Woods Hole - GFD
Summer School
July 29, 1990

The Free Energy "Principle", Negative Energy Modes and Stability

TWO Goals: (1) tell something about few dof Hamiltonian dynamics

(2) describe my philosophy for viewing fluid and plasma dynamical systems as Ham. systems - clarifies and generalizes things that occur over and over in these systems, in particular negative energy modes & stab.

Mention before NEM; Von Laue 1905 →

Non Goal: The talk is intended to be more pedagogical than technical.

Overview

I. Few DOF Ham. Systems (FEP - NEM - Stab.)
stability, H. Chaos, KAM thm., SOS, Arnold Diffusion
Example: ch. ptcle on a mountain

II. Noncanonical Ham. Mech. (me 1979 → Sophos Lie)(Dirac)
generalization of Poisson bracket that in ∞ dim fits the Eulerian Description of Classical Media (ideal) fluids - Vlasov

III. N.C. Ham. Field theory
Gardner Bracket

IV. Vlasov Equation

V. Dissipation

I. Few DOF Ham. Systems.

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

A. Stability

$$\dot{q}_i = 0 = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = 0 = -\frac{\partial H}{\partial q_i}$$

$\Rightarrow$ critical point, fixed point, equilibrium point

stationary point : $(q_0, p_0) \equiv z_0$

Notation $\quad \underline{z} = (\underline{q}, \underline{p}) \qquad z^i \begin{cases} q & \text{first } N \\ p & \text{second } N \end{cases} \qquad i = 1, \dots 2N$

$\underline{z} = z_0 + \delta \underline{z}$ Taylor expand $H \Rightarrow$

$$\delta^2 H = \sum_{ij} a_{ij}\, \delta z^i \delta z^j$$

$$\delta \dot{q}_i = \frac{\partial \delta^2 H}{\partial \delta p_i} \qquad \delta \dot{p}_i = -\frac{\partial \delta^2 H}{\partial \delta q_i}$$

$\Longleftrightarrow$

$$\delta \dot{z}^i = J_c^{ij} \frac{\partial \delta^2 H}{\partial \delta z^j} \qquad (J_c) = \begin{bmatrix} O_N & I_N \\ -I_N & O_N \end{bmatrix}$$

Two approaches:

(1) Spectral Problem

$\delta z \sim e^{i\omega t} \Rightarrow \omega$

(2) Liapunov Stability

use const. of motion

H - built-in

<u>Standard Case</u> (T + V form $\longrightarrow$ easy & special)

$$H = \frac{p^2}{2m} + V(q)$$

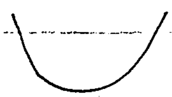
$$p_0 = 0 \ , \quad q_0 \text{ s.t. } \left.\frac{\partial V}{\partial q}\right|_{q_0} = 0$$

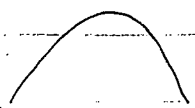
$$\Longrightarrow \quad \delta^2 H = \frac{(\delta p)^2}{2m} + \frac{1}{2}\frac{\partial^2 V}{\partial q^2}(q_0)(\delta q)^2$$

Lagranges Theorem: (Necessary & Sufficient Cond.)

$$\left.\frac{\partial^2 V}{\partial q^2}\right|_{q_0} > 0 \quad \Longrightarrow \quad \text{stable} \qquad \text{(i)}$$

$$< 0 \quad \Longrightarrow \quad \text{unstable} \qquad \text{(ii)}$$

(i) 

 (ii)

<u>General Case</u> Dirichlet's Theorem

$$H(z) \neq T + V \quad \Longrightarrow$$

$$\delta^2 H = \sum_{i,j} a_{ij}\, \delta z^i \delta z^j$$

$$\frac{\partial^2 H}{\partial z^i \partial z^j}(z_0) \;\begin{cases} \nearrow \text{definite} \Longrightarrow \text{Stable} \\ \searrow \text{indef.} \Longrightarrow \text{"Nothing"} \end{cases}$$

The case of interest is when $\delta^2 H$ is indef. In this case two possibilities

(i) linear stability
(ii) linear instability

Actually we are interested in case (i); i.e.

$$\boxed{\delta^2 H \text{ indef.} \qquad \text{linear stability}}$$

This implies a NEM. It is a case where

(i) Equil is not a min energy state (taut)

(ii) Free Energy is "available"

(iii) Occurs for rotating systems: gyroscopic forces
(Mech. literature)

(iv) Systems (PDE's) for which dynamical equil Z_0 $\neq$ thermal equil.

Old Question (we address here):

Is Dirichlet necessary and sufficient for real stability; i.e. including N.L. terms.

Digression on Stab. Def.

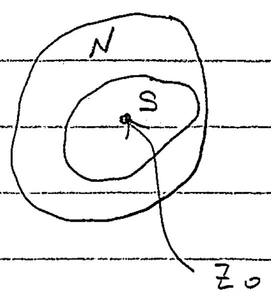
Stability Definition (chicken scratch)

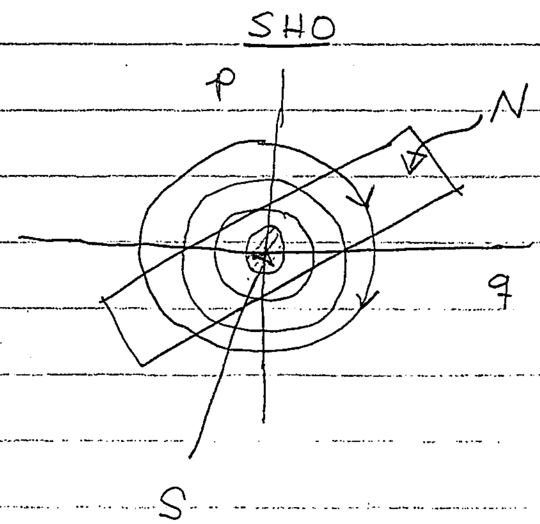
An equilibrium point z_0 is stable if $\forall$ nbhds N of z_0 $\exists$ exists a subnbhd of z_0 $S \subset N$, s.t. if $z(t=0) \in S \Rightarrow z(t) \in N \ \forall t$.

linear stability: $z(t) = z_0 + \delta z(t)$; $\delta z(t)$ determined by linear dynamics

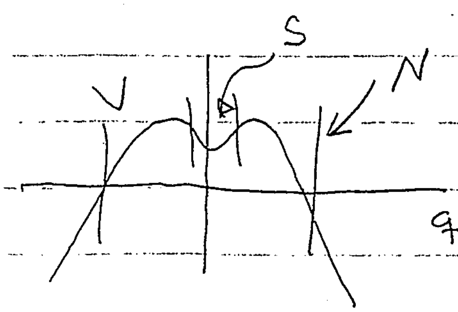
nonlinear stability: $z(t)$ determined by nonlinear dynamics

phase space





Finite Amp. Unstable, but Stable



N as small as you like yet
linear stab. $\not\Rightarrow$ nonlinear stab.

Example - A particle on a Mountain

(a) Ptle on Harmonic Mountain

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \underbrace{\frac{eB}{2c}\left(\dot{y}x - \dot{x}y\right)}_{\underline{A}\cdot\underline{v}} + \frac{k}{2}\left(x^2 + y^2\right)$$

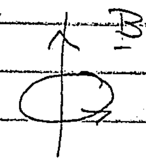
$V(x, y)$ = inverted parabola

Scale and Legendre Trans $\Rightarrow$

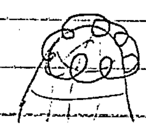
$$H = \frac{p_1^2 + p_2^2}{2} + \omega_L\left(q_2 p_1 - q_1 p_2\right) + \frac{1}{2}\left(\omega_L^2 - \omega_0^2\right)\left(q_1^2 + q_2^2\right)$$

Two freq.

$$\omega_L = \frac{eB}{2mc}$$



$$\omega_0 = \sqrt{k/m}$$



$k \sim \frac{\partial^2 V}{\partial q^2}$

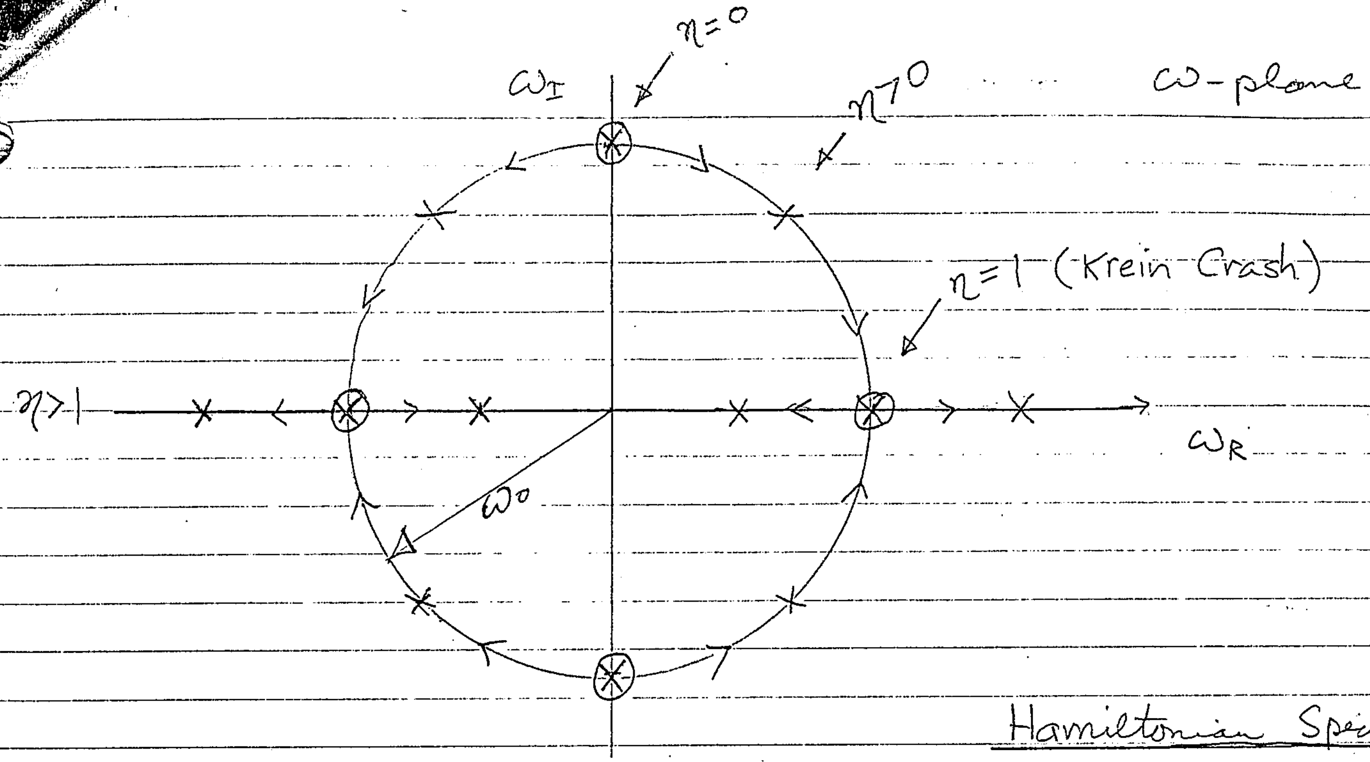
Linear Prob. = Prob.

equil $q_i = p_i = 0 \Longrightarrow$

$\delta z \sim e^{i\omega t}$

$$\frac{\omega}{\omega_0} = \pm\left(\sqrt{\eta - 1} \pm \sqrt{\eta}\right)$$

$$\eta = \frac{\omega_L^2}{\omega_0^2} \sim B^2$$

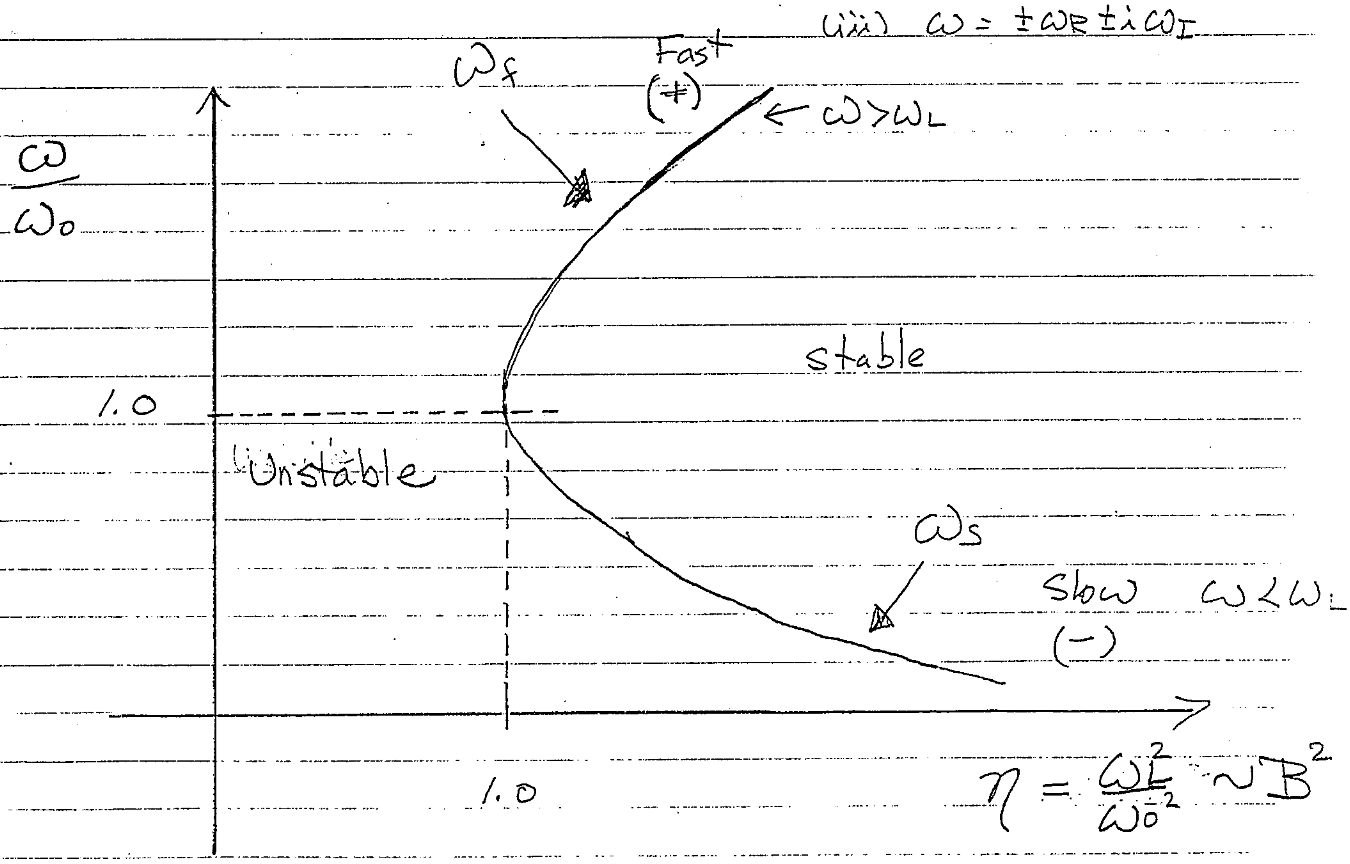


Hamiltonian Spectra

(i) $\omega = \pm i\omega_I$

(ii) $\omega = \pm \omega_R$ stable

(iii) $\omega = \pm\omega_R \pm i\omega_I$



Hamiltonian Hopf Bifurcation (inverse)

Negative Energy Mode - Hamiltonian Def.

For $\eta > 1$ the system has a negative energy mode.

$\exists$ a canonical trans. to coords (P_i, Q_i) s.t.

$$H = -\frac{1}{2}\omega_s(P_1^2 + Q_1^2) + \frac{1}{2}\omega_f(P_2^2 + Q_2^2)$$

Thus (Recall Doublet)

$$\boxed{\delta^2 H \text{ indef.} \qquad \text{linearly stable} \Longleftrightarrow NEM}$$

Sylvester's theorem $\Rightarrow$ invariant def.

Krein's theorem ($\eta > 1 \rightarrow \eta < 1$)

When eigenvalues collide on the ω_R axis $(\omega_R \neq 0)$ they can only result in instability if their signatures are different.

(b) Particle on Anharmonic Mountain – Integrable

add a general cubic potential $V_3(\underline{q}) \Rightarrow$

$$H = -\frac{1}{2}\omega_s(P_1^2+Q_1^2) + \frac{1}{2}\omega_f(P_2^2+Q_2^2) + V_3(\underline{Q},\underline{P})$$

split V_3 into two pieces

$$H = H_{int} + H'$$

where

$$\boxed{H_{int} = -\frac{1}{2}\omega_s(P_1^2+Q_1^2) + \frac{1}{2}\omega_f(P_2^2+Q_2^2) + \frac{\alpha}{2}\left[Q_2(Q_1^2-P_1^2) - 2Q_1P_1P_2\right]}$$

In the case of $O(3)$ resonance ($\exists\ \eta$ value)

$$\boxed{\omega_f = 2\omega_s}$$

Cherry's Hamiltonian (1927)

$$\Rightarrow \quad Q_1 = \frac{\sqrt{2}}{\varepsilon-\alpha t}\sin(\omega_1 t+\gamma) \qquad P_1 = \frac{\sqrt{2}}{\varepsilon-\alpha t}\cos(\omega_1 t+\gamma)$$

$$Q_2 = \frac{-1}{\varepsilon-\alpha t}\sin(2\omega_1 t+2\gamma) \qquad P_2 = \frac{-1}{\varepsilon-\alpha t}\cos(2\omega_1 t+2\gamma)$$

Explosive Growth for I.C.'s in arbitrarily small nbhd. of equil $\Rightarrow$ Nonlinear Instability

How Integrable? $J_i = (Q_i^2+P_i^2)/2 \Rightarrow$

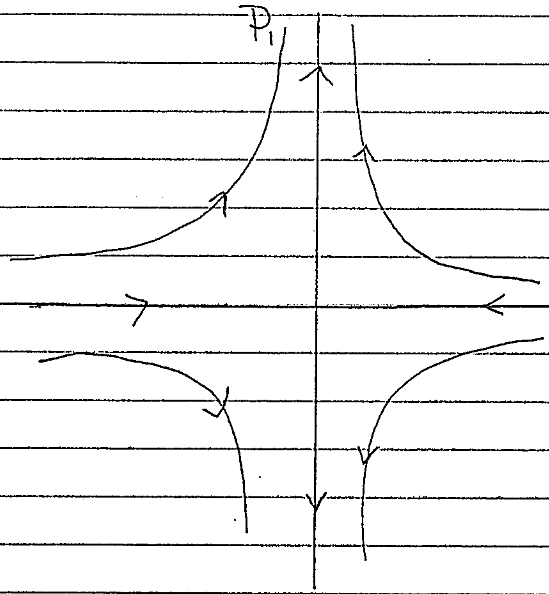
$$H_{int} = -\omega_s J_1 + \omega_f J_2 + \alpha J_1\sqrt{J_2}\cos(\theta_1+2\theta_2)$$

$$I = 2J_1 - J_2 = \text{const.}$$

Surface of Section

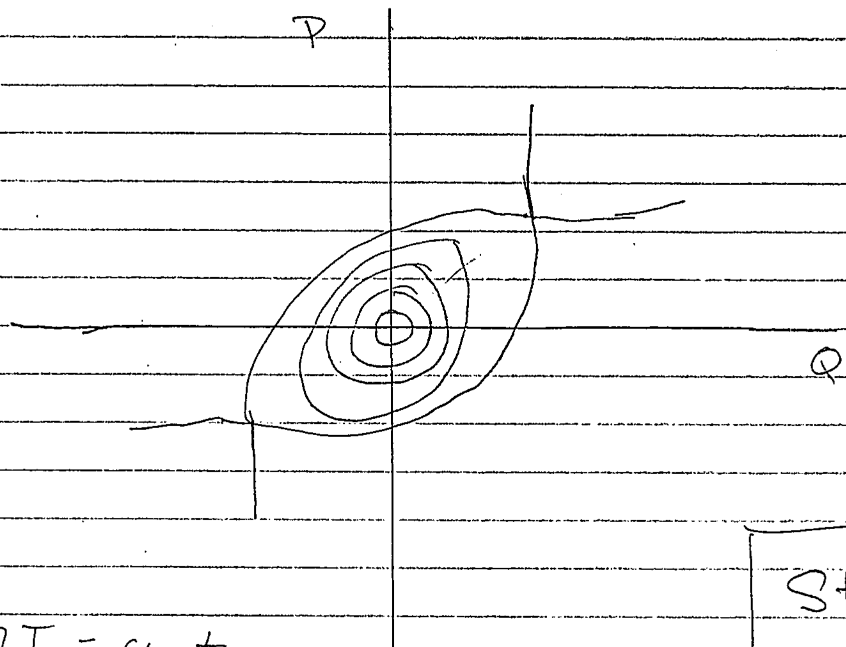
$Q_2 = 0$, $H = \text{const} \Rightarrow P_2$ pick (P_1, Q_1)

Resonant



Not to be confused w/ phase plane for SHO!

Nonresonant



$\boxed{\text{Stable!}}$

$H + \lambda I = \text{const}$

$\Rightarrow$ 4 ellipsoid

Mimic of Nonintegrable Cubic Mountain

Instead of studying o.d.e. we mimic the problem by an

Area preserving or <u>Symplectic map</u>

because (1) easier, particularly for higher dimensional systems (2) genericness ?

<u>Cubic Map</u> $T: \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{q' = -p \qquad p' = -t p + q - p^3}$$

This map is derivable from a generating function

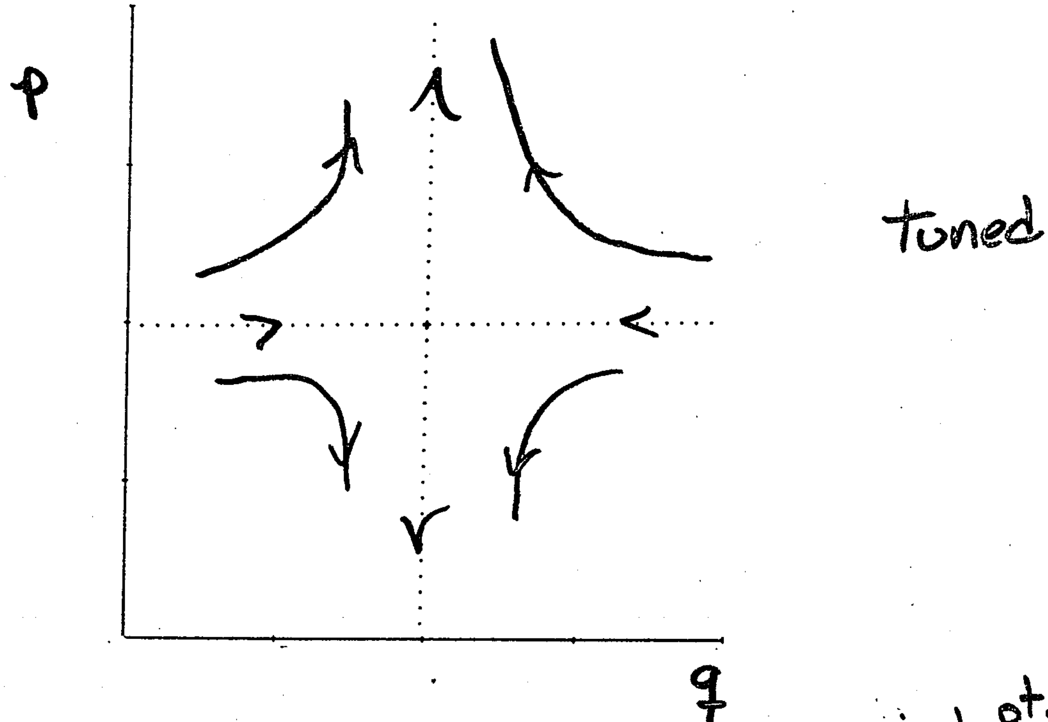
$$F(q, q') = q q' - \frac{t q^2}{2} + \frac{q^4}{4}$$

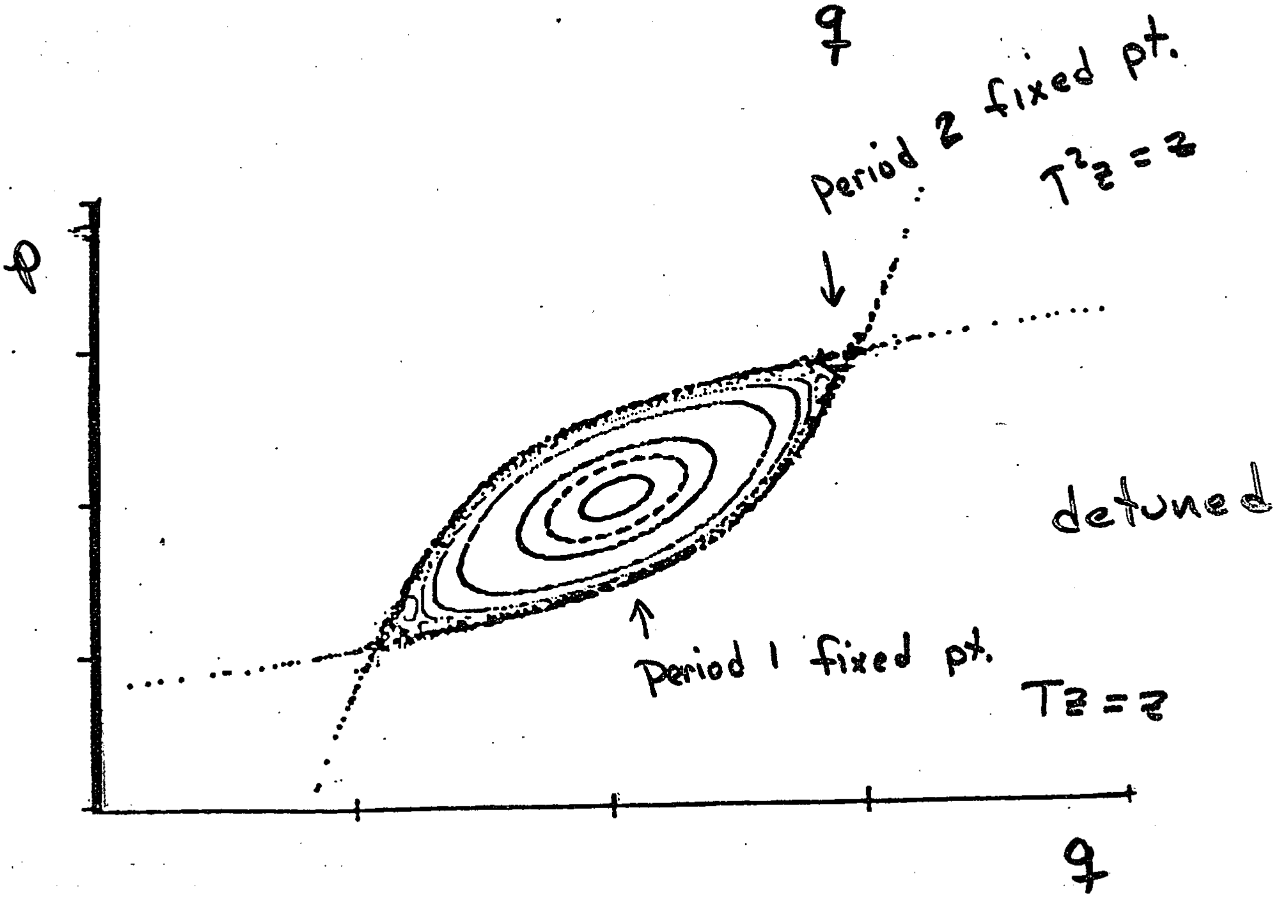
$$p' = -\frac{\partial F}{\partial q'} \qquad p = \frac{\partial F}{\partial q} .$$

This area preserving map has an inverse tangent bifurcation at

trace $t = -2$.

Surfaces of Section for Mimic of Cubic Nonintegrable Mountain



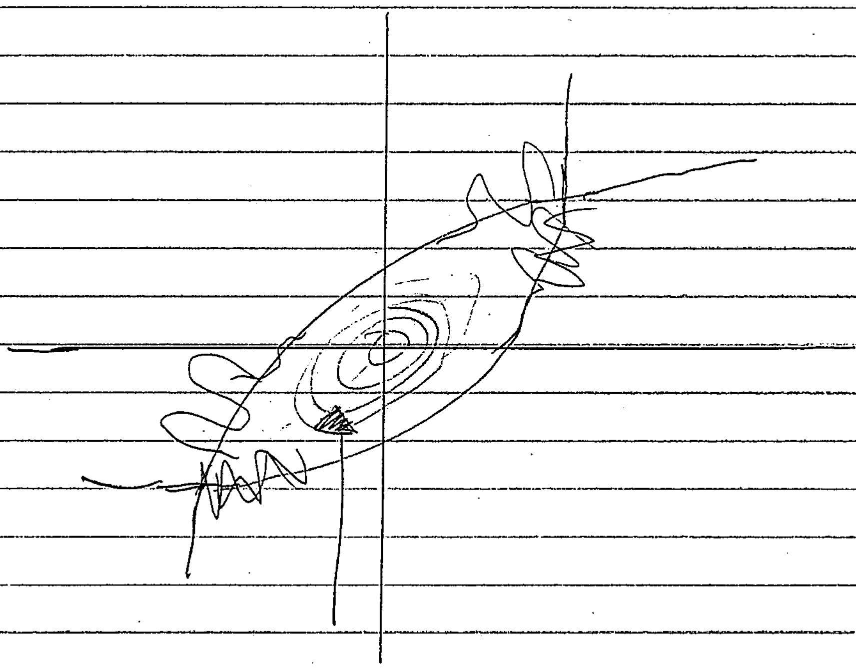


Linearization about f.p. $\Rightarrow$ hyperbolic or elliptic.

(c) PHe on Anharmonic Mountain - Nonintegrable

What happens when we keep the rest of V_3?

Can no longer "write down" the surface of section - must numerically integrate



KAM
Surfaces $\mathcal{M}(1)$ as $Q, P \to 0$ (Moser)

$\Rightarrow$ Stable

Show Mimic (two trans.)
(1) cubic map (Symplectic) p. 12.

(2) SOS p. 13

(d) Nonintegrable Cubic Mountain w/ Earthquake

$V_3(\underline{q}, \underline{p}; t)$

explicit time dependence (Periodic)

SOS obtained now by plotting Q_i, P_i at period of T.

Some orbits drift across "islet" by

Arnold Diffusion, until $\to \infty$.

4 - Dim Symplectic map.

Show

~~(1) DOF & Maps (15)~~ ?

(2) SOS (15)

(3) FEP (16)

Comment on Dissipation.

$\Rightarrow$ Dirichlet Necessary for Stability & "Sufficient"

Cubic Mountain with Earthquake Map

Add explicit time dependence to the cubic part of the potential. This is a two degree of freedom nonautonomous system.

Recall:

(A) 2 degree of freedom autonomous

$\longrightarrow$ 1 degree of freedom nonauton.

(B) 3 degree of freedom autonomous

$\longrightarrow$ 2 degree of freedom nonauton.

(A) = area preserving map of the plane

(B) = 4 dimensional symplectic map

Generating Function:

$$F = F_{\text{cubic}} + F_{?} + F_{\text{coupling}}$$

The 4d symplectic map is the generic 3 degree of freedom system. Our map will have a negative energy mode. Example. Three wave ~~non resonant~~ interaction.

4 Dimensional Symplectic Map (mimic)

(anharmonic mountain with earthquake)

Generating Function: $F = QQ' + qq' + \frac{\tau Q^2}{2} - \frac{\tau q^2}{2} + \frac{Q^3}{3} + \frac{q^4}{4}$

Coupling : $\rightarrow$ $\underline{+aqQ}$

coupled quadratic & cubic area preserving maps.

$$\frac{\partial F}{\partial Q'} = P' \qquad \frac{\partial F}{\partial Q} = \pm P \qquad \frac{\partial F}{\partial q'} = -p' \qquad \frac{\partial F}{\partial q} = p$$

$$P' = Q \qquad\qquad p' = -q$$

$$Q' = \tau Q + Q^2 + P + aq \qquad\qquad q' = p + \tau q + q^3 + aQ$$

Orbit A

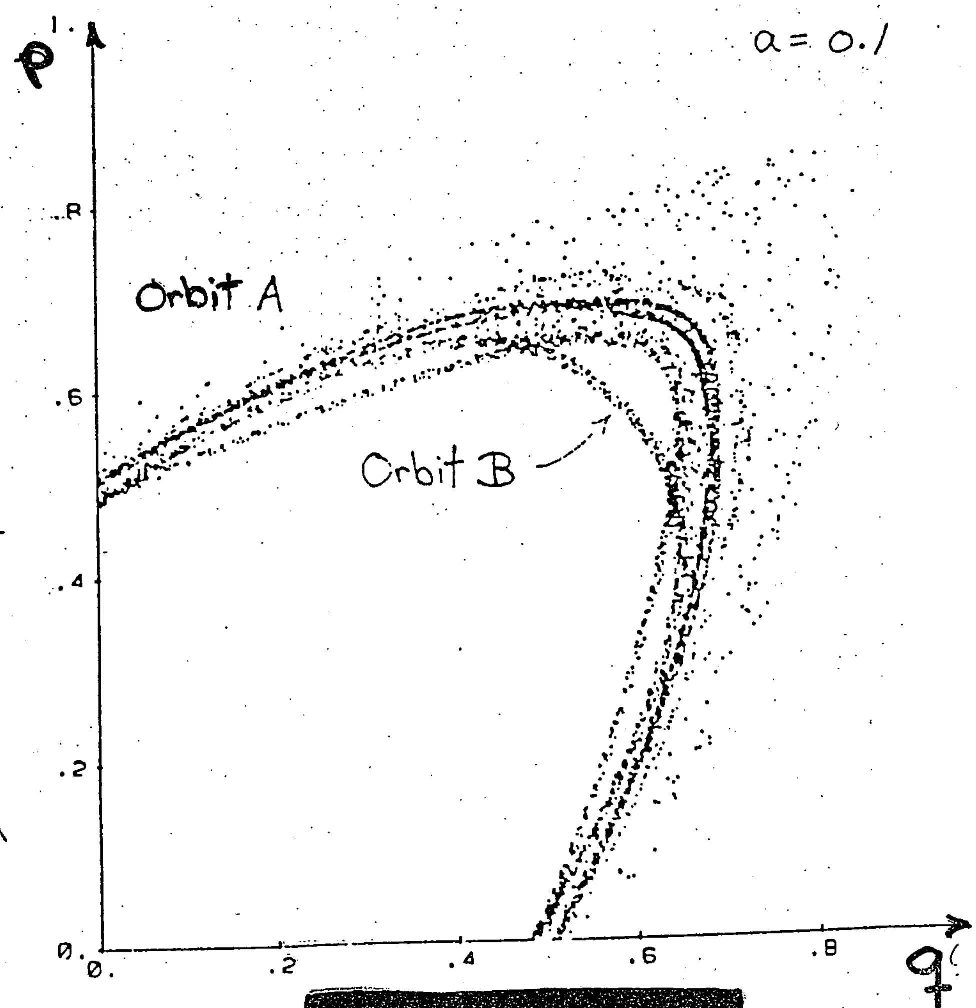
$(q, p, Q, P) = (.65, .65, 0, 0)$

No movement in 10 million iterations. (5×10^3 plotted)

Orbit B

$(q, p, Q, P) = (.623\ \ , .623\ldots, 0, 0)$

2 million iterations. The first 5×10^3 map out separatrix lying completely inside A. Suddenly the orbit jumps outside A, jumps again and then $\longrightarrow \infty$. The last 5×10^3 are plotted.



Free Energy Principle

If the free energy, $\delta^2 F$, is indefinite, then there are two possibilities:

(1) The system has linear (spectral) instability. Bad.

(2) The spectrum is stable, but there is a negative energy mode.

Also Bad.

In Case (2):

* <u>Nonlinear</u> instability (Not finite amplitude) e.g. explosive growth, slow growth followed by fast, . . .

* Dissipated negative energy modes can be <u>structurally</u> unstable.

II. Noncanonical Ham. Mech. (Gen. Mech.)

In order to introduce NCHM we write first Ham's eqs. as follows:

$$\frac{dz^i}{dt} = [z^i, H] = J_c^{ij} \frac{\partial H}{\partial z^j} \qquad i,j = 1, \dots N$$

Σ notation

Recall

$$\underline{z} = (\underline{q}, \underline{p}) \qquad (J_c) = \begin{bmatrix} O_N & I_N \\ -I_N & O_N \end{bmatrix}$$

$$[f, g] = \frac{\partial f}{\partial z^i} J_c^{ij} \frac{\partial g}{\partial z^j}$$

$(J_c) \equiv 2^{nd}$ order contravariant tensor

Cosymplectic Form

<u>Canonical Trans.</u> preserve the form of P.B. $\Longleftrightarrow$ form invariance of J_c. E.G. $\tilde{z} = \tilde{z}(z)$

$$\frac{\partial \tilde{z}^l}{\partial z^i} J_c^{ij} \frac{\partial \tilde{z}^m}{\partial z^j} = \underbrace{\tilde{J}^{lm}(\tilde{z})}_{\substack{\text{trans. } 2^{nd} \\ \text{Contra}}} \equiv \underbrace{J_c^{lm}}_{\substack{\text{form} \\ \text{invaria}}}$$

noncanonical trans. → trans. 2nd Contra

canonical trans. → form invaria

<u>Devious trans.</u> Eq. $\longrightarrow$ tricky trans. $\longrightarrow$ You ?

physical grounds $\Rightarrow$ energy?

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} \Rightarrow [f,g] = \frac{\partial f}{\partial z^i} J^{ij} \frac{\partial g}{\partial z^j}$$

Can get back if $J \Rightarrow [f,g]$ satisfies

(1) $[f,g] = -[g,f] \Leftrightarrow J^{ij} = -J^{ji}$

(2) $[f,[g,h]] + \circlearrowleft = 0$ (Jacobi) <u>Lie Algebra</u>

$$\Leftrightarrow S^{ijk} = J^{il}\frac{\partial J^{jk}}{\partial z^l} + \circlearrowleft = 0$$

(3) $\det J \neq 0$

Darboux 1850 ish $\Rightarrow \exists$ coords such that

$$J \rightarrow J_c$$

<u>Noncanonical Mechanics</u> - Emphasize the algebra.

$$\dot{z}^i = J^{ij}\frac{\partial H}{\partial z^j} \qquad J(z)$$

w/ (1) & (2) but we <u>drop</u> (3)!!

$$\left\{ \begin{array}{l} [f,g] = \dfrac{\partial f}{\partial z^i} J^{ij} \dfrac{\partial g}{\partial z^j} \qquad i,j = 1,2 \ldots M \\ \text{on } M \text{ dim Space} \end{array} \right.$$

Not necessarily even!!

odd $\Rightarrow \det J = 0$

<u>Poisson Manifold</u>

Built-In Invariants – Casimirs (Lie's Distinguished elements)

Def $[f, C] = 0 \quad \forall f$

indep of ham. Kinematic statement

$\frac{\partial f}{\partial z^i} J^{ij} \frac{\partial C}{\partial z^j} = 0 \quad \forall f \Rightarrow$

$J^{ij} \frac{\partial C}{\partial z^j} = 0$

Nontrivial sol $\Rightarrow \det J = 0$

If Rank of $J = 2N < M$ then

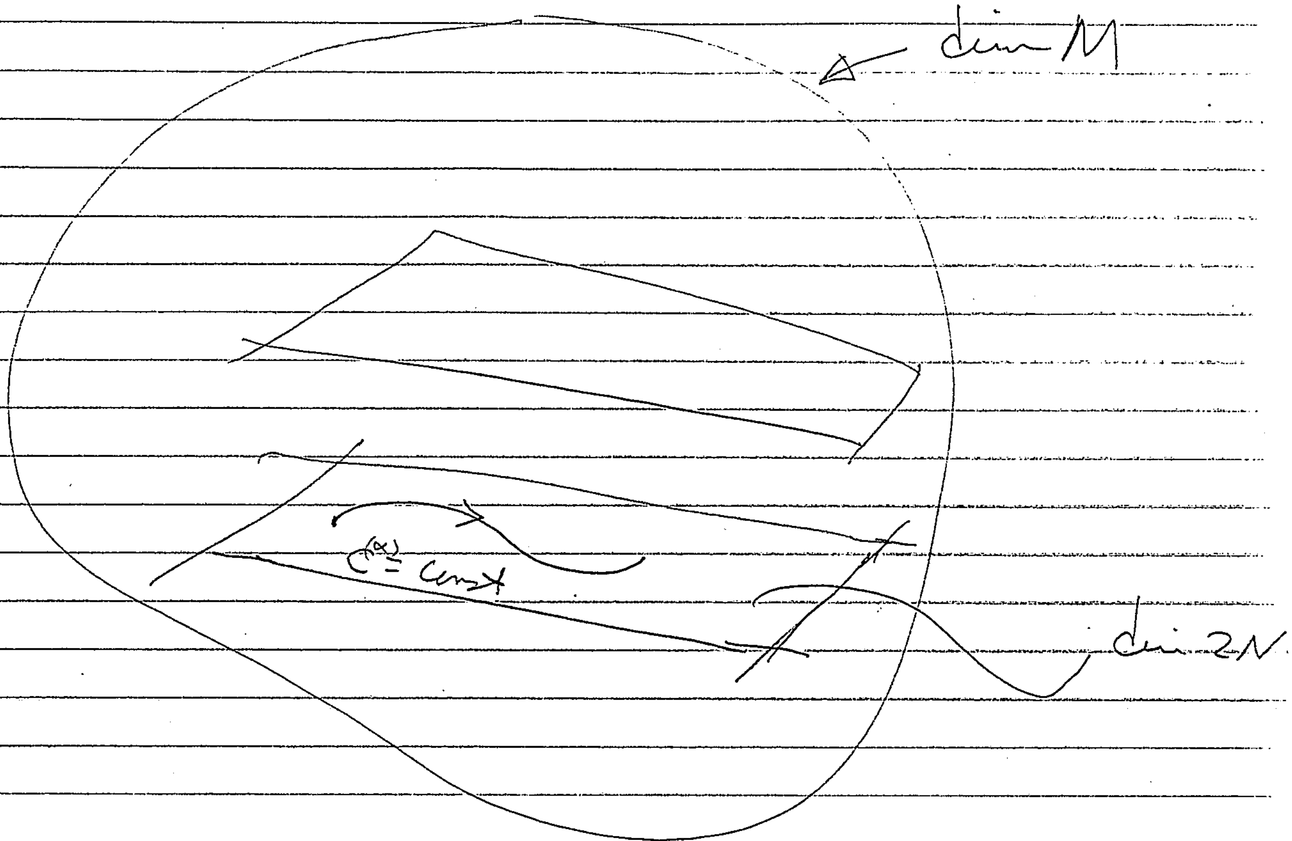
Corank $= M - 2N$ and $\exists$ this many
null eigenvectors

Jacobi $\Rightarrow \exists$ this many C's. s.t. their
grads. are null e-vectors; i.e. $\frac{\partial C^{(\alpha)}}{\partial z^i}$ $(\alpha) = 1, 2 \cdots M-2N$

Darboux The theorem $\Rightarrow$

$$J \rightarrow \left(\begin{array}{cc|c} O_N & I_N & \\ -I_N & O_N & O \\ \hline \multicolumn{2}{c|}{O} & O \end{array} \right)$$

Phase Space



Phase space foliated by symplectic leaves.

Equil & Stability

$$\dot{z}^i = [z^i, H] = [z^i, \underbrace{H + C}_{\equiv F}] = J^{ij} \frac{\partial F}{\partial z^j}$$

extremize H subject to const. C's. $\Rightarrow$

$$\dot{z}^i = 0 \Leftarrow \frac{\partial F}{\partial z^j} = 0 \Rightarrow z_0$$

$\frac{\partial H}{\partial z^i} = 0$ { Vacuum state / thermal equil } { fluid $H = \int \frac{\rho v^2}{2} + B^2$; $\rho = \text{const}$; $v = 0$; $B = 0$ }

$\frac{\partial F}{\partial z^i} = 0$ dynamical equil z_0

Stability

F - surface near z_0

$$\delta^2 F = \frac{1}{2} \frac{\partial^2 F}{\partial z^i \partial z^j}(z_0)\, \delta z^i \delta z^j$$

Would like to say

$$\boxed{\text{stable} \ \ \& \ \ \delta^2 F\big|_c \text{ indef} \Rightarrow \underset{?}{\text{NEM}}}$$

Question: Is δz arbitrary? $\exists$ $\delta z_{\parallel}$ & $\delta z_{\perp}$.

in leaf, other

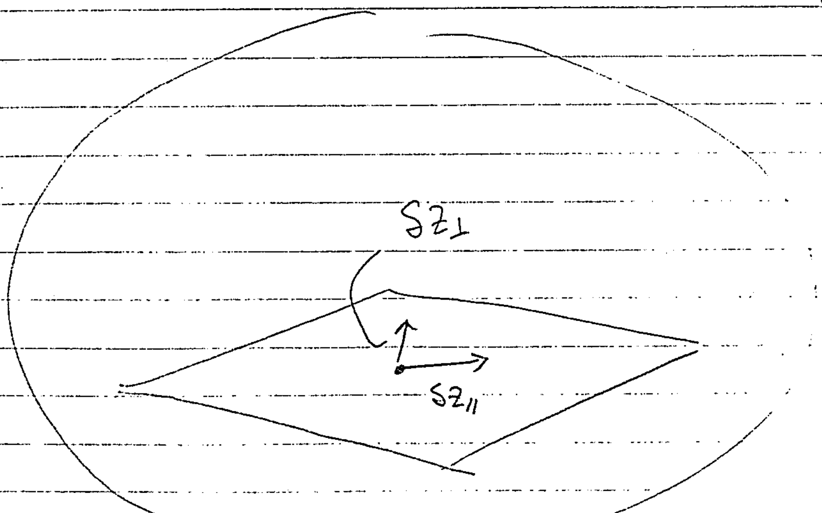
Sometimes the C's are robust constants, e.g. incompressibility or Liouville's theorem. Suppose δz comes about via $H \Rightarrow H + H_{ext}(t)$

$$\dot{z} = J \frac{\partial H_{ext}}{\partial z} \Rightarrow \delta z_{\parallel} \text{ only.}$$

one should evaluate $\delta^2 F \Big|_{\text{constant surface}}$

constant surface $C^{(\alpha)}$'s = const.

add $\big|_c$ above



III. N.C. Field Theory (PDE's)

$$\dot{z}^i = J^{ij} \frac{\partial H}{\partial z^j} \longrightarrow \frac{\partial \psi^i}{\partial t} = J^{ij} \frac{\delta H}{\delta \psi^j}$$

$\psi^i(\underline{x}, t)$ field variable $\quad H[\psi] =$ functional

J^{ij} operator $\quad$ e.g. $\int \frac{1}{2} \rho v^2 d^3x$

$\sum_i \omega_i J_i + \dots$

$\frac{\delta H}{\delta \psi^i}$ functional derivative

$$\frac{d}{d\varepsilon} H[\psi + \varepsilon \delta\psi] \Big|_{\varepsilon=0} = \int \frac{\delta H}{\delta \psi} \delta\psi \, d\underline{x} = \left\langle \frac{\delta H}{\delta \psi}, \delta\psi \right\rangle$$

$$\frac{d}{d\varepsilon} f(\underline{x} + \varepsilon \delta \underline{x}) = \frac{\partial f}{\partial x^i} \delta x_i = \nabla f \cdot \delta \underline{x}$$

Example (Gardner)

KdV: $\quad u_t(x,t) = -u u_x - u_{xxx}$

$$\{F, G\} = \int - \frac{\delta F}{\delta u} \frac{d}{dx} \frac{\delta G}{\delta u} \, dx$$

$$H[u] = \int dx \left(\frac{u^3}{6} - \frac{u_x^2}{2} \right)$$

$$\frac{\delta H}{\delta u} = \frac{u^2}{2} - \frac{d}{dx}\left(-u_x\right) = \frac{u^2}{2} + u_{xx}$$

IV Vlasov - Poisson Equation

$f(x, v, t) \equiv$ phase space den. or dist. fn.

$$\frac{\partial f}{\partial t} = -v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \Phi[x; f]}{\partial x}\frac{\partial f}{\partial v}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = -4\pi e \int f(x, v)\, dv \qquad -\frac{\partial \Phi}{\partial x} = E$$

Energy Expression for NEM

(1) $\delta^2 F = \frac{1}{16\pi} |E(k,\omega)|^2 \omega \frac{\partial \mathcal{E}(k,\omega)}{\partial \omega} \qquad \omega(k)$

Von Laue $\longrightarrow$ Landau & Lifshitz

Can show $\Longleftrightarrow$ Ham. def.

(2) Constants $\Rightarrow$ $\delta^2 F$ $\left\{ \begin{array}{l} \text{analogous to} \\ \text{Arnold theorem (preceded} \end{array} \right.$

also related to Ham. def.

Gardner, Kruskal, Oberman, Newcomb. by 10 yrs.

Equil. prob.

(3) $\delta^2 F\big|_c$ correct way

$\int \frac{E^2}{8\pi}$

$/\!/$

<u>Bracket</u>

$$H[f] = \int \frac{mv^2}{2} f \, dx\,dv + \int \frac{e}{2} f(x,v)\,\Phi(x;f)\,dx\,dv$$

$$C[f] = \int \mathcal{F}(f)\,dx\,dv$$ Liouville's theorem

$$\frac{\delta H}{\delta f} = \frac{mv^2}{2} + e\phi \equiv \mathcal{E}$$

$$\{F,G\} = \int f \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] dx\,dv$$

$$[f,g] = \frac{1}{m}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial x}\right)$$

$$= \int + \frac{\delta F}{\delta f}\left(-[f,\cdot\,]\right)\frac{\delta G}{\delta f}$$

$$J = -[f,\cdot\,]$$

<u>Equil</u>

$$F = H + C$$

$$\frac{\delta F}{\delta f} = 0 \Rightarrow \mathcal{F}'(f_0) + \mathcal{E} = 0$$

$$\Rightarrow f_0 = \mathcal{F}'^{-1}(-\mathcal{E})$$ <u>Monotonic</u>! only

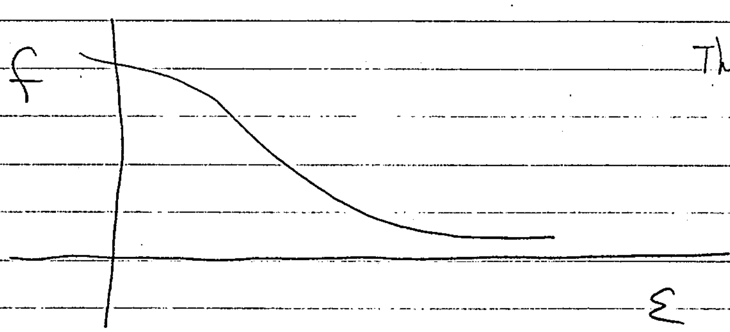
Equil. Prob.

$$\delta^2 F = \frac{1}{2}\int \mathcal{J}''(f_0)\,\delta f^2\,dx\,dv + \frac{1}{8\pi}\int (\delta E)^2 dx$$

$$\frac{\partial f}{\partial \varepsilon}\mathcal{J}''(f) = -1$$

$$\Rightarrow \quad \delta^2 F = \frac{1}{2}\int \frac{(\delta f)^2}{\left(-\frac{\partial f}{\partial \varepsilon}\right)} + \frac{1}{8\pi}\int (\delta E)^2 dx$$

$$\frac{\partial f}{\partial \varepsilon} < 0 \quad \Rightarrow \quad \text{stable}$$



Thermal Equil

$f \sim e^{-v^2/v_\theta^2}$

$\varepsilon = \frac{mv^2}{2}$

What about case where $\partial f/\partial \varepsilon$ not monotonic?

<u>General Equil.</u>

Clearly

$$\frac{\partial f}{\partial t} = 0 \quad \Rightarrow \quad f_0(\varepsilon)$$

$\uparrow$ arb.

How can these be handled?

Extremize H subject to <u>all</u> C's being Constant, w/o using Lag. multiplier.

$$\delta C[f;\delta f] = \int \frac{\partial \mathcal{F}(f)}{\partial f}\,\delta f\,dx\,dv = 0$$

Achieved if $\delta f = [G, f_0]$

$\uparrow$ arb. generating fn.

$$\delta F = \int \mathcal{E}\,\delta f\,dx\,dv + \int \mathcal{F}'\delta f$$

$$= \int \{ \mathcal{E}[G, f_0] + \cancel{\mathcal{F}'(f_0)[G, f_0]}_{\,0} \}\,dx\,dv$$

$$= \int G\,[f_0, \mathcal{E}]\,dx\,dv = 0 \quad \forall G \implies$$

$$[f_0, \mathcal{E}] = 0 \implies f_0(\mathcal{E}) \quad \text{arb.}!$$

Now one can calculate the energy to 2nd order if one δf s.t. $\delta^2 C = 0$ to 2nd order.

$$\delta^{(1)} f = [G^{(1)}, f_0]$$

$$\delta^{(2)} f = [G^{(2)}, f_0] + \frac{1}{2}[G^{(1)}, [G^{(1)}, f_0]]$$

Always for Ham. Systems in terms of 1st order Quantities

$\Longrightarrow$

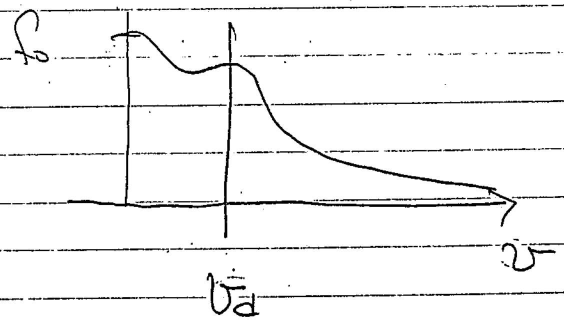
$$\delta^2 F = \int -\frac{1}{2}[G, \varepsilon]^2 \frac{\partial f_0}{\partial \varepsilon}\, dx\, dv + \int \frac{\delta E^2}{8\pi}\, dx$$

$$\boxed{\delta^2 F\Big|_c = \int \frac{1}{2}[G, f_0][\varepsilon, G]\, dx\, dv + \int \frac{\delta E^2}{8\pi}\, dx}$$

$G \equiv G^{(1)}$

Penrose Criterion

$\varepsilon = \frac{m v^2}{2}$



NEM

$$\left\{ \begin{array}{c} 0 < v_d < v_* \\ \text{spectrally stable} \end{array} \right\} \quad f_0'(v_d) = 0$$

$$\delta^2 F = \int \frac{\delta E^2}{8\pi}\, dx - \frac{m}{2} \int \left(\frac{\partial G}{\partial x}\right)^2 v \frac{\partial f_0}{\partial v}\, dv\, dx$$

indef. $\boxed{\text{Stable ? Probably Not}}$

$\boxed{\text{FEP}}$

All ideal Fluid & Plasma Models

V

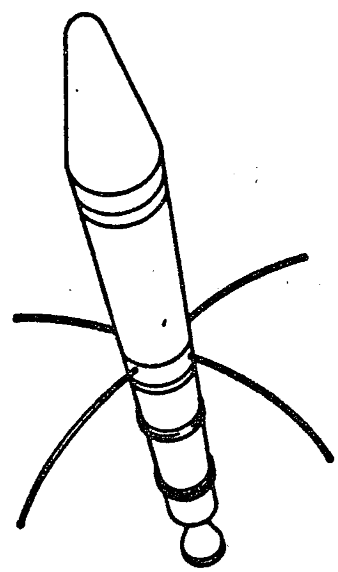
Dissipation of Negative Energy Modes $\Rightarrow$ Spectral Instability

Appealing Intuitive Idea: take energy out $\Rightarrow$ $|E|$ bigger $\Rightarrow$ growth.

Kelvin-Tait Theorem (1879)

Attitude Stabilization of Spacecraft (1958)

Explorer I $\longrightarrow$



Plasma Physics

Greene and Coppi (1965)
FLR stabilization

⋮

Lore

Kelvin - Tait Theorem

$$A_{ij}\ddot{x}_j + G_{ij}\dot{x}_j + K_{ij}x_j = -D_{ij}\dot{x}_j$$

Hamiltonian

dissipation

A symmetric positive definite - mass

G antisymmetric - Gyroscopic term

K symmetric - spring constant

Completeness

D symmetric positive definite

all eigenvalues positive.

Is this generally implied by the 2nd Law? No!

Simple Example of

Dissipation of Negative Energy Modes

Dissipation $\Rightarrow \quad \frac{dH}{dt} < 0$

2nd Law of Thermodynamics

$$H = \frac{q_+^2 + p_+^2}{2} - \frac{(q_-^2 + p_-^2)}{2}$$

$$\dot{q}_+ = p_+ \qquad \dot{q}_- = -p_-$$

$$\dot{p}_+ = -q_+ - \nu_+ p_+ \qquad \dot{p}_- = q_- + \nu_- p_-$$

$$\sim e^{i\omega t} \qquad \text{small } \nu_i > 0_{\,i=\pm}$$

$$\omega_+ \to \omega_+ + i\frac{\nu_+}{2} \qquad \omega_- \to \omega_- - i\frac{\nu_-}{2}$$

damping

growth

Boltzmann Equation - Completeness?

$$\frac{df}{dt} = \left.\frac{\partial f}{\partial t}\right)_c$$

$f \equiv$ phase space density

$$F = E + \lambda \underbrace{\int f \ln f}_{\equiv\, S_T}$$

S_T & $\left.\frac{\partial f}{\partial t}\right)_c$ matched so as

$\longrightarrow \quad \delta F = 0 \Rightarrow f_M$ Thermal Equilibrium

$\frac{dF}{dt} \leq 0$ asymptotic stability

Suppose

$$F = E + S_A \qquad S_A \neq S_T$$

$\delta F = 0 \Rightarrow f_e \neq f_m$ maintained by stationary source

S_A & $\left.\frac{\partial f}{\partial t}\right)_c$ unmatched

Maybe not

GENERAL CASE

Consider a noncanonical Hamiltonian System

$$\dot{z}^i = J^{ij}(z) \frac{\partial F}{\partial z^j}(z) + D^i(z)$$

Hamiltonian

dissipation

e.g. viscosity

"Second Law" ($F = H + C$)

$$\frac{dH}{dt} = \frac{\partial H}{\partial z^i} D^i \leq 0$$

<u>Linearize</u> about an equilibrium $z = z_e + y$

Equil. $$0 = J^{ij}(z_e) \frac{\partial F}{\partial z^j}(z_e) + D^i(z_e) + S^i$$

Source term e.g. tok. E-field

$$\frac{\partial F}{\partial z^j}(z_e) = 0$$

$$S^i = -D^i(z_e)$$

1st order

$$\dot{y}^i = J^{ij} \frac{\partial^2 F}{\partial z^j \partial z^\ell} y^\ell + \frac{\partial D^i}{\partial z^\ell} y^\ell$$

Case of interest: Neg. Energy Mode

$*$ $\delta^2 F = \frac{1}{2} \frac{\partial^2 F}{\partial z^i \partial z^\ell} y^i y^\ell$ <u>indefinite</u>

$*$ $\dot{y} = J \frac{\partial^2 F}{\partial z^2} y + \boxed{\frac{\partial D}{\partial z} y}$ <u>real spectrum w/ $D=0$</u>

Dissipation means

$$\frac{d}{dt} \delta^2 F \le 0 ,$$

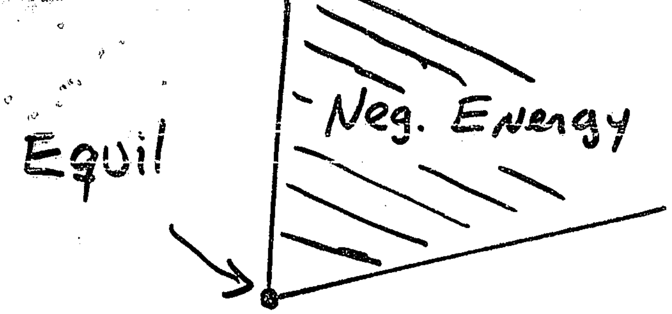
But

$$\frac{d}{dt} \delta^2 F = \frac{\partial^2 F}{\partial z^i \partial z^j} \frac{\partial D^j}{\partial z^\ell} y^i y^\ell = ?$$

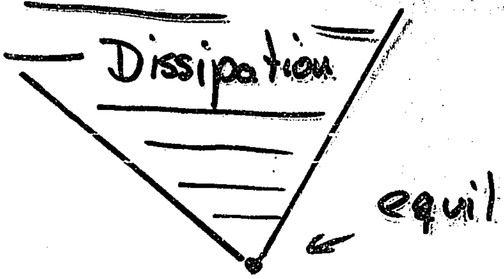
In general

$\frac{dH}{dt} = \frac{\partial H}{\partial z^i} D^i \le 0$ implies nothing about

$\frac{d}{dt} \delta^2 F$!



Positive energy



$\frac{d}{dt}\,\delta^2 F > 0$

For stability

$$\{N.E.\} \cap \{\text{Dissip}\} = \phi \quad ?$$

Generically a negative energy mode will get dissipated.

* must check
sometimes not.
compare w/ dissip.
even if slow → matter under extreme conds.

Caveats

* Nonlinear Instability may be slow
* Energy is not frame invariant
  - Usually signature is - otherwise check
* Hidden Invariants
  - besides energy, momentum
    unlikely for physical systems
* Negative energy modes may not
  get dissipated - Generic ?
* For P.D.E.'s dissipation may give rise to new modes - Singular modes not present in the ideal system. Perhaps most dangerous. (tearing modes - Rayleigh vs. Orr-Sommerfeld Eqs.)