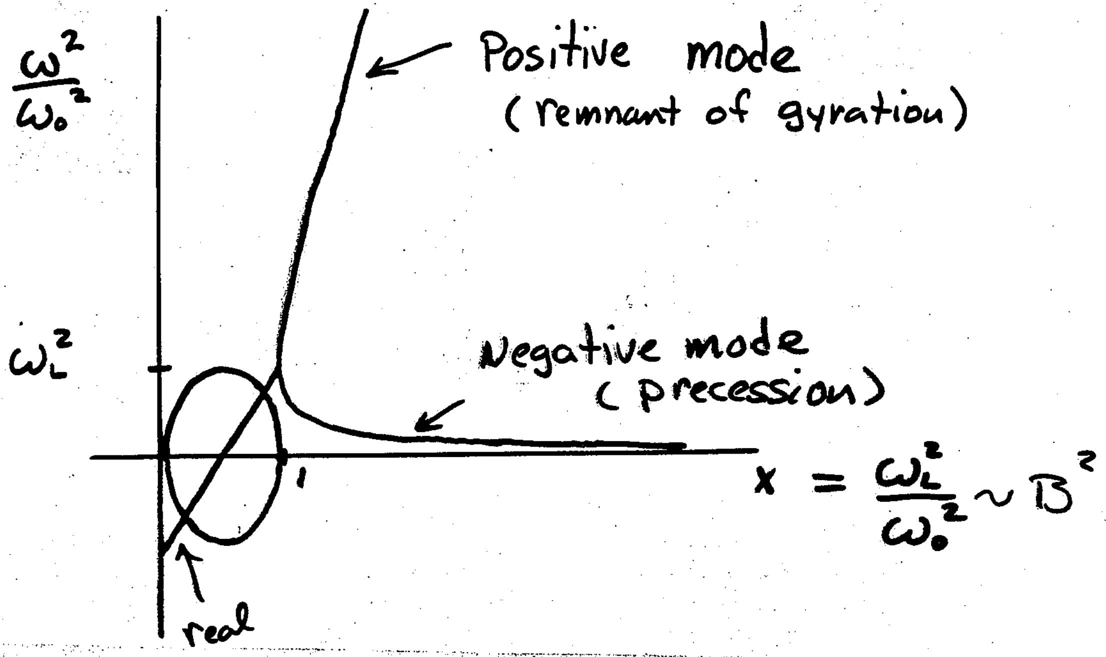
Two frequencies : $\omega_L = \dfrac{eB}{2mc}$, $\omega_0 = \sqrt{\dfrac{k}{m}}$

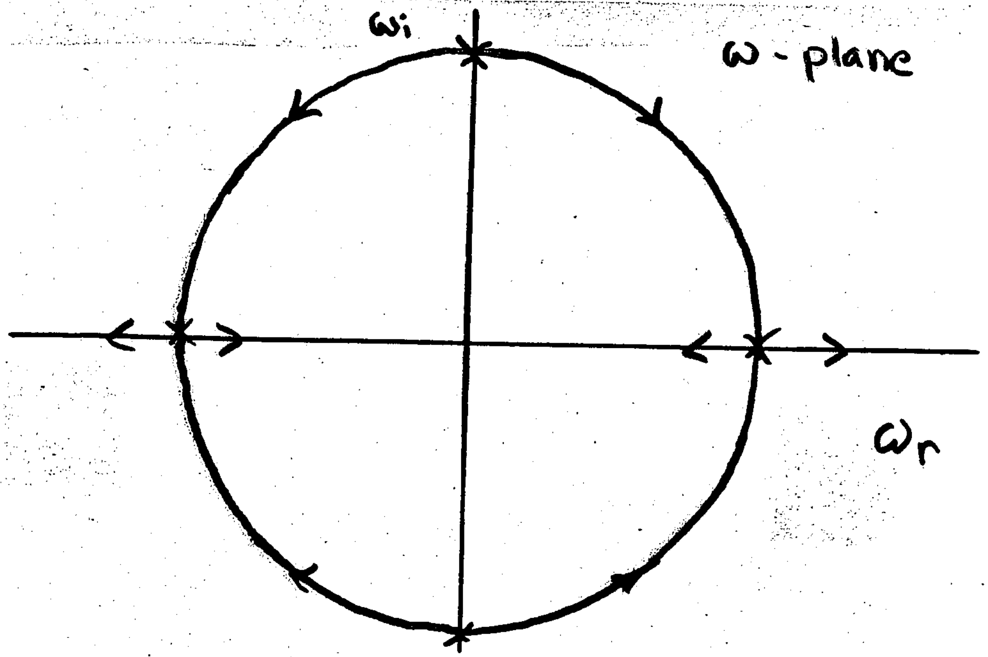
Hamiltonian:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \omega_L (y p_x - x p_y)$$

$$+ \frac{1}{2} m (\omega_L^2 - \omega_0^2)(x^2 + y^2) + \mathcal{O}(3)$$

Eigenfrequencies





Backwards Krein Crash (ωr ⇒ stable)

ANHARMONIC POTENTIAL

add an arbitrary cubic to the Hamiltonian.

Canonically transform to <u>linear</u> action-angle variables $\Rightarrow$

$$H = \underset{\substack{\nearrow \\ \text{integrable}}}{H_0} + \underset{\substack{\nwarrow \\ \text{nonintegrable} \\ \text{perturbation} \\ \mathcal{O}(3)}}{H_1}$$

$$H_0 = \omega_1 J_1 + \omega_2 J_2 + \bar{\alpha} J_1 J_2^{1/2} \cos(2\theta_1 + \theta_2)$$

$\mathcal{O}(3)$ resonance occurs when

$$2|\omega_1| = |\omega_2| \qquad\qquad 9\omega_0^2 = 8\omega_L^2$$

In terms of p's & q's

$$H_0 = \frac{1}{2}(p_1^2 + q_1^2) - (p_2^2 + q_2^2) + \frac{\alpha}{2}\Big[q_2(q_1^2 - p_1^2) - 2q_1 p_1 p_2\Big]$$

Cherry (1925)

Negative Energy Mode

* real spectrum (stable)

* indefinite (free) energy

$$\delta^2 H = \left.\frac{\partial^2 H}{\partial z^i \partial z_j}\right|_e \delta z^i \delta z^j$$

$$= H_2 = \sum_i \omega_i J_i$$

← not all the same sign

Resonant Nonlinear Instability

* $O(3)$ resonance ; $\omega_1 = 1$, $\omega_2 = -2$

$$2\omega_1 + \omega_2 = 0$$

* Negative energy mode

⇒ Explosive growth

$$q_1 = \frac{\sqrt{2}}{\alpha(t-\varepsilon)} \sin(t+\gamma) \qquad p_1 = \frac{\sqrt{2}}{\alpha(t-\varepsilon)} \cos(t+\gamma)$$

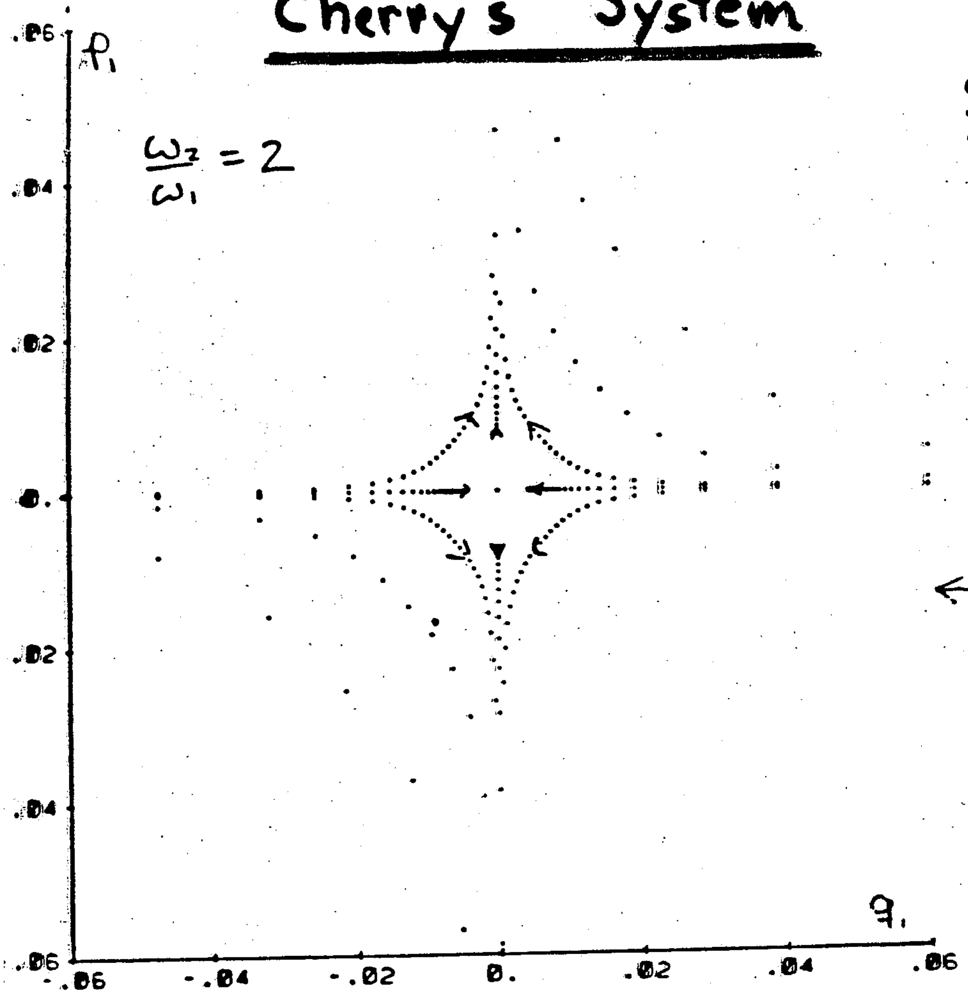
$$q_2 = \frac{1}{\alpha(t-\varepsilon)} \sin(2t+\gamma) \qquad p_2 = \frac{-1}{\alpha(t-\varepsilon)} \cos(2t+\gamma)$$

Cherry's System

Courtesy C Kuery

Surfaces of Section

$$H = \frac{\omega_1}{2}\left(p_1^2 + q_1^2\right) - \frac{\omega_2}{2}\left(p_2^2 + q_2^2\right) + \frac{\alpha}{2}\left[q_2\left(q_1^2 - p_1^2\right) - 2 q_1 p_1 p_2\right]$$

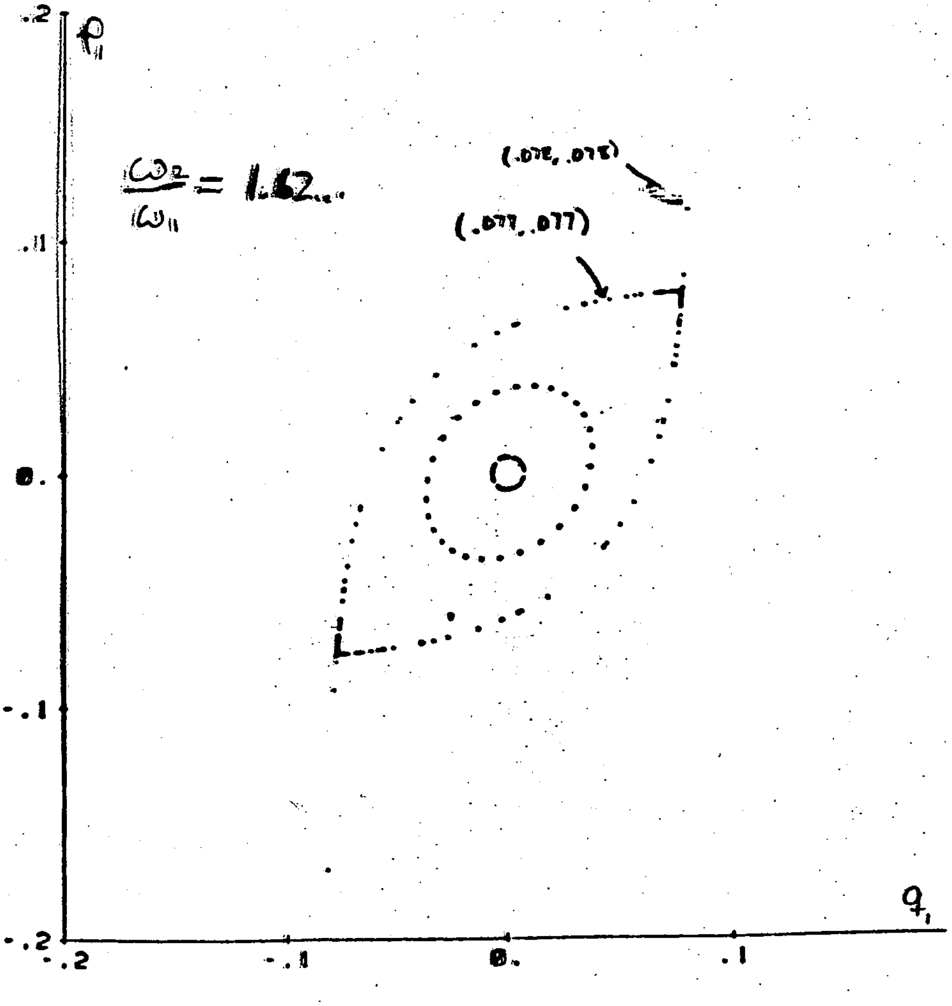


← Tuned O(3) resonance

↖ Surfaces of section ↙

$q_2 = 0$

Detuned resonance →



Orbits outside eyelet → ∞

Both are integrable

Generalization of Cherry

Most systems are not integrable and may be detuned. So we consider the following Hamiltonian:

$$H = \frac{1}{2}\omega_1(p_1^2 + q_1^2) - \frac{1}{2}\omega_2(p_2^2 + q_2^2)$$
$$+ \frac{\alpha}{2}\left[q_2(-p_1^2 + q_1^2) - \underset{\uparrow}{(1+\varepsilon)}\, q_1 p_1 p_2\right]$$

<u>non integrable perturbation</u>
or General cubic

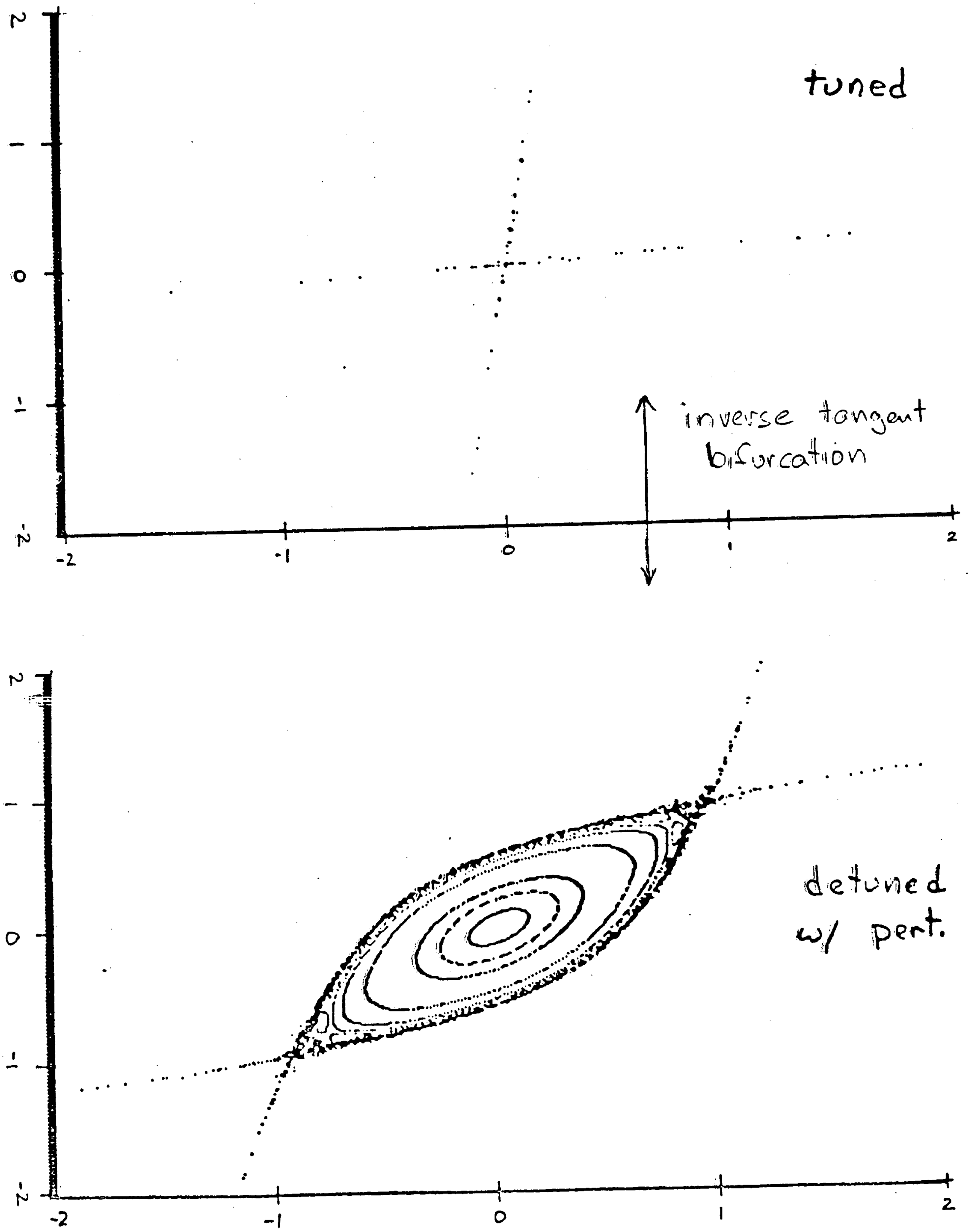
Since differential equations are more difficult to analyze and Arnold diffusion may take a long time we study a map that mimics features of the above system. This can be done with a cubic (not quadratic) map

$$\boxed{\begin{aligned} x' &= -y \\ y' &= -t\,y + x - y^3 \end{aligned}}$$

Cubic
Area Preserving
Map

This <u>area preserving</u> map has an inverse tangent bifurcation at trace $t = -2$ $(Res = 0)$.

Cubic Map (mimic)



Warm Fluid Two-Stream Instability

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha} \frac{dp_\alpha}{dx}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \quad , \quad \frac{dE}{dx} = 4\pi e(n_i - n_e)$$

$$H = \int dx \left\{ \frac{1}{2}\rho_i v_i^2 + \frac{1}{2}\rho_e v_e^2 + \rho_i U_i + \rho_e U_e + \frac{E^2}{8\pi}(n_\alpha) \right\}$$

$$\frac{\delta H}{\delta v_\alpha} = \rho_\alpha v_\alpha = 0$$

$$\frac{\delta H}{\delta \rho_\alpha} = \frac{v_\alpha^2}{2} + h_\alpha + \frac{e_\alpha}{m_\alpha}\phi = 0$$

$$\Longrightarrow \quad \rho_\alpha = \text{const.} \quad v_\alpha = 0$$

$\nabla H = 0 \Rightarrow$ uninteresting equil.

Need constraints. Why? $\int v\, dx, \int n\, dx$

Understanding comes from Noncanonical Formalism. $\left\{\begin{array}{l}\text{conservation}\\ \text{of phase space}\\ \text{volume}\end{array}\right.$

Noncanonical Hamiltonian Systems &

THE PERTURBED ENERGY & $\delta^2 F$

$$\dot{z}^i = J^{ij}(z)\frac{\partial F}{\partial z^j} = [z^i, \underset{\overset{\parallel}{\underline{F}}}{H+C}]$$

Linearize : $z = z_e + \delta z$

Equilibrium: $\dfrac{\partial F(z_e)}{\partial z^j} = 0$

Generally $\dfrac{\partial H(z_e)}{\partial z^j} \neq 0$ or yields trivial equilibria.

$$\delta \dot{z}^i = J^{ij}(z_e)\,\frac{\partial^2 F(z_e)}{\partial z^j \partial z^k}\,\delta z^k$$

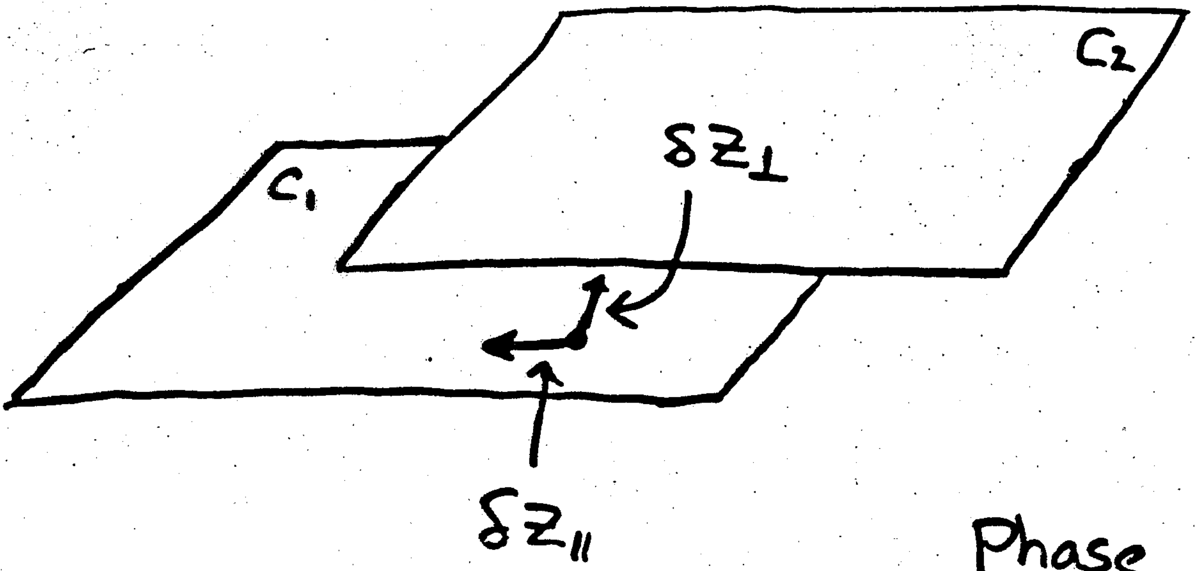
$$= \{\delta z^i, \frac{\delta^2 F}{2}\}_L$$

$$\uparrow$$

<u>Perturbed</u> <u>Hamiltonian</u> $\dfrac{\delta^2 F}{2}$

Not $\dfrac{\delta^2 H}{2}$! What is the energy.

$\delta^2 F/2$ = Free Energy



Phase Space

We add a source term and pull the system away from equilibrium.

$\delta z_\parallel$ is the only relevant part

Since $\delta z_\perp$ changes the equil.

Thermodynamic analogy: $\underset{\text{input}}{\underset{\uparrow}{dW}} = \underset{\delta z_\parallel}{\underset{\uparrow}{dU}} + \underset{\delta z}{\underset{\uparrow}{TdS}}$

Let $H \rightarrow H + H_{ext}(t)$

$H_{ext} = z^j S_j(t) \quad (= q\, F_{ext}(t))$

$$\boxed{\Delta H_c = -\int_0^t \dot{z}^j S_j(t)\, dt = \frac{\delta^2 F}{2}}$$

Two-Stream Instability (warm ions & electrons)

$$\frac{\partial v_\alpha}{\partial t} + v_\alpha \frac{\partial v_\alpha}{\partial x} = \frac{e_\alpha}{m_\alpha} E - \frac{1}{\rho_\alpha}\frac{\partial p_\alpha}{\partial x}$$

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial}{\partial x}(n_\alpha v_\alpha) = 0 \qquad \frac{\partial E}{\partial x} = 4\pi e(n_i - n_e)$$

equil. n_{0i}, n_{0e}, v_D ← drifting electrons

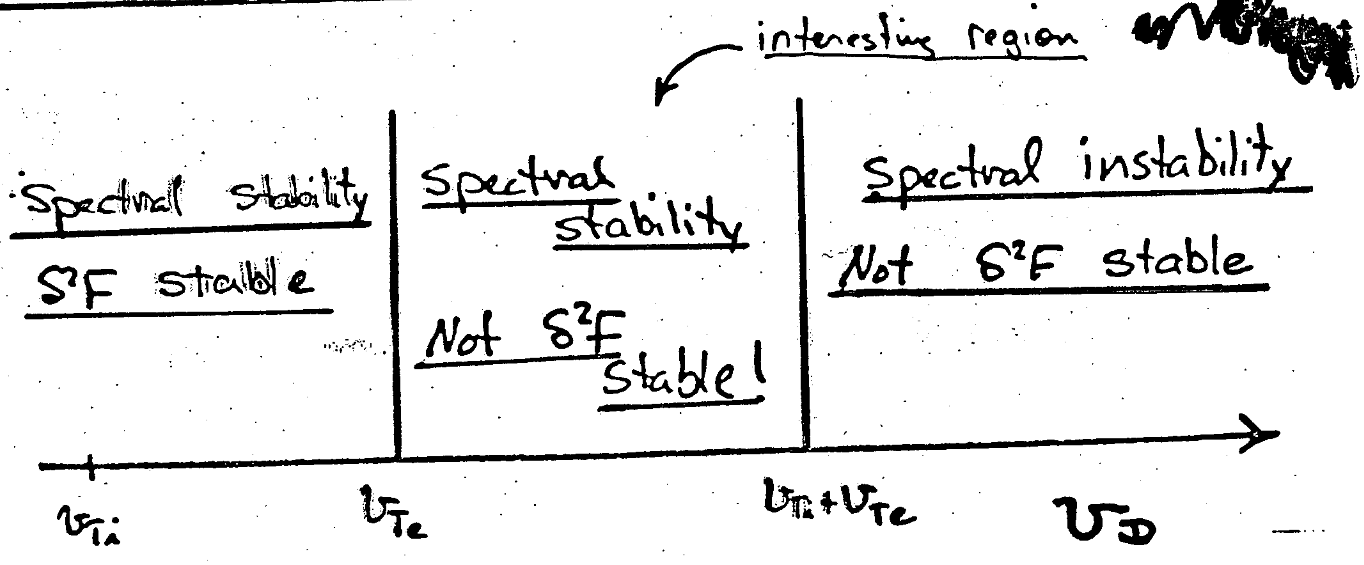
spectral stability condition given via

$$\boxed{0 = 1 - \frac{\omega_{pi}^2}{\omega^2 - k^2 v_{Ti}^2} - \frac{\omega_{pe}^2}{(\omega - k v_D)^2 - k^2 v_{Te}^2} = \varepsilon(k,\omega)}$$

Threshold: $v_D > v_{Ti} + v_{Te}$ ⇒ **instability**

$\delta^2 F$:

Threshold: $v_D < v_{Te}$ ⇒ $\delta^2 F$ positive definite



Noncanonical Variables $\longrightarrow$

Canonical Variables + Fourier Trans.

$\Longrightarrow$

$$H = \sum_{k}^{\infty} \omega_k J_k + \mathcal{O}(J^{3/2})$$

In the band $v_{Te} < v_D < v_{Ti} + v_{Te}$

$\exists$ ω_k's < 0.

Pick out "1" resonant triad + resonant driving term $\Longrightarrow$

$$J \sim \frac{1}{t_0 - t}$$

Explosive Growth.

Detune resonance $\Rightarrow$?

Coherent 3-Wave resonance (detuned)

"$=$" 4 dimensional Symplectic Map

2 Degree of freedom Autonomous

$\longrightarrow$ 1 degree of freedom Nonautonomous

$=$ Area preserving map

3 Degree of freedom autonomous

$\longrightarrow$ 2 degree of freedom Nonautonomous

$=$ 4 dim. symp. map

Generating function :

$$F = \underset{\uparrow}{F_{\text{cubic}}} + \underset{\nearrow}{F_{?}} + F_{\text{coupling}}$$

area preservers

4 Dimensional Symplectic Map (mimic)

(anharmonic mountain with earthquake)

Generating Function: $F = QQ' + qq' + \frac{\tau Q^2}{2} - \frac{t q^2}{2} + \frac{Q^3}{3} + \frac{q^4}{4}$

Coupling : $\rightarrow \underline{\underline{+ a q Q}}$

coupled <u>quadratic</u> & <u>cubic</u>
area preserving maps.

$$\frac{\partial F}{\partial Q'} = P' \qquad \frac{\partial F}{\partial Q} = -P \qquad \frac{\partial F}{\partial q'} = -p' \qquad \frac{\partial F}{\partial q} = p$$

$$P' = Q \qquad\qquad p' = -q$$

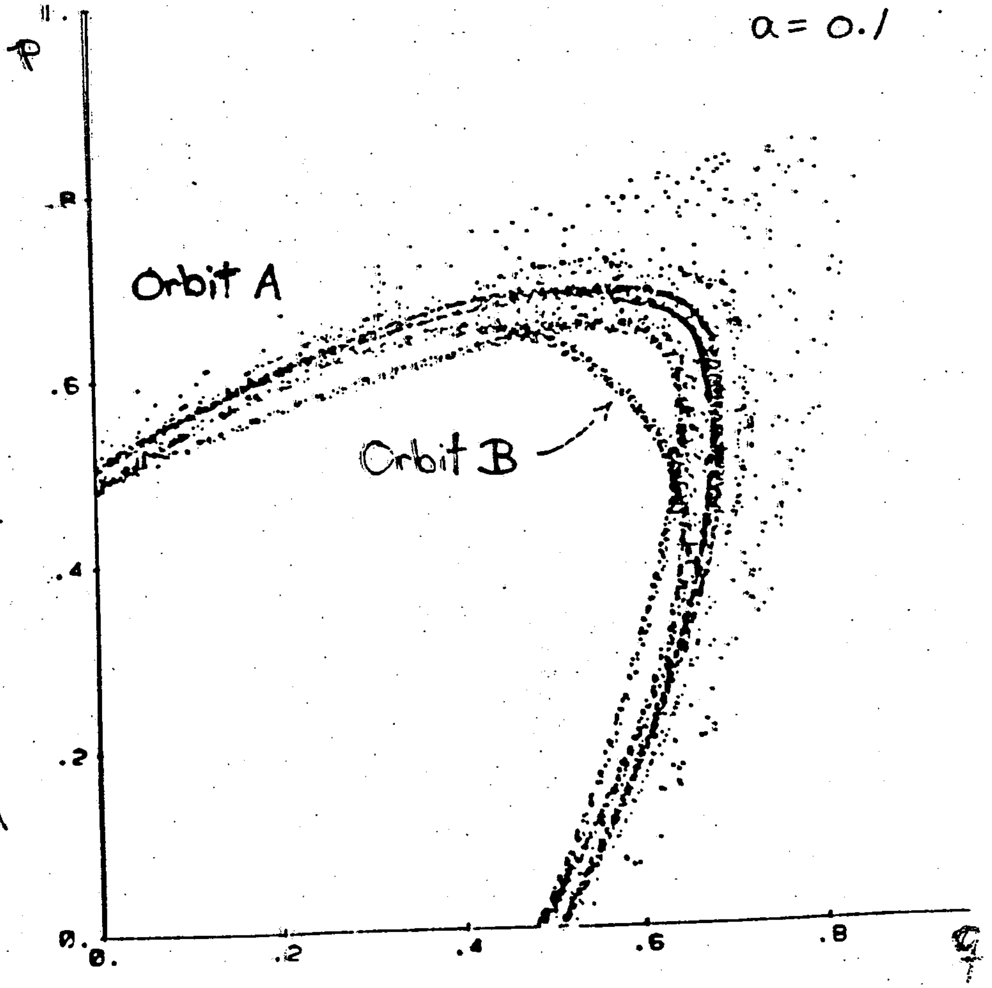
$$Q' = \tau Q + Q^2 + P + \underline{aq} \qquad\qquad q' = p + tq + q^3 + \underline{aQ}$$

Orbit A

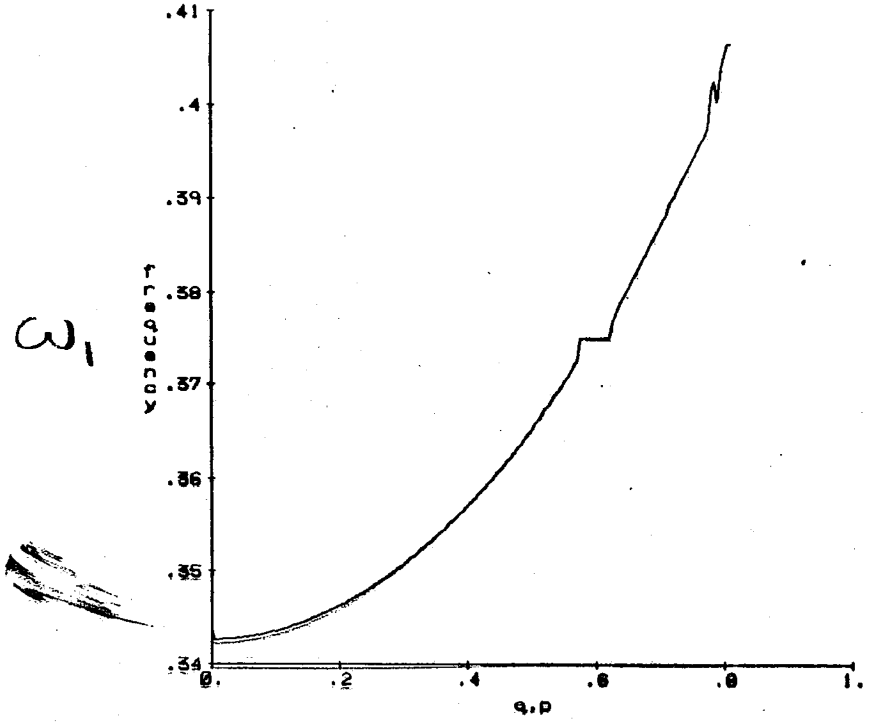
$(q,p,Q,P) = (.65, .65, 0, 0)$
No movement in <u>10 million</u> iterations. (5×10^3 plotted)

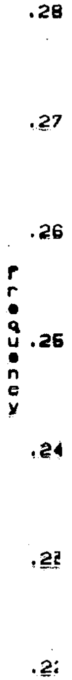
Orbit B

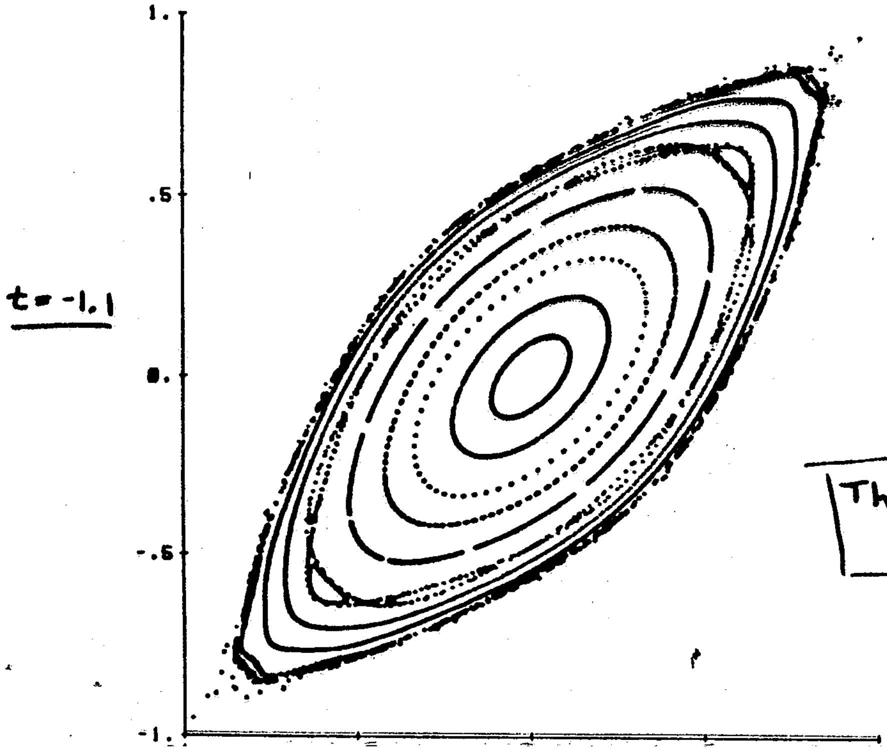
$(q,p,Q,P) = (.623\ , .623\ldots, 0, 0)$
<u>2 million</u> iterations. The first 5×10^3 map out separatrix lying <u>completely</u> inside A. Suddenly the orbit jumps outside A, jumps again and then $\longrightarrow \infty$. The last 5×10^3 are plotted.



Courtesy C Kueny



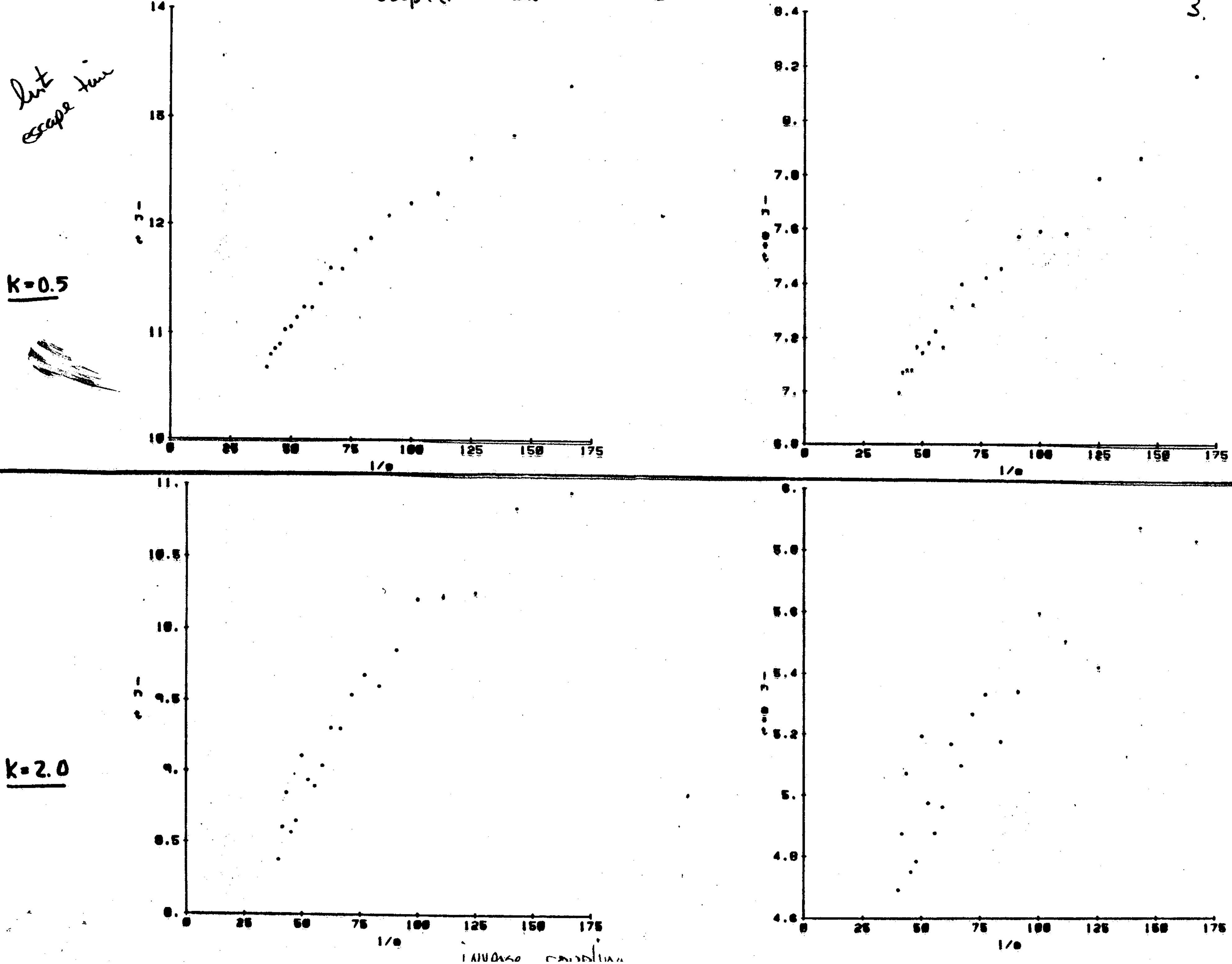




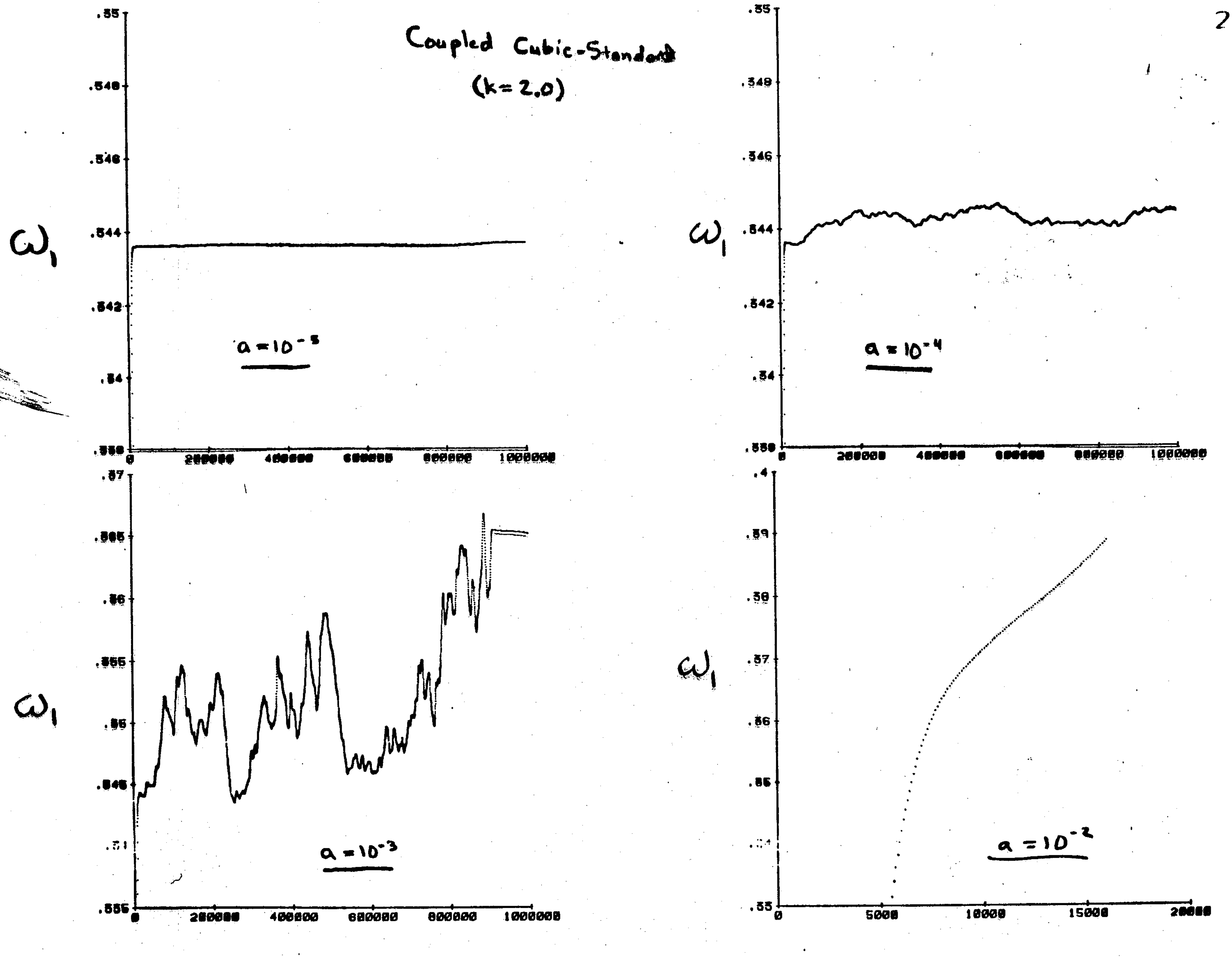
The Cubic Map

Coupled Cubic - Standard

ln t - escape time



Coupled Cubic-Standard
(k= 2.0)



# of iterations

Dissipation of Negative Energy Modes

Dissipation $\Rightarrow \quad \frac{dH}{dt} < 0$

2nd Law of Thermodynamics

$$H = \frac{q_+^2 + p_+^2}{2} - \frac{(q_-^2 + p_-^2)}{2}$$

$$\dot{q}_+ = p_+ \qquad\qquad \dot{q}_- = -p_-$$

$$\dot{p}_+ = -q_+ - \nu_+ p_+ \qquad\qquad \dot{p}_- = q_- - \nu_- p_-$$

$$\sim e^{i\omega t} \qquad\qquad \text{small } \nu_i > 0_{\; i=+,-}$$

$$\omega_+ \to \omega_+ + \underset{\underset{\text{damping}}{\nearrow}}{i\frac{\nu_+}{2}} \qquad\qquad \omega_- \to \omega_- - \underset{\underset{\text{growth}}{\nearrow}}{i\frac{\nu_-}{2}}$$

Boltzmann Equation

$$\frac{df}{dt} = \left.\frac{\partial f}{\partial t}\right)_c \qquad f \equiv \text{phase space density}$$

$$F = E + \lambda \underset{\equiv\, S_T}{\int f \ln f}$$

S_T & $\left.\frac{\partial f}{\partial t}\right)_c$ matched so as

$\longrightarrow$ $\delta F = 0 \Rightarrow f_M$ Thermal Equilibrium

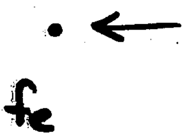
$\frac{dF}{dt} \leq 0$ asymptotic stability

Suppose

$$F = E + S_A \qquad S_A \neq S_T$$

$$\delta F = 0 \Rightarrow f_e \neq f_m$$

S_A & $\left.\frac{\partial f}{\partial t}\right)_c$ Unmatched



? Maybe not

GENERAL CASE

Consider a noncanonical Hamiltonian System

$$\dot{z}^i = J^{ij}(z)\,\frac{\partial F}{\partial z^j}(z) + \underset{\substack{\uparrow\\ \text{dissipation}}}{D^i(z)}$$

Second Law ($F = H + C$)

$$\frac{dH}{dt} = \frac{\partial H}{\partial z^i} D^i \le 0$$

Linearize about an equilibrium $z = z_e + y$

Equil. $$0 = J^{ij}(z_e)\,\frac{\partial F}{\partial z^j}(z_e) + D^i(z_e) + \underset{\substack{\uparrow\\ \text{Source}\\ \text{term}}}{S^i}$$

$$\frac{\partial F}{\partial z^j}(z_e) = 0$$

$$S^i = -D^i(z_e)$$

1st order $$\dot{y}^i = J^{ij}\frac{\partial^2 F}{\partial z^j \partial z^\ell} y^\ell + \frac{\partial D^i}{\partial z^\ell} y^\ell$$

Case of interest: Neg. Energy Mode

* $\delta^2 F = \frac{1}{2} \frac{\partial^2 F}{\partial z^i \partial z^\ell} y^i y^\ell$ <u>indefinite</u>

* $\dot{y} = J \frac{\partial^2 F}{\partial z^2} y + \boxed{\frac{\partial D}{\partial z} y}$ <u>Real Spectrum</u> <u>w/ D=0</u>

Dissipation means

$$\frac{d}{dt} \delta^2 F \leq 0 \quad ,$$

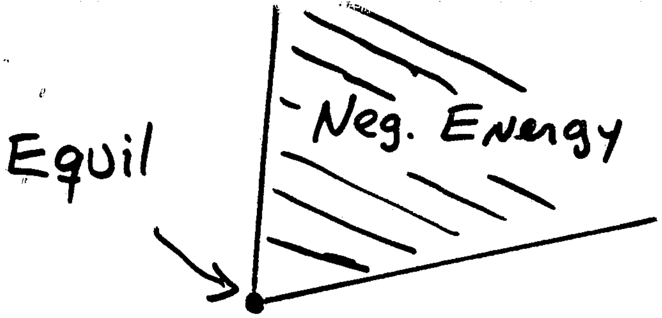
But

$$\frac{d}{dt} \delta^2 F = \frac{\partial^2 F}{\partial z^i \partial z^j} \frac{\partial D^j}{\partial z^\ell} y^i y^\ell = ?$$

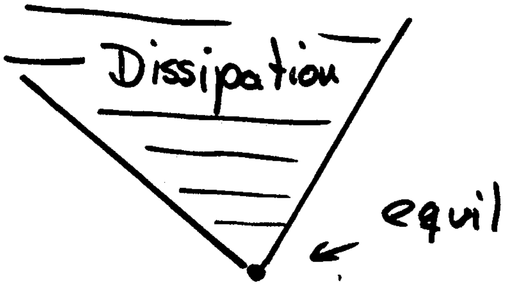
In general

$\frac{dH}{dt} = \frac{\partial H}{\partial z^i} D^i \leq 0$ implies nothing about

$\frac{d}{dt} \delta^2 F$!



Positive energy



$$\frac{d}{dt}\,\delta^2 F > 0$$

For stability

$$\{N.E.\} \cap \{\text{Dissip}\} = \phi$$

?

Generically a negative energy mode will get dissipated.

* Nonlinear Instability may be slow
 - Won't know until investigate -

* Hidden Invariants
 - besides energy, momentum, Casimirs
 unlikely, at least for physical systems.

* Negative Energy Modes may not get
dissipated - Generic ?

* For P.D.E.'s dissipation may
give rise to new modes -
Singular modes not present in the
ideal system. Perhaps most dangerous.
(Rayleigh vs. Orr-Sommerfeld Eqs.)
 - Should investigate -

* Who cares about infinitesimal
pertubations ? How Small ?

Conclusion

$\delta^2 F$ is sufficient & "Necessary" for stability.

If there is free energy in a system & No reason it cannot be tapped, then *it can happen & probably will.*